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M2 Mathématiques & Applications
UE (ANEDP, COCV): Analyse et contrôle de systèmes quantiques
Corrigé du Contrôle des connaissances
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Propagator of a damped and driven quantum harmonic oscillator

We consider a quantum harmonic oscillator with annihilation operator $\boldsymbol{a}$, photon number operator $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, pulsation ω, damping time $1/\kappa$. This oscillator is driven by a coherent drive of complex amplitude u and pulsation $\omega_d = \omega_c - \Delta$ (Δ being the detuning between the drive of pulsation ω_d and the oscillator of pulsation ω_c). Its density operator ρ obeys to the following Lindbald master equation:

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] - \imath\Delta[\boldsymbol{N}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}(\boldsymbol{N}\rho + \rho\boldsymbol{N})\right).$$

We denote by $\rho(t)$ the solution starting from an initial density operator $\rho_0 = \rho(0)$.

1. Assume $u = 0$, $\Delta = 0$ and $\kappa > 0$.
   (a) What is the limit of $\rho(t)$ for t tending to $+\infty$?
   (b) Show that $\rho(t)$ admits the following expression (do not consider convergence issues for the series)

$$\rho(t) = \sum_{n=0}^{+\infty} \left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} \boldsymbol{a}^n \, \rho_0 \, (\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}.$$

2. Assume $u = 0$, $\Delta \neq 0$ and $\kappa > 0$.
   (a) What is the limit of $\rho(t)$ for t tending to $+\infty$?
   (b) Show that $\rho(t)$ admits the following expression

$$\rho(t) = \sum_{n=0}^{+\infty} \left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}} \boldsymbol{a}^n \, \rho_0 \, (\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}}.$$

3. Assume $u \neq 0$, $\Delta = 0$ and $\kappa > 0$.
   (a) What is the limit of $\rho(t)$ for t tending to $+\infty$?
   (b) Show that $\rho(t)$ admits the following expression

$$\rho(t) = \sum_{n=0}^{+\infty} \left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) \boldsymbol{D}_\alpha \left(e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} \boldsymbol{a}^n\right) \boldsymbol{D}_{-\alpha} \, \rho_0 \, \boldsymbol{D}_\alpha \left((\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\right) \boldsymbol{D}_{-\alpha}$$

   where $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger - \alpha^*\boldsymbol{a}}$ is the displacement of complex amplitude α. What is the expression of α versus u and κ.

4. Assume $u \neq 0$, $\Delta \neq 0$ and $\kappa > 0$.
   (a) To what kind of frame corresponds the above Lindblad master equation ?
   (b) What is the limit of $\rho(t)$ for t tending to $+\infty$?
   (c) Show that $\rho(t)$ admits the following expression

$$\rho(t) = \sum_{n=0}^{+\infty} \left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) \boldsymbol{D}_\alpha \left(e^{-(\frac{\kappa}{2}+\imath\Delta)t\boldsymbol{N}} \boldsymbol{a}^n\right) \boldsymbol{D}_{-\alpha}\, \rho_0\, \boldsymbol{D}_\alpha \left((\boldsymbol{a}^\dagger)^n e^{-(\frac{\kappa}{2}-\imath\Delta)t\boldsymbol{N}}\right) \boldsymbol{D}_{-\alpha}.$$

   What is here the expression of α versus u, Δ and κ ?

Dissipation induced dephasing

We consider a harmonic oscillator coupled dispersively to a single qubit. In the rotating frame of the qubit and the cavity the Hamiltonian is given by

$$\boldsymbol{H}_{\text{disp}} = -\frac{\chi}{2}\boldsymbol{\sigma_z} \otimes \boldsymbol{a}^\dagger \boldsymbol{a}.$$

Furthermore, we assume the cavity to be dissipative so that the total dynamics of the system for the density matrix $\boldsymbol{\rho}$ is given by

$$\frac{d}{dt}\boldsymbol{\rho} = -i[\boldsymbol{H}_{\text{disp}}, \boldsymbol{\rho}] + \kappa(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}).$$

In the sequel, we will assume that $\chi \gg \kappa$.

