Quantum Systems: Dynamics and Control[1]

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Outline

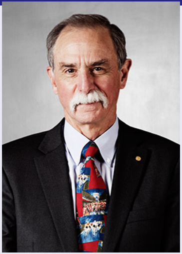
Controlling quantum degrees of freedom

Some applications

- Nuclear Magnetic Resonance (NMR) applications;
- Quantum chemical synthesis;
- High resolution measurement devices (e.g. atomic/optic clocks);
- Quantum communication;
- Quantum computation .

Physics Nobel prize 2012

Serge Haroche

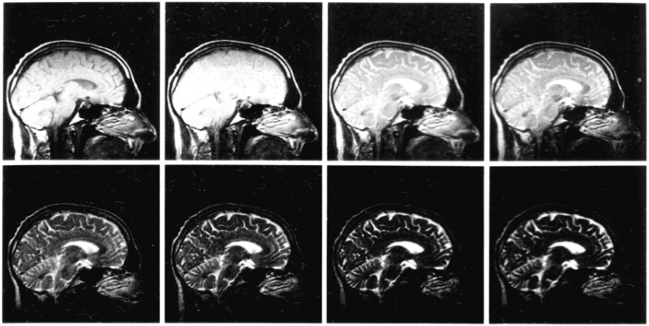


David J. Wineland

Nobel prize: ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems.

Nuclear Magnetic Resonance

- Control of an ensemble of spins with a dispersion (uncertainty) in parameters (frequency, coupling strengths);
- Time-optimal control to beat the relaxation of the spins;
- . . .



Improving the contrast in Magnetic Resonance Imaging (MRI).

Atomic clocks

SI second is defined to be "the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom".

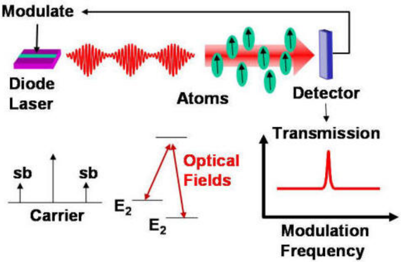
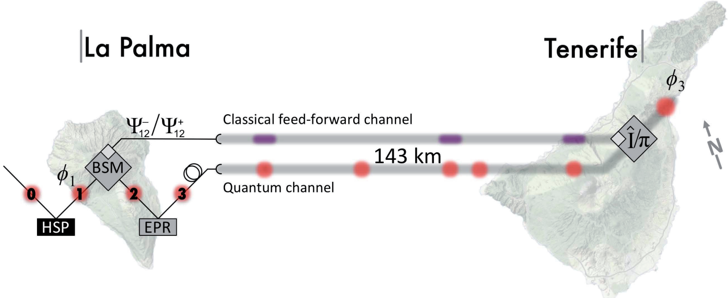


Figure: http://tf.nist.gov/ofm/smallclock/OverallDesign.htm

The goal is to modulate the laser frequencies to reach the maximum transmission (minimum absorbtion).

Quantum communication

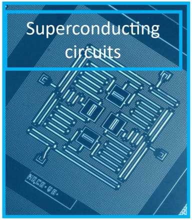
- Secure communication enforced by laws of quantum physics;



Quantum teleportation between two canary islands.
Courtesy of X. Ma *et. al.* Nature (2012)

- Quantum repeaters for long-distance (>100km) communication: requires a quantum memory where quantum information is stabilized (protected) against various noise sources.

