Modelling, simulation and feedback of open quantum systems

Pierre Rouchon
Centre Automatique et Systèmes
Mines ParisTech
PSL Research University

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Outline of the lectures (Autumn 2018 at Mines ParisTech)

Lect. 1& 2 (Oct. 24: 11:10-13:00 and 14:00–15:50) Feedback for classical and for quantum systems; the first experimental realization of a quantum-state feedback (LKB photon box); three quantum features (Schrödinger; collapse of the wave packet, tensor product); Quantum Non Demolition (QND) measurement of photons (ideal Markov model and Matlab simulations).

Lect. 3 (Nov. 7: 11:10–13:00) Models of the LKB photon box: entanglement between the probe-qubit and the photons; qubit-measurement back-action on the photons; measurement errors and imperfections; decoherence as unread fictitious measurements; discrete-time Markov chain; Kraus map; quantum trajectories and realistic Matlab simulations of the photon box

Lect. 4 (Nov. 14: 11:10–13:00) Feedback stabilization of photon-number state: resonant interaction, the Lyapunov feedback scheme, closed-loop simulations / experimental data.

Lect. 5 (Nov. 21: 9:00–10:50) The LPA super-conduction qubit under continuous-time measurements (counting versus homo/hetero-dyne measurements): continuous-time stochastic master equation (Poisson versus Wiener); Lindblad master equation; QND measurement of a qubit and Matlab simulations.

Lect. 6 (Nov. 28: 11:10–13:00) Feedback stabilization of the excited state of a qubit; quantum-state feedback based on QND measurement;closed-loop simulations.

Main references

- S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford University Press, 2006.
- H.M. Wiseman and G.J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, 2009.
- M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- C.W. Gardiner and P. Zoller *Quantum Noise*. Springer, 2010.

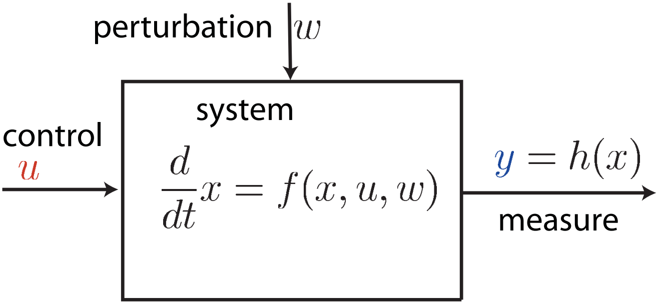
Outline

Notion of Feedback

Discrete-time systems: The LKB photon-box

Continuous-time systems: qubit in circuit QED

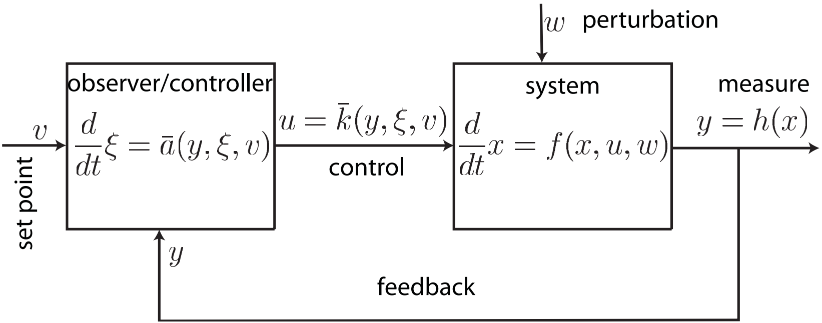
Continuous diffusive-jump SME



For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$



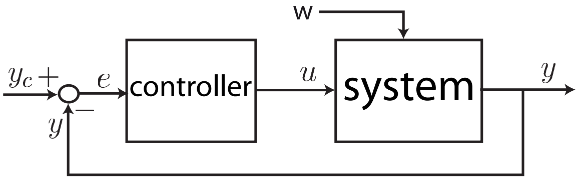
Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point $v = y^c$

$$u = -K_p(y - y^c) - K_d \frac{d}{dt}(y - y^c) - K_{\text{int}} \int (y - y^c)$$

with the positive gains $(K_p, K_d, K_{\text{int}})$ tuned as follows ($0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\sqrt{\omega^2 + \Omega_0^2}, \quad , K_{\text{int}} = \epsilon(\omega^2 + \Omega_0^2)^{3/2}.$$