1. Write the system in the rotating frame of $\boldsymbol{H}_{\text{disp}}$ (i.e. give the dynamics of $\boldsymbol{\xi} = \exp(i\boldsymbol{H}_{\text{disp}}t)\boldsymbol{\rho}\exp(-i\boldsymbol{H}_{\text{disp}}t)$). Simplify the dynamics using the rotating wave approximation and knowing that $\chi \gg \kappa$.

2. Consider the system initialized in the separable state $\rho_q \otimes \rho_c$ with $\rho_q = |\psi_q\rangle\langle\psi_q|$ and an arbitrary cavity state ρ_c. Furthermore take $|\psi_q\rangle = c_g|g\rangle + c_e|e\rangle$. What is the steady state of the above simplified system towards which the solution converges? Interpret the result.
   (Hint: start by writing $\boldsymbol{\xi}(t) = |e\rangle\langle e| \otimes \boldsymbol{\xi}_{ee}(t) + |g\rangle\langle g| \otimes \boldsymbol{\xi}_{gg}(t) + |e\rangle\langle g| \otimes \boldsymbol{\xi}_{eg}(t) + |g\rangle\langle e| \otimes \boldsymbol{\xi}_{ge}(t)$.)

3. Interpret the result.

Do and undo an entangled state between two harmonic oscillators

We consider two harmonic oscillators of annihilation operators $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, photon-number operators $\boldsymbol{N}_1 = \boldsymbol{a}_1^\dagger\boldsymbol{a}_1$ and $\boldsymbol{N}_2 = \boldsymbol{a}_2^\dagger\boldsymbol{a}_2$, interacting sequentially with a qubit of ground state $|g\rangle$ and excited state $|e\rangle$.

- Firstly the qubit interacts with oscillator 1 according to the unitary operator (resonant interaction with vacuum Rabi angle θ_1):

$$\boldsymbol{U}_1 = |g\rangle\langle g|\, \cos(\theta_1\sqrt{\boldsymbol{N}_1}) + |e\rangle\langle e|\, \cos(\theta_1\sqrt{\boldsymbol{N}_1 + 1}) - |e\rangle\langle g|\boldsymbol{a}_1\, \frac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}} + |g\rangle\langle e|\, \frac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\, \boldsymbol{a}_1^\dagger.$$

- After its interaction with oscillator 1, the same qubit interacts then with oscillator 2 according to the unitary operator (resonant interaction with vacuum Rabi angle θ_2):

$$\boldsymbol{U}_2 = |g\rangle\langle g| \cos(\theta_2\sqrt{\boldsymbol{N}_2}) + |e\rangle\langle e| \cos(\theta_2\sqrt{\boldsymbol{N}_2+1}) - |e\rangle\langle g|\boldsymbol{a}_2 \frac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}} + |g\rangle\langle e| \frac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}} \boldsymbol{a}_2^\dagger.$$

- After its interaction with oscillator 2, the same qubit is measured according to its energy operator $|e\rangle\langle e| - |g\rangle\langle g|$. The measurement outcome is denoted by $y \in \{g, e\}$.

Before the interaction with the qubit, the wave function of the composite system made of the two oscillators is denoted by $|\psi\rangle$. Its admits the following expression in the photon-number basis of each oscillators

$$|\psi\rangle = \sum_{n_1,n_2\geq 0} \psi_{n_1,n_2}|n_1 n_2\rangle \text{ with } \sum_{n_1,n_2\geq 0} |\psi_{n_1,n_2}|^2 = 1.$$

Just after qubit measurement, the wave function of the two oscillators is denoted by $|\psi\rangle_+$.

1. Before its interactions with the two oscillators, the qubit is prepared in $|e\rangle$.
   (a) Express $|\psi\rangle_+$ with respect to $|\psi\rangle$ and measurement outcome y. What are the probabilities to detect y knowing $|\psi\rangle$.
   (b) Assume in this question that $\theta_1 = \pi/4$, $\theta_2 = \pi/2$ and $|\psi\rangle = |00\rangle$. What are $|\psi\rangle_+$ and the probabilities to detect y. Interpret the result.