Technologies for quantum simulation and computation[5]



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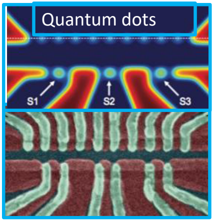

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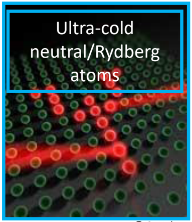

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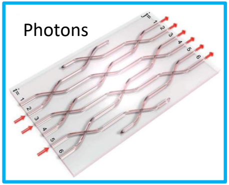

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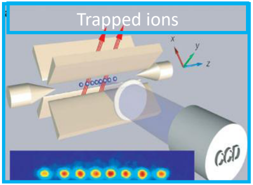

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Requirement:

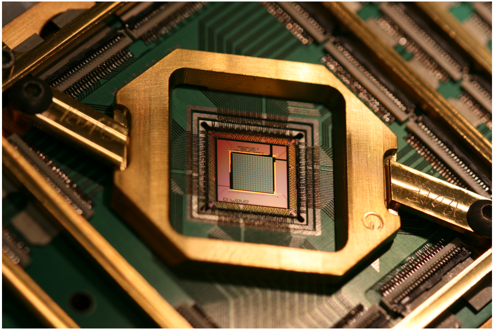
Scalable modular architecture

Control software from the very beginning.

[5] Courtesy of Walter Riess, IBM Research - Zurich.

Quantum computation: towards quantum electronics

D-Wave machine: machines to solve certain huge-dimensional optimization problems (state space of dimension 2^{100}).



Major challenge: Fragility of quantum information versus external noise.

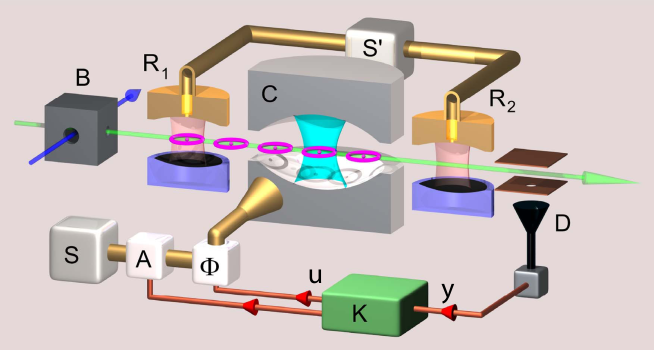
Quantum error correction

We protect quantum information by stabilizing a manifold of quantum states.

Outline

The first experimental realization of a **quantum state feedback**

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche (Nobel Prize 2012), J.M.Raimond and M. Brune.[6]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).
Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[6] Courtesy of Igor Dotsenko. **Sampling period $\Delta t \approx 80\mu s$.**

C. Sayrin et. al., Nature 477, 73-77, Sept. 2011.

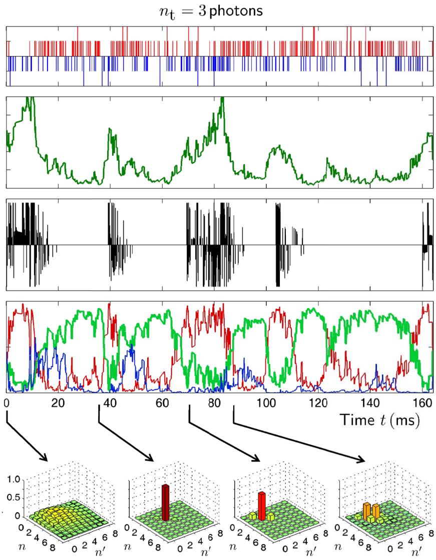
Decoherence due to finite photon life time around 70 ms)

Detection efficiency 40%
Detection error rate 10%
Delay 4 sampling periods

Model includes cavity decoherence, measurement imperfections, delays (Bayes law).

Truncation to 9 photons

Stabilization around 3-photon state



Models of open quantum systems are based on three features[7]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H},\rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Entanglement and tensor product for composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.
3. Randomness and irreversibility induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}(\rho\boldsymbol{P}_\mu)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y\rho\boldsymbol{P}_y}{\text{Tr}(\rho\boldsymbol{P}_y)}$$

[7] S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \;\middle|\; (c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