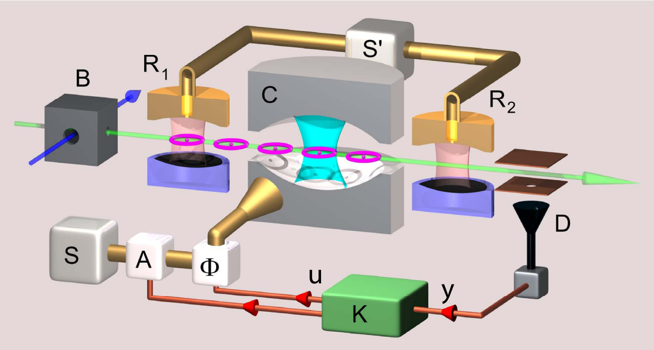
A typical stabilizing feedback-loop for a classical system



Two kinds of stabilizing feedbacks for quantum systems

1. Measurement-based feedback: **controller is classical**; measurement back-action on the system $\mathcal{S}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.
2. Coherent/autonomous feedback and reservoir engineering: the system $\mathcal{S}$ is coupled to **the controller, another quantum system**; the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state.

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche (Nobel Prize 2012), J.M.Raimond and M. Brune.[1]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).
Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[1] Courtesy of Igor Dotsenko. **Sampling period 80 μs.**

Three quantum features emphasized by the LKB photon box[2]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho\boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y\rho\boldsymbol{P}_y}{\text{Tr}\left(\rho\boldsymbol{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[2] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

Composite system built with an harmonic oscillator and a qubit.

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

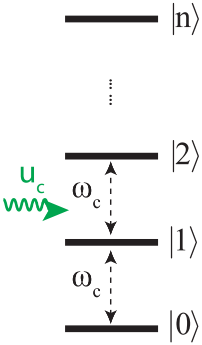
  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is a qu-bit, a 2-level system (idem 1/2 spin system) : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:**

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \Psi_{ng}|n\rangle \otimes |g\rangle + \Psi_{ne}|n\rangle \otimes |e\rangle, \qquad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n \in \mathbb{N}}$.

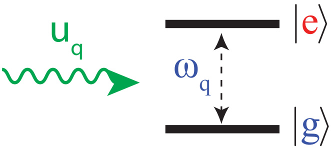
S: quantum harmonic oscillator (spring system)

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \; (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho, \; \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\; |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p, \; \frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).



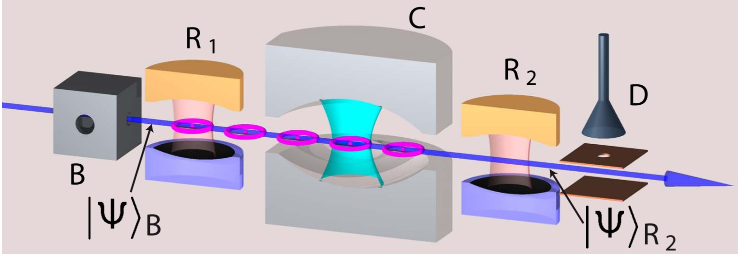
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}: \; |\alpha\rangle = \sum_{n\geq 0} \left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath \sqrt{2} x \Im \alpha} e^{-\frac{(x - \sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \,\middle|\, (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

The Markov ideal model (1)