2. Before its interactions with the two oscillators, the qubit is prepared in $|g\rangle$.
   (a) Express $|\psi\rangle_+$ with respect to $|\psi\rangle$ and measurement outcome y. What are the probabilities to detect y knowing $|\psi\rangle$.
   (b) Assume that $|\psi\rangle = |00\rangle$ and θ_1, θ_2 arbitrary. What are $|\psi\rangle_+$ and the probabilities to detect y. Interpret the result.
   (c) We assume in this question that $\theta_1 = \pi/2$, $\theta_2 = \pi/4$ and $|\psi\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$. What are $|\psi\rangle_+$ and the probabilities to detect y. Interpret the result according to question 1b.

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1. (a) $\rho(t)$ converges towards vacuum, i.e., $|0\rangle\langle 0|$ where $|0\rangle$ is the 0-photon quantum state $\boldsymbol{a}^\dagger\boldsymbol{a}|0\rangle = 0$.

   (b) Derivation with respect to t of the term indexed by n yields

$$\begin{aligned}\frac{d}{dt}\left[\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\right] = &\\ \kappa e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^{n-1}}{(n-1)!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} &\\ -\tfrac{\kappa}{2}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\boldsymbol{N}e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} &\\ -\tfrac{\kappa}{2}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{N} &\end{aligned}$$

   Assume that $\rho(t)$ is given by the series. Then

$$\begin{aligned}\frac{d}{dt}\rho = \sum_{n=0}^{+\infty} \kappa e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^{n+1}\,\rho_0\,(\boldsymbol{a}^\dagger)^{n+1} e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} &\\ -\tfrac{\kappa}{2}\boldsymbol{N}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} &\\ -\tfrac{\kappa}{2}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{N} &\\ = \sum_{n=0}^{+\infty} \kappa e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^{n+1}\,\rho_0\,(\boldsymbol{a}^\dagger)^{n+1} e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}} - \tfrac{\kappa}{2}(\boldsymbol{N}\rho + \rho\boldsymbol{N}). &\end{aligned}$$

   Using $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+1)\boldsymbol{a}$ and $f(\boldsymbol{N})\boldsymbol{a}^\dagger = \boldsymbol{a}^\dagger f(\boldsymbol{N}+1)$ for any function f, we get

$$\begin{aligned}\boldsymbol{a}\rho\boldsymbol{a}^\dagger = \sum_{n=0}^{+\infty}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\boldsymbol{a}e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^\dagger &\\ = \sum_{n=0}^{+\infty} e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^{n+1}\,\rho_0\,(\boldsymbol{a}^\dagger)^{n+1} e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^\dagger. &\end{aligned}$$

   Thus $\frac{d}{dt}\rho = \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \frac{1}{2}(\boldsymbol{N}\rho + \rho\boldsymbol{N})\right)$.

2. (a) $\rho(t)$ still converges to vacuum.

(b) The computations are slightly more complex than those of previous question. Since

$$\begin{aligned}\frac{d}{dt}\left[\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}}\right] = \\ \kappa e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^{n-1}}{(n-1)!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}} \\ -\left(\tfrac{\kappa}{2}+\imath\Delta\right)\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\boldsymbol{N}e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}} \\ -\left(\tfrac{\kappa}{2}-\imath\Delta\right)\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^n\,\rho_0\,(\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{N}\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{dt}\rho == \sum_{n=0}^{+\infty}\kappa e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^{n+1}\,\rho_0\,(\boldsymbol{a}^\dagger)^{n+1} e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}} \\ -\imath\Delta\boldsymbol{N}\rho+\imath\Delta\rho\boldsymbol{N}-\tfrac{\kappa}{2}(\boldsymbol{N}\rho+\rho\boldsymbol{N}).\end{aligned}$$

With

$$\boldsymbol{a}\rho\boldsymbol{a}^\dagger = \sum_{n=0}^{+\infty} e^{-\kappa t}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right) e^{-\left(\frac{\kappa}{2}+\imath\Delta\right)t\boldsymbol{N}}\boldsymbol{a}^{n+1}\,\rho_0\,(\boldsymbol{a}^\dagger)^{n+1} e^{-\left(\frac{\kappa}{2}-\imath\Delta\right)t\boldsymbol{N}}$$

we conclude that $\frac{d}{dt}\rho = -\imath\Delta[\boldsymbol{N},\rho]+\kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger-\frac{1}{2}(\boldsymbol{N}\rho+\rho\boldsymbol{N})\right)$.