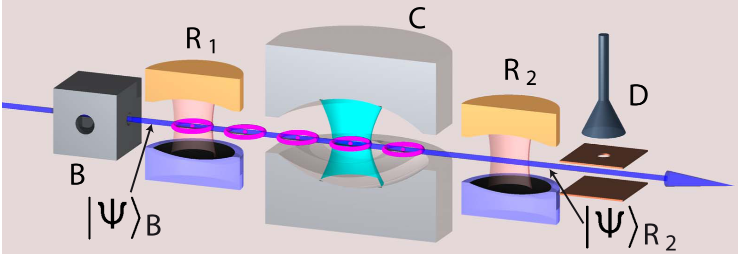
  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity

- Meter M is a qu-bit, a 2-level system (idem 1/2 spin system) : $\mathcal{H}_M = \mathcal{H}_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$

- **State of the full system** $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M = \mathcal{H}_c \otimes \mathcal{H}_a$:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} c_{ng}|n\rangle \otimes |g\rangle + c_{ne}|n\rangle \otimes |e\rangle, \quad c_{ne}, c_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n\in\mathbb{N}}$.



- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

The Markov model (2)

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \left\langle \psi | \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu | \psi \right\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \dfrac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle}} & \text{with } y_k = g, \text{ probability } \mathbb{P}_g = \left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle; \\ \dfrac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle}} & \text{with } y_k = e, \text{ probability } \mathbb{P}_e = \left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle. \end{cases}$$

Outline

Reference books

1. Cohen-Tannoudji, C.; Diu, B. & Laloë, F.: Mécanique Quantique Hermann, Paris, 1977, I& II (*quantum physics: a well known and tutorial textbook*)
2. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006. (*quantum physics: spin/spring systems, decoherence, Schrödinger cats, entanglement.*)
3. C. Gardiner, P. Zoller: The Quantum World of Ultra-Cold Atoms and Light I& II. Imperial College Press, 2009. (*quantum physics, measurement and control*)
4. Barnett, S. M. & Radmore, P. M.: Methods in Theoretical Quantum Optics Oxford University Press, 2003. (*mathematical physics: many useful operator formulae for spin/spring systems*)
5. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976. (*mathematical physics: functional analysis aspects when the Hilbert space is of infinite dimension*)
6. Gardiner, C. W.: Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences [3rd ed], Springer, 2004. (*tutorial introduction to probability, Markov processes, stochastic differential equations and Ito calculus.*)
7. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000. (*tutorial introduction with a computer science and communication view point*)

Outline of the lectures

Jan21 (PR)	Introduction: motivating applications; LKB photon-box as prototype of open quantum system; spring system (harmonic oscillator, spectral decomposition, annihilation/creation operators, coherent state and displacement).
Jan28 (AS)	spin system (qubit, Pauli matrices); composite spin/spring system (tensor product, resonant/dispersive interaction, underlying PDE's).
Feb04 (PR)	Averaging and rotating waves approximation (first/second order perturbation expansion, asymptotic stability)
Feb11 (AS)	Open-loop control via averaging techniques (Rabi oscillations for a qubit, Law-Eberly method for a spin/spring system)
Feb18 (PR)	Adiabatic control (qubit with Bloch vector coordinates, STIRAP) and optimal control (monotone numerical algorithm).
Feb25 (AS)	Measurement back-action, POVM and discrete-time model of open quantum system: LKB photon box, measurement imperfection, why density operator instead of wave-function, Kraus map (quantum channel).
Mar03 (AS)	Discrete-time open-quantum systems: LKB photon box, (QND) measurement, open-loop asymptotic behavior, measurement-based feedback, Lyapunov stabilization, quantum filtering.
Mar10 (PR)	Continuous-time open-quantum system: Ito calculus, homodyne (QND) measurement, open-loop asymptotic behaviour, Lyapunov stabilization, quantum filtering.
Mar17 (PR)	Decoherence as unread measurements performed by the environment: continuous-time formulation, Lindblad differential equation (damped harmonic oscillator, convergence and asymptotic stability, Wigner function and PDE formulations).
Mar24 (AS)	Stabilization by reservoir engineering as tailored decoherence. Application: quantum error correction by measurement-based feedback and by reservoir engineering. Outlook on related math.techniques (adiabatic elimination).