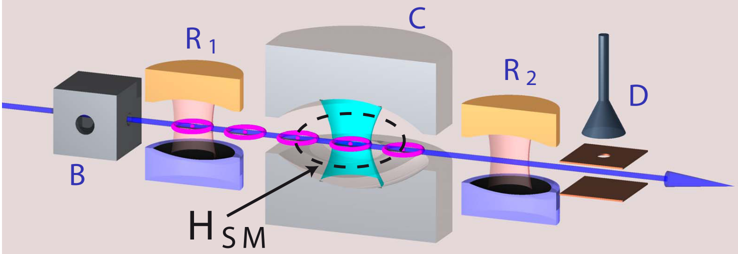


- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

The Markov ideal model (2)



The unitary propagator $\boldsymbol{U}_{SM}$ is derived from Jaynes-Cummings Hamiltonian $\boldsymbol{H}_{SM}$ in the interaction frame.

Two kinds of qubit/cavity Hamiltonians:

resonant, $\boldsymbol{H}_{SM}/\hbar = i\big(\Omega(vt)/2\big)\,\big(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma_-} - \boldsymbol{a} \otimes \boldsymbol{\sigma_+}\big)$,

dispersive, $\boldsymbol{H}_{SM}/\hbar = \big(\Omega^2(vt)/(2\delta)\big)\,\boldsymbol{N} \otimes \boldsymbol{\sigma_z}$,

where $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity, Ω_0 vacuum Rabi pulsation, w radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation ω_q and cavity pulsation ω_c ($|\delta| \ll \Omega_0$).

Dispersive and resonant Jaynes-Cummings propagators

The solution of $\imath \frac{d}{dt}\boldsymbol{U} = -\frac{i}{\hbar}\boldsymbol{H}_{SM}(t)\boldsymbol{U}$, with $\boldsymbol{U}_0 = \boldsymbol{I}$ reads

- for $\boldsymbol{H}_{SM}(t)/\hbar = i\, f(t)\big(\boldsymbol{a}^\dagger \otimes |g\rangle\langle e| - \boldsymbol{a} \otimes |e\rangle\langle g|\big)$ (resonant)

$$\boldsymbol{U}_t = \cos\left(\tfrac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right) \otimes |g\rangle\langle g| + \cos\left(\tfrac{\theta_t}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right) \otimes |e\rangle\langle e| \\ - \boldsymbol{a}\frac{\sin\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}} \otimes |e\rangle\langle g| + \frac{\sin\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger \otimes |g\rangle\langle e|.$$

- for $\boldsymbol{H}_{SM}(t)/\hbar = f(t)\ \boldsymbol{N} \otimes (|e\rangle\langle e| - |g\rangle\langle g|)$ (dispersive)

$$\boldsymbol{U}(t) = \exp\left(i\theta(t)\boldsymbol{N}\right) \otimes |g\rangle\langle g| + \exp\left(-i\theta(t)\boldsymbol{N}\right) \otimes |e\rangle\langle e|.$$

where $\theta(t) = \int_0^t f(\tau)\, d\tau$.

Just before detector D the quantum state is **entangled**:

$$|\boldsymbol{\Psi}\rangle_{R_2} = (\boldsymbol{M}_g|\psi\rangle) \otimes |g\rangle + (\boldsymbol{M}_e|\psi\rangle) \otimes |e\rangle$$

Just after outcome y, the state becomes **separable** [3]:

$$|\boldsymbol{\Psi}\rangle_D = \left(\frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle\right) \otimes |y\rangle.$$

Outcome y obtained with probability $\mathbb{P}_y = \langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle$..

Quantum trajectories (Markov chain, stochastic dynamics):

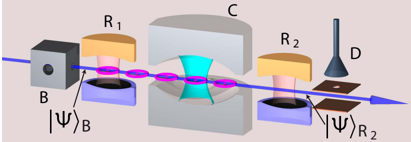
$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}}|\psi_k\rangle, & y_k = g \text{ with probability } \langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}}|\psi_k\rangle, & y_k = e \text{ with probability } \langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle; \end{cases}$$

with state $|\psi_k\rangle$ and measurement outcome $y_k \in \{g, e\}$ at time-step k:

[3] Measurement operator $\boldsymbol{O} = \boldsymbol{I}_S \otimes (|e\rangle\langle e| - |g\rangle\langle g|)$.