3. (a) $\rho(t)$ converges to the coherent state $|\alpha\rangle$ of amplitude $\alpha = 2u/\kappa$.

(b) With the changement of frame $\rho\mapsto\xi=\boldsymbol{D}_{-\alpha}\rho\boldsymbol{D}_\alpha$, the Lindblad equation becomes $\frac{d}{dt}\xi=\kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger-\frac{1}{2}(\boldsymbol{N}\xi+\xi\boldsymbol{N})\right)$ with $\xi_0=\boldsymbol{D}_{-\alpha}\rho_0\boldsymbol{D}_\alpha$ (use $\boldsymbol{D}_{-\alpha}\boldsymbol{a}\boldsymbol{D}_\alpha=\boldsymbol{a}+\alpha$ and $\boldsymbol{D}_{-\alpha}\boldsymbol{N}\boldsymbol{D}_\alpha=(\boldsymbol{a}^\dagger+\alpha^*)(\boldsymbol{a}+\alpha)=\boldsymbol{N}+\alpha^*\boldsymbol{a}+\alpha\boldsymbol{a}^\dagger+|\alpha|^2$) . Since

$$\xi(t)=\sum_{n=0}^{+\infty}\left(\tfrac{(1-e^{-\kappa t})^n}{n!}\right)\left(e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\boldsymbol{a}^n\right)\,\xi_0\,\left((\boldsymbol{a}^\dagger)^n e^{-\left(\frac{\kappa}{2}\right)t\boldsymbol{N}}\right)$$

we get the formula for $\rho(t)=\boldsymbol{D}_\alpha\xi(t)\boldsymbol{D}_{-\alpha}$ with $\alpha=2u/\kappa$.

4. (a) The frame corresponds to the drive frame, i.e. a frame rotating at pulsation ω_d and defined by the unitary transformation $e^{-\imath\omega_d t\boldsymbol{N}}$.

(b) $\rho(t)$ converges to the coherent state $|\alpha\rangle$ of amplitude $\alpha=u/(\kappa/2+\imath\Delta)$. This results from the fact that with the changement of frame $\rho\mapsto\xi=\boldsymbol{D}_{-\alpha}\rho\boldsymbol{D}_\alpha$, the Lindblad equation becomes $\frac{d}{dt}\xi=-\imath\Delta[\boldsymbol{N},\xi]+\kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger-\frac{1}{2}(\boldsymbol{N}\xi+\xi\boldsymbol{N})\right)$ and $\xi(t)\mapsto|0\rangle\langle 0|$. Thus $\rho(t)=\mapsto\boldsymbol{D}_\alpha\xi(t)\boldsymbol{D}_{-\alpha}$ converges towards $\boldsymbol{D}_\alpha|0\rangle\langle 0|\boldsymbol{D}_{-\alpha}=|\alpha\rangle\langle\alpha|$.

(c) Just use the series of 2b for ξ to obtain after a coherent displacement of amplitude $\alpha=u/(\kappa/2+\imath\Delta)$, the series for ρ.

Dissipation induced dephasing

1. In this frame $\boldsymbol{a}$ become

$$e^{it\boldsymbol{H}_{\rm disp}}\boldsymbol{a}e^{-it\boldsymbol{H}_{\rm disp}}=\boldsymbol{a}(e^{it\frac{\chi}{2}}|e\rangle\langle e|+e^{-it\frac{\chi}{2}}|g\rangle\langle g|).$$

Therefore the Lindblad equation becomes

$$\frac{d}{dt}\boldsymbol{\xi} = \kappa \left((|e\rangle\langle e| \otimes \boldsymbol{a})\boldsymbol{\xi}(|e\rangle\langle e| \otimes \boldsymbol{a}^\dagger) + (|g\rangle\langle g| \otimes \boldsymbol{a})\boldsymbol{\xi}(|g\rangle\langle g| \otimes \boldsymbol{a}^\dagger) - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a} \right)$$
$$+ \kappa \left(e^{it\chi}(|e\rangle\langle e| \otimes \boldsymbol{a})\boldsymbol{\xi}(|g\rangle\langle g| \otimes \boldsymbol{a}^\dagger) + e^{-it\chi}(|g\rangle\langle g| \otimes \boldsymbol{a})\boldsymbol{\xi}(|e\rangle\langle e| \otimes \boldsymbol{a}^\dagger) \right).$$