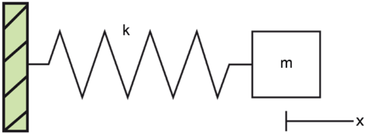
Outline

Harmonic oscillator

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

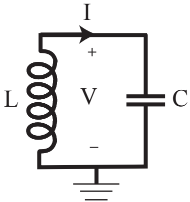
$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Mechanical oscillator



Frictionless spring: $\frac{d^2}{dt^2}x = -\frac{k}{m}x.$

Electrical oscillator:



LC oscillator:

$$\frac{d}{dt}I = \frac{V}{L}, \ \frac{d}{dt}V = -\frac{I}{C}, \quad \left(\frac{d^2}{dt^2}I = -\frac{1}{LC}I\right).$$

Quantum regime

$k_B T \ll \hbar\omega$ where $\hbar \simeq 1.054 \cdot 10^{-34}$ *Js* and $k_B \simeq 1.38 \cdot 10^{-23}$ J/K. Typically for the photon box experiment in these lectures, $\omega = 51 GHz$ and $T = 0.8K$.

$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation (from now on we always assume units such that $\hbar = 1$)

$$i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where $\boldsymbol{H}$ results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}\boldsymbol{X}$ and p by momentum operator $\sqrt{2}\boldsymbol{P} = -i\frac{\partial}{\partial x}$. $\boldsymbol{H}$ is a Hermitian operator on $L^2(\mathbb{R},\mathbb{C})$, with its domain to be given.

PDE model: $i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$

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[8]Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Average position $\langle \boldsymbol{X} \rangle_t = \langle \psi | \boldsymbol{X} | \psi \rangle$ and momentum $\langle \boldsymbol{P} \rangle_t = \langle \psi | \boldsymbol{P} | \psi \rangle$:

$$\langle \boldsymbol{X} \rangle_t = \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle \boldsymbol{P} \rangle_t = -\tfrac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation $\boldsymbol{a}$ and creation operators $\boldsymbol{a}^\dagger$ (domains to be given):

$$\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad \boldsymbol{a}^\dagger = \boldsymbol{X} - i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[\boldsymbol{X}, \boldsymbol{P}] = \tfrac{i}{2} \boldsymbol{I}, \quad [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = \omega \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right).$$

Set $\boldsymbol{X}_\lambda = \frac{1}{2} \left(e^{-i\lambda} \boldsymbol{a} + e^{i\lambda} \boldsymbol{a}^\dagger \right)$ for any angle λ:

$$\left[\boldsymbol{X}_\lambda, \boldsymbol{X}_{\lambda + \frac{\pi}{2}} \right] = \tfrac{i}{2} \boldsymbol{I}.$$

Spectrum of Hamiltonian $\boldsymbol{H} = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$:

$$E_n = \omega(n+\frac{1}{2}), \ \ \psi_n(x) = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \ \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Spectral decomposition of $\boldsymbol{a}^\dagger \boldsymbol{a}$ using $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$:

- If $|\psi\rangle$ is an eigenstate associated to eigenvalue λ, then $\boldsymbol{a}|\psi\rangle$ and $\boldsymbol{a}^\dagger|\psi\rangle$ are also eigenstates associated to $\lambda - 1$ and $\lambda + 1$.
- $\boldsymbol{a}^\dagger \boldsymbol{a}$ is semi-definite positive.
- The ground state $|\psi_0\rangle$ is necessarily associated to eigenvalue 0 and is given by the Gaussian function $\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$.

$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$: spectrum of $\boldsymbol{a}^\dagger \boldsymbol{a}$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photons (phonons): the eigenstate of $\boldsymbol{a}^\dagger \boldsymbol{a}$ associated to the eigenvalue n ($|n\rangle \sim \psi_n(x)$):

$$\boldsymbol{a}^\dagger \boldsymbol{a}|n\rangle = n|n\rangle, \quad \boldsymbol{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ is called 0-photon state or vacuum state.