Quantum Non Demolition (QND) measurement of photons [4]

$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}\left(|\psi\rangle\otimes|g\rangle\right)$$



$$\boldsymbol{U}_{R_1} = \boldsymbol{I}_S \otimes \left(\left(\frac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{-|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)$$

$$\boldsymbol{U}_C = e^{-i\frac{\phi_0}{2}\boldsymbol{N}}\otimes|g\rangle\langle g| + e^{i\frac{\phi_0}{2}\boldsymbol{N}}\otimes|e\rangle\langle e|$$

$$\boldsymbol{U}_{R_2} = \boldsymbol{U}_{R_1}$$

$$\boldsymbol{U}_{R_1}\left(|\psi\rangle\otimes|g\rangle\right) = \tfrac{1}{\sqrt{2}}\left(|\psi\rangle\otimes|g\rangle + |\psi\rangle\otimes|e\rangle\right)$$

$$\boldsymbol{U}_C\boldsymbol{U}_{R_1}\left(|\psi\rangle\otimes|g\rangle\right) = \tfrac{1}{\sqrt{2}}\left(\left(e^{-i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes|g\rangle + \left(e^{i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes|e\rangle\right)$$

$$|\boldsymbol{\Psi}\rangle_{R_2} = \tfrac{1}{2}\left(\left(e^{-i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes(|g\rangle+|e\rangle) + \left(e^{i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes(-|g\rangle+|e\rangle)\right)$$
$$= \left(-i\sin(\tfrac{\phi_0}{2}\boldsymbol{N})|\psi\rangle\right)\otimes|g\rangle + \left(\cos(\tfrac{\phi_0}{2}\boldsymbol{N})|\psi\rangle\right)\otimes|e\rangle$$

Thus $\boldsymbol{M}_g = -i\sin(\frac{\phi_0}{2}\boldsymbol{N})$ and $\boldsymbol{M}_e = \cos(\frac{\phi_0}{2}\boldsymbol{N})$.

Quantum Monte-Carlo simulations with MATLAB:

`IdealModelPhotonBoxWaveFunction.m`

[4] M. Brune, . . . : Manipulation of photons in a cavity by dispersive atom-field coupling: quantum non-demolition measurements and generation of "Schrödinger cat" states . Physical Review A, 45:5193-5214, 1992.

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle\right) \otimes |y\rangle.$$

Markov process (density matrix formulation $\rho \sim |\psi\rangle\langle\psi|$)

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)}, & \text{with probability } \mathbb{P}_g = \mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)}, & \text{with probability } \mathbb{P}_e = \mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right). \end{cases}$$

Kraus map: $\mathbb{E}\left(\rho_+/\rho\right) = \boldsymbol{K}(\rho) = \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger$.

LKB photon-box: Markov process with detection efficiency

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$.

- Detection efficiency: the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for y: $y = g$ if detection in g, $y = e$ if detection in e and $y = 0$ if no detection.

The only possible update is based on ρ: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the outcome $y \in \{g, e, 0\}$.

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\text{Tr}(\boldsymbol{M}_g \rho \boldsymbol{M}_g)} & \text{if } y = g\text{, probability } \eta \, \text{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g\right) \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\text{Tr}(\boldsymbol{M}_e \rho \boldsymbol{M}_e)} & \text{if } y = e\text{, probability } \eta \, \text{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e\right) \\ \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger & \text{if } y = 0\text{, probability } 1 - \eta \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.