After the 1st order RWA (keeping only the secular terms in the first line) we find:

$$\frac{d}{dt}\boldsymbol{\xi} = \kappa \left((|e\rangle\langle e| \otimes \boldsymbol{a})\boldsymbol{\xi}(|e\rangle\langle e| \otimes \boldsymbol{a}^\dagger) + (|g\rangle\langle g| \otimes \boldsymbol{a})\boldsymbol{\xi}(|g\rangle\langle g| \otimes \boldsymbol{a}^\dagger) - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a} \right).$$

2. We start by writing

$$\boldsymbol{\xi} = |e\rangle\langle e| \otimes \boldsymbol{\xi}_{ee}(t) + |g\rangle\langle g| \otimes \boldsymbol{\xi}_{gg}(t) + |e\rangle\langle g| \otimes \boldsymbol{\xi}_{eg}(t) + |g\rangle\langle e| \otimes \boldsymbol{\xi}_{ge}(t)$$

where $\boldsymbol{\xi}_{gg}, \boldsymbol{\xi}_{ee}, \boldsymbol{\xi}_{ge}, \boldsymbol{\xi}_{eg}$ all live on the Hilbert space of the harmonic oscillator. Furthermore, $\boldsymbol{\xi}_{gg}, \boldsymbol{\xi}_{ee}$ are positive semi-definite trace-class and Hermitian operators with $\mathrm{Tr}\left(\boldsymbol{\xi}_{gg}\right) + \mathrm{Tr}\left(\boldsymbol{\xi}_{ee}\right) = 1$. Also, $\boldsymbol{\xi}_{ge} = \boldsymbol{\xi}_{eg}^\dagger$. In order to find the dynamics satisfied by each of these operators, we multiply the above equation by $\langle g|$ or $\langle e|$ on the left and by $|g\rangle$ or $|e\rangle$ on the right. Therefore

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{\xi}_{gg} &= \kappa(\boldsymbol{a}\boldsymbol{\xi}_{gg}\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi}_{gg} - \frac{1}{2}\boldsymbol{\xi}_{gg}\boldsymbol{a}^\dagger\boldsymbol{a}),\\
\frac{d}{dt}\boldsymbol{\xi}_{ee} &= \kappa(\boldsymbol{a}\boldsymbol{\xi}_{ee}\boldsymbol{a}^\dagger - \frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi}_{ee} - \frac{1}{2}\boldsymbol{\xi}_{ee}\boldsymbol{a}^\dagger\boldsymbol{a}),\\
\frac{d}{dt}\boldsymbol{\xi}_{ge} &= \frac{d}{dt}\boldsymbol{\xi}_{eg}^* = -\kappa(\frac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi}_{ge} + \frac{1}{2}\boldsymbol{\xi}_{ge}\boldsymbol{a}^\dagger\boldsymbol{a}).
\end{aligned}$$

First, we note that $\mathrm{Tr}\left(\boldsymbol{\xi}_{gg}(t)\right)$ et $\mathrm{Tr}\left(\boldsymbol{\xi}_{ee}(t)\right)$ remain constant and furthremore $\mathrm{Tr}\left(\boldsymbol{\xi}_{gg}(0)\right) = |c_g|^2$ and $\mathrm{Tr}\left(\boldsymbol{\xi}_{ee}(0)\right) = |c_e|^2$. Therefore following the result of the course

$$\boldsymbol{\xi}_{gg}(t) \to |c_g|^2|0\rangle\langle 0| \quad \text{and} \quad \boldsymbol{\xi}_{ee}(t) \to |c_e|^2|0\rangle\langle 0| \qquad \text{as } t \to \infty.$$

Let us now study the dynamics of $\boldsymbol{\xi}_{ge}$. We start by writing $\boldsymbol{\xi}_{ge} = c_{mn}|m\rangle\langle n|$. We therefore have

$$\frac{d}{dt}c_{mn} = -\kappa\frac{(m+n)}{2}c_{mn}.$$

Therefore for all $(m, n) \neq (0, 0)$, $c_{mn} \to 0$ and thus

$$\boldsymbol{\xi}_{ge}(t) \to \langle 0|\boldsymbol{\xi}_{ge}(0)|0\rangle = c_g^* c_e \langle 0|\boldsymbol{\rho}_c|0\rangle \qquad \text{as } t \to \infty$$