The operator $\boldsymbol{a}$ (resp. $\boldsymbol{a}^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Hilbert space of quantum system: $\mathcal{H} = \{\sum_n c_n|n\rangle \mid (c_n) \in l^2(\mathbb{C})\} \sim L^2(\mathbb{R}, \mathbb{C})$.
Domain of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^1(\mathbb{C})\}$.
Domain of $\boldsymbol{H}$ or $\boldsymbol{a}^\dagger \boldsymbol{a}$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^2(\mathbb{C})\}$.

$$h^k(\mathbb{C}) = \{(c_n) \in l^2(\mathbb{C}) \mid \sum n^k |c_n|^2 < \infty\}, \qquad k = 1, 2.$$

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$, ($\mathbb{H} = \frac{\omega}{2}(p^2 + x^2) + \sqrt{2}ux$)

$$\boldsymbol{H} = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\tfrac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator $\boldsymbol{D}_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a}} = e^{2i\Im\alpha\boldsymbol{X} - 2i\Re\alpha\boldsymbol{P}}$$

From Baker-Campbell Hausdorf formula, for all operators $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$e^{\boldsymbol{A}}\boldsymbol{B}e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \tfrac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \tfrac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots$$

we get the Glauber formula[9] when $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$:

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\, e^{\boldsymbol{B}}\, e^{-\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]}.$$

[9] Take s derivative of $e^{s(\boldsymbol{A}+\boldsymbol{B})}$ and of $e^{s\boldsymbol{A}}\, e^{s\boldsymbol{B}}\, e^{-\frac{s^2}{2}[\boldsymbol{A},\boldsymbol{B}]}$.

Harmonic oscillator: identities resulting from Glauber formula

With $\boldsymbol{A} = \alpha \boldsymbol{a}^\dagger$ and $\boldsymbol{B} = -\alpha^* \boldsymbol{a}$, Glauber formula gives:

$$\boldsymbol{D}_\alpha = e^{-\frac{|\alpha|^2}{2}} \, e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} = e^{+\frac{|\alpha|^2}{2}} \, e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger}$$

$$\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}_{-\alpha} \boldsymbol{a}^\dagger \boldsymbol{D}_\alpha = \boldsymbol{a}^\dagger + \alpha^* \boldsymbol{I}.$$

With $\boldsymbol{A} = 2i\Im\alpha \boldsymbol{X} \sim i\sqrt{2}\Im\alpha x$ and $\boldsymbol{B} = -2\imath\Re\alpha \boldsymbol{P} \sim -\sqrt{2}\Re\alpha \frac{\partial}{\partial x}$, Glauber formula gives[10]:

$$\boldsymbol{D}_\alpha = e^{-i\Re\alpha\Im\alpha} \, e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha \frac{\partial}{\partial x}}$$

$$(\boldsymbol{D}_\alpha |\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha} \, e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

Exercise: Prove that, for any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have[11]

$$\boldsymbol{D}_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} \boldsymbol{D}_\alpha \boldsymbol{D}_\beta$$

$$\boldsymbol{D}_{\alpha+\epsilon} \boldsymbol{D}_{-\alpha} = \left(1 + \frac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right) \boldsymbol{I} + \epsilon \boldsymbol{a}^\dagger - \epsilon^* \boldsymbol{a} + \boldsymbol{O}(|\epsilon|^2)$$

$$\left(\frac{d}{dt}\boldsymbol{D}_\alpha\right) \boldsymbol{D}_{-\alpha} = \left(\frac{\alpha \frac{d}{dt}\alpha^* - \alpha^* \frac{d}{dt}\alpha}{2}\right) \boldsymbol{I} + \left(\frac{d}{dt}\alpha\right) \boldsymbol{a}^\dagger - \left(\frac{d}{dt}\alpha^*\right) \boldsymbol{a}.$$

[10] Note that the operator $e^{-r\partial/\partial x}$ corresponds to a translation of x by r.