- With pure state $\rho = |\psi\rangle\langle\psi|$, we have
$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger$$
when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$.
- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0,1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0,1]$ (given by the contrast of the Ramsey fringes).

Bayes law: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the imperfect detection y.

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{if } y = g, \text{ prob. } \text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{if } y = e, \text{ prob. } \text{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.

We get

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)}, & \text{with prob. } \mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} & \text{with prob. } \mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

Key point:

$$\mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right) \text{ and } \mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.
Generalization by merging a Kraus map $\boldsymbol{K}(\rho) = \sum_\mu \boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger$ where $\sum_\mu \boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu = \boldsymbol{I}$ with a left stochastic matrix $(\eta_{\mu',\mu})$:

$$\rho_+ = \frac{\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)} \quad \text{when we detect } y = \mu'.$$

The probability to detect $y = \mu'$ knowing ρ is $\mathrm{Tr}\left(\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)$.

Cavity decay (decoherence) seen as unread measurements

The cavity mirrors play the role of a detector with two possible outcomes:

- zero photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_0 = \boldsymbol{I} - \frac{\Delta T}{2} \boldsymbol{L}_{-1}^\dagger \boldsymbol{L}_{-1}$, probability $\approx$ Tr $\left(\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger \right)$ with back action $\rho_{t+\Delta T} \approx \frac{\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger}{\text{Tr}\left(\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger \right)}$.
- one photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_{-1} = \sqrt{\Delta T} \boldsymbol{L}_{-1}$, probability $\approx$ Tr $\left(\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger \right)$ with back action $\rho_{t+\Delta T} \approx \frac{\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger}{\text{Tr}\left(\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger \right)}$

where

$$\boldsymbol{L}_{-1} = \sqrt{\frac{1}{T_{cav}}} \boldsymbol{a}$$

is the Lindbald operator associated to cavity damping (see bellow the continuous time models) with T_{cav} the photon life time and $\Delta T \ll T_{cav}$ the sampling period ($T_{cav} = 100\ ms$ and $\Delta T \approx 100\ \mu s$ for the LKB photon Box).

Cavity decoherence: cavity decay, thermal photon(s)

Three possible outcomes:

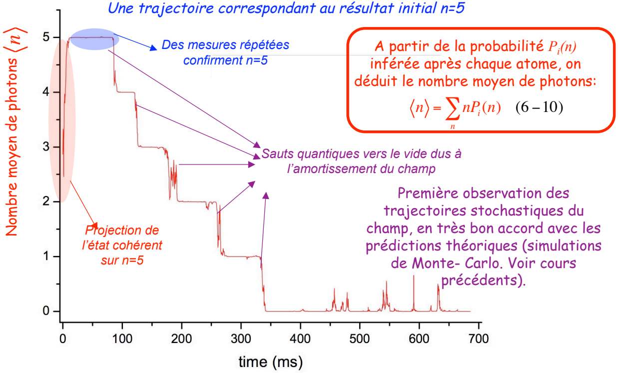
- zero photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_0 = \boldsymbol{I} - \frac{\Delta T}{2}\boldsymbol{L}_{-1}^\dagger \boldsymbol{L}_{-1} - \frac{\Delta T}{2}\boldsymbol{L}_1^\dagger \boldsymbol{L}_1$, probability $\approx$ $\text{Tr}\left(\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger}{\text{Tr}\left(\boldsymbol{M}_0 \rho_t \boldsymbol{M}_0^\dagger\right)}$.
- one photon annihilation during ΔT: Kraus operator $\boldsymbol{M}_{-1} = \sqrt{\Delta T}\boldsymbol{L}_{-1}$, probability $\approx$ $\text{Tr}\left(\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger}{\text{Tr}\left(\boldsymbol{M}_{-1} \rho_t \boldsymbol{M}_{-1}^\dagger\right)}$
- one photon creation during ΔT: Kraus operator $\boldsymbol{M}_1 = \sqrt{\Delta T}\boldsymbol{L}_1$, probability $\approx$ $\text{Tr}\left(\boldsymbol{M}_1 \rho_t \boldsymbol{M}_1^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\boldsymbol{M}_1 \rho_t \boldsymbol{M}_1^\dagger}{\text{Tr}\left(\boldsymbol{M}_1 \rho_t \boldsymbol{M}_1^\dagger\right)}$