Calling $r := \langle 0|\boldsymbol{\rho}_c|0\rangle \leq 1$, we therefore obtain

$$\boldsymbol{\xi}(t) \to \left(|c_g|^2|g\rangle\langle g| + |c_e|^2|e\rangle\langle e| + rc_g^*c_e|g\rangle\langle e| + rc_e^*c_g|e\rangle\langle g|\right) \otimes |0\rangle\langle 0|.$$

3. While the cavity state decays to the vacuum state $|0\rangle\langle 0|$, the qubit state converges to a state that is less pure than the initial state. More precisely, even if we start with a pure state, as soon as $r = \langle 0|\boldsymbol{\rho}_c|0\rangle < 1$ (i.e. $\boldsymbol{\rho}_c$ is not the vacuum state) the steady qubit state is a mixed state. This is the dephasing (decoherence) of the qubit caused by its coupling to a dissipative cavity.

Do and undo an entangled state between two harmonic oscillators

1. (a) Before interaction with oscillator 1, the wave function is $|e\rangle \otimes |\psi\rangle$. After interaction with oscillator 1, it becomes

$$\boldsymbol{U}_1|e\rangle \otimes |\psi\rangle = |e\rangle \otimes \cos(\theta_1\sqrt{\boldsymbol{N}_1+1})|\psi\rangle + |g\rangle \otimes \tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger|\psi\rangle.$$

After interaction with oscillator 2, it reads

$$\begin{aligned}
&\boldsymbol{U}_2\boldsymbol{U}_1|e\rangle \otimes |\psi\rangle = \\
&|e\rangle \otimes \cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\cos(\theta_1\sqrt{\boldsymbol{N}_1+1})|\psi\rangle + |g\rangle \otimes \tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger \cos(\theta_1\sqrt{\boldsymbol{N}_1+1})|\psi\rangle \\
&\quad + |g\rangle \otimes \cos(\theta_2\sqrt{\boldsymbol{N}_2})\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger|\psi\rangle - |e\rangle \otimes \boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger|\psi\rangle \\
&\quad = |g\rangle \otimes \left(\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger \cos(\theta_1\sqrt{\boldsymbol{N}_1+1}) + \cos(\theta_2\sqrt{\boldsymbol{N}_2})\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger\right)|\psi\rangle \\
&\quad + |e\rangle \otimes \left(\cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\cos(\theta_1\sqrt{\boldsymbol{N}_1+1}) - \boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger\right)|\psi\rangle.
\end{aligned}$$

With

$$\begin{aligned}
\boldsymbol{M}_g &= \tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger \cos(\theta_1\sqrt{\boldsymbol{N}_1+1}) + \cos(\theta_2\sqrt{\boldsymbol{N}_2})\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger \\
\boldsymbol{M}_e &= \cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\cos(\theta_1\sqrt{\boldsymbol{N}_1+1}) - \boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\,\boldsymbol{a}_1^\dagger
\end{aligned}$$

we have $\boldsymbol{U}_2\boldsymbol{U}_1|e\rangle \otimes |\psi\rangle = |g\rangle \otimes \boldsymbol{M}_g|\psi\rangle + |e\rangle \otimes \boldsymbol{M}_e|\psi\rangle$. Measurement of the qubit gives then the following Markov chain

$$|\psi\rangle_+ = \begin{cases} \frac{\boldsymbol{M}_g|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle}}, & \text{if } y = g \text{ with proba. } \langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle; \\ \frac{\boldsymbol{M}_e|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle}}, & \text{if } y = e \text{ with proba. } \langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle. \end{cases}$$

(b) When $\theta_1 = \pi/4$ and $\theta_2 = \pi/2$ we have

$$\boldsymbol{M}_g|00\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \text{ and } \boldsymbol{M}_e|00\rangle = 0.$$

Thus $y = g$ with probability 1. This corresponds to a deterministic preparation of the entangled state $|\psi\rangle_+ = \frac{|01\rangle+|10\rangle}{\sqrt{2}}$ between the two oscillators.