[11] Use the formula $\frac{d}{dt}\boldsymbol{E}(t) = \left(\int_0^1 e^{s\boldsymbol{A}(t)} \left(\frac{d}{dt}\boldsymbol{A}(t)\right) e^{-s\boldsymbol{A}(t)} ds\right) \boldsymbol{E}(t)$ where $\boldsymbol{E}(t) = e^{\boldsymbol{A}(t)}$ for any operator $\boldsymbol{A}(t)$ depending smoothly on t.

Harmonic oscillator: lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)\right)|\psi\rangle$. Set $\langle \boldsymbol{a}\rangle = \langle\psi|\boldsymbol{a}|\psi\rangle$. Then

$$\frac{d}{dt}\langle \boldsymbol{a}\rangle = -i\omega\langle \boldsymbol{a}\rangle - iu.$$

From $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P}$, we have $\langle \boldsymbol{a}\rangle = \langle \boldsymbol{X}\rangle + i\langle \boldsymbol{P}\rangle$ where $\langle \boldsymbol{X}\rangle = \langle\psi|\boldsymbol{X}|\psi\rangle \in \mathbb{R}$ and $\langle \boldsymbol{P}\rangle = \langle\psi|\boldsymbol{P}|\psi\rangle \in \mathbb{R}$. Consequently

$$\frac{d}{dt}\langle \boldsymbol{X}\rangle = \omega\langle \boldsymbol{P}\rangle, \quad \frac{d}{dt}\langle \boldsymbol{P}\rangle = -\omega\langle \boldsymbol{X}\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t}\boldsymbol{D}_{\langle \boldsymbol{a}\rangle_t}|\chi\rangle$ with

$$\theta_t = \int_0^t \left(\omega|\langle \boldsymbol{a}\rangle|^2 + u\Re(\langle \boldsymbol{a}\rangle)\right), \quad D_{\langle \boldsymbol{a}\rangle_t} = e^{\langle \boldsymbol{a}\rangle_t \boldsymbol{a}^\dagger - \langle \boldsymbol{a}\rangle_t^* \boldsymbol{a}},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation[12]

$$i\frac{d}{dt}|\chi\rangle = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle \boldsymbol{a}\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

[12] The time-varying change of frame $|\psi\rangle = \boldsymbol{U}|\chi\rangle$ where $\frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{A}\boldsymbol{U}$ with $\boldsymbol{A}^\dagger \equiv \boldsymbol{A}$, transforms $\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle$ into $|\chi\rangle = -i(\boldsymbol{U}^\dagger(\boldsymbol{H} - \boldsymbol{A})\boldsymbol{U})|\chi\rangle$.

Coherent states

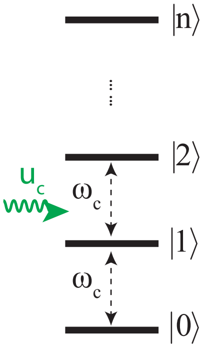
$$|\alpha\rangle = \boldsymbol{D}_\alpha|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigenstate of $\boldsymbol{a}$: $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[13]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum.

[13] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.

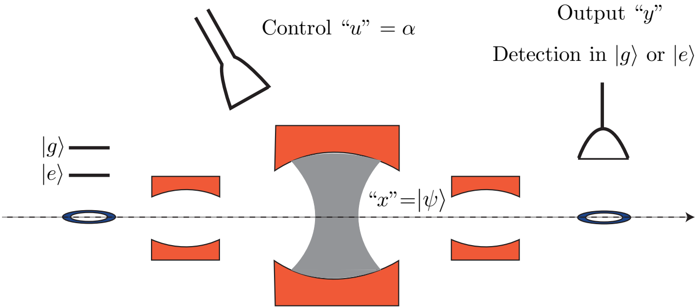
Summary for the quantum harmonic oscillator

- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^\dagger\boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath\boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_c\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a}+\boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2}u_c$).



- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}:\ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}}e^{\imath\sqrt{2}x\Im\alpha}e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

Control of quantum harmonic oscillator: LKB photon-box



Simple schematic of LKB experiment for control of cavity field