where

$$\boldsymbol{L}_{-1} = \sqrt{\frac{1+n_{th}}{T_{cav}}}\boldsymbol{a}, \quad \boldsymbol{L}_1 = \sqrt{\frac{n_{th}}{T_{cav}}}\boldsymbol{a}^\dagger$$

are the Lindbald operators associated to cavity decoherence : T_{cav} the photon life time, $\Delta T \ll T_{cav}$ the sampling period and n_{th} is the average of thermal photon(s) (vanishes with the environment temperature) ($n_{th} \approx 0.05$ for the LKB photon box).

Valeur moyenne du nombre de photons le long d'une longue séquence de mesure: observation d'une trajectoire stochastique



See the quantum Monte Carlo simulations of the Matlab script: RealisticModelPhotonBox.m.

[5] From Serge Haroche, Collège de France, notes de cours 2007/2008.

Feedback: Kraus operators depend on the control input u[6]

$u = 0$: dispersive interaction with

$$\boldsymbol{M}_g(0) = \cos\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right) \text{ and } \boldsymbol{M}_e(0) = \sin\left(\frac{\phi_0 \boldsymbol{N} + \phi_R}{2}\right),$$

$u = 1$: resonant interaction with atom prepared in $|e\rangle$

$$\boldsymbol{M}_g(1) = \frac{\sin\left(\frac{\theta_0}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger \text{ and } \boldsymbol{M}_e(1) = \cos\left(\frac{\theta_0}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right)$$

$u = -1$: resonant interaction with atom prepared in $|g\rangle$

$$\boldsymbol{M}_g(-1) = \cos\left(\frac{\theta_0}{2}\sqrt{\boldsymbol{N}}\right) \text{ and } \boldsymbol{M}_e(-1) = -\boldsymbol{a}\frac{\sin\left(\frac{\theta_0}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}$$

$(\phi_0, \phi_R, \theta_0)$ are constant parameters.

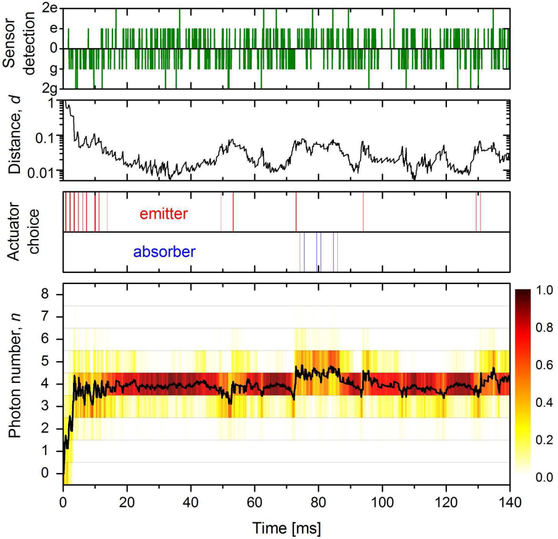
[6] Zhou, X.; Dotsenko, I.; Peaudecerf, B.; Rybarczyk, T.; Sayrin, C.; S. Gleyzes, J. R.; Brune, M.; Haroche, S. Field locked to Fock state by quantum feedback with single photon corrections. Physical Review Letter, 2012, 108, 243602.