2. (a) Before interaction with oscillator 1, the wave function is $|g\rangle \otimes |\psi\rangle$. After interaction with oscillator 1, it becomes

$$\boldsymbol{U}_1|g\rangle \otimes |\psi\rangle = |g\rangle \otimes \cos(\theta_1\sqrt{\boldsymbol{N}_1})|\psi\rangle - |e\rangle \otimes \boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}|\psi\rangle.$$

After interaction with oscillator 2, it reads

$$\begin{aligned}
&\boldsymbol{U}_2\boldsymbol{U}_1|g\rangle \otimes |\psi\rangle = \\
&\quad |g\rangle \otimes \cos(\theta_2\sqrt{\boldsymbol{N}_2})\cos(\theta_1\sqrt{\boldsymbol{N}_1})|\psi\rangle - |e\rangle \otimes \boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\cos(\theta_1\sqrt{\boldsymbol{N}_1})|\psi\rangle \\
&\quad - |e\rangle \otimes \cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}|\psi\rangle - |g\rangle \otimes \tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}|\psi\rangle \\
&\quad = |g\rangle \otimes \left(\cos(\theta_2\sqrt{\boldsymbol{N}_2})\cos(\theta_1\sqrt{\boldsymbol{N}_1}) - \tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\right)|\psi\rangle \\
&\quad - |e\rangle \otimes \left(\boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\cos(\theta_1\sqrt{\boldsymbol{N}_1}) + \cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}\right)|\psi\rangle.
\end{aligned}$$

With

$$\boldsymbol{M}_g = \cos(\theta_2\sqrt{\boldsymbol{N}_2})\cos(\theta_1\sqrt{\boldsymbol{N}_1}) - \tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\,\boldsymbol{a}_2^\dagger\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}$$
$$\boldsymbol{M}_e = -\boldsymbol{a}_2\,\tfrac{\sin(\theta_2\sqrt{\boldsymbol{N}_2})}{\sqrt{\boldsymbol{N}_2}}\cos(\theta_1\sqrt{\boldsymbol{N}_1}) - \cos(\theta_2\sqrt{\boldsymbol{N}_2+1})\boldsymbol{a}_1\,\tfrac{\sin(\theta_1\sqrt{\boldsymbol{N}_1})}{\sqrt{\boldsymbol{N}_1}}$$

we have $\boldsymbol{U}_2\boldsymbol{U}_1|e\rangle\otimes|\psi\rangle = |g\rangle\otimes\boldsymbol{M}_g|\psi\rangle + |e\rangle\otimes\boldsymbol{M}_e|\psi\rangle$. Measurement of the qubit gives then the following Markov chain

$$|\psi\rangle_+ = \begin{cases} \frac{\boldsymbol{M}_g|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle}}, & \text{if } y=g \text{ with proba. } \langle\psi|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi\rangle; \\ \frac{\boldsymbol{M}_e|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle}}, & \text{if } y=e \text{ with proba. } \langle\psi|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi\rangle. \end{cases}$$

(b) We have

$$\boldsymbol{M}_g|00\rangle = |00\rangle \text{ and } \boldsymbol{M}_e|00\rangle = 0.$$

Starting with qubit and oscillators in ground state, it is impossible to have any exchange of energy between them. We recover here the fact that $y = g$ with probability 1 and $|\psi\rangle_+ = |00\rangle$.

(c) With $\theta_1 = \pi/2$, $\theta_2 = \pi/4$ we have

$$\boldsymbol{M}_g\tfrac{|10\rangle+|01\rangle}{\sqrt{2}} = \tfrac{|01\rangle-|01\rangle}{2} = 0 \text{ and } \boldsymbol{M}_e\tfrac{|10\rangle+|01\rangle}{\sqrt{2}} = -\tfrac{|00\rangle+|00\rangle}{2} = -|00\rangle$$

Thus $y = e$ with probability 1 with the deterministic result $-|00\rangle$. With initial qubit-state $|g\rangle$ and these values of θ_1 and θ_2 , we undo the entangled state $\frac{|10\rangle+|01\rangle}{\sqrt{2}}$ obtained in 1b and recover the vacuum state of the oscillators (defined up to a global phase).