- Compute u_k as a function of ρ_k such that ρ_k converges towards the goal $|\bar{n}\rangle\langle\bar{n}|$.
- Lyapunov function $V(\rho) = \text{ Tr}\left((N-\bar{n})^2\rho\right)$ for example: when $u_k = 0$ we have

$$\mathbb{E}\left(V(\rho_{k+1} \;/\; \rho_k, u_k = 0\right) = V(\rho_k) \quad \text{(martingale)}$$

- Lyapunov control: choose u_k in $\{0, 1, -1\}$ at each step k in order to minimize $u \mapsto \mathbb{E}\left(V(\rho_{k+1} \;/\; \rho_k, u\right)$.
- In closed-loop, $\mathbb{E}\left(V(\rho_{k+1}\right)$ will be decreasing. It is reasonable to guess that $V(\rho_k)$ tends to 0, i.e., that ρ_k converges to $|\bar{n}\rangle\langle\bar{n}|$.

Closed-loop experimental results



Zhou et al. Field locked to Fock state by quantum feedback with single photon corrections. Physical Review Letter, 2012, 108, 243602.

See the closed-loop quantum Monte Carlo simulations of the Matlab script: `RealisticFeedbackPhotonBox.m`.

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\boldsymbol{K}_{y_k}(\rho_k)}{\mathrm{Tr}\big(\boldsymbol{K}_{y_k}(\rho_k)\big)}, \text{ with proba. } p_{y_k}(\rho_k) = \mathrm{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)$$

where each $\boldsymbol{K}_y$ is a linear completely positive map admitting the expression

$$\boldsymbol{K}_y(\rho) = \sum_\mu \boldsymbol{M}_{y,\mu}\rho\boldsymbol{M}_{y,\mu}^\dagger \quad \text{with} \quad \sum_{y,\mu} \boldsymbol{M}_{y,\mu}^\dagger \boldsymbol{M}_{y,\mu} = \boldsymbol{I}.$$

$\boldsymbol{K} = \sum_y \boldsymbol{K}_y$ corresponds to a Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_y \boldsymbol{K}_y(\rho_k).$$

Quantum filtering (Belavkin quantum filters)

data: initial quantum state ρ_0, past measurement outcomes $\boldsymbol{y_l}$ for $l \in \{0, \ldots, k-1\}$;

goal: estimation of ρ_k via the recurrence (quantum filter)

$$\rho_{l+1} = \frac{\boldsymbol{K}_{\boldsymbol{y_l}}(\rho_l)}{\mathrm{Tr}\left(\boldsymbol{K}_{\boldsymbol{y_l}}(\rho_l)\right)}, \quad l = 0, \ldots, k-1.$$

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\boldsymbol{K}_{y_k}(\rho_k)}{\text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)}, \text{ with proba. } p_{y_k}(\rho_k) = \text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)$$

associated to Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_y \boldsymbol{K}_y(\rho_k)$$

Continuous-time models are stochastic differential systems

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \text{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$

driven by Wiener process[7] $dW_{\nu,t} = dy_{\nu,t} - \sqrt{\eta_\nu}\, \text{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\,\rho_t\right) dt$
with measures $y_{\nu,t}$, detection efficiencies $\eta_\nu \in [0, 1]$ and
Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

[7] and/or Poisson processes, see next slides.

Given a SDE

$$dX_t = F(X_t, t)dt + \sum_\nu G_\nu(X_t, t)dW_{\nu,t},$$

we have the following chain rule summarized by the heuristic formulae:

$$dW_{\nu,t} = O(\sqrt{dt}), \quad dW_{\nu,t}dW_{\nu',t} = \delta_{\nu,\nu'}dt.$$

Ito's rule Defining $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t}(G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt + \sum_\nu \frac{\partial f}{\partial X}\Big|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\mathbb{E}\left(\frac{d}{dt} f_t \,\middle|\, X_t \right) = \mathbb{E}\left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t}(G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

With a single imperfect measure $dy_t = \sqrt{\eta}\, \text{Tr}\left((L + L^\dagger)\,\rho_t\right)\, dt + dW_t$ and detection efficiency $\eta \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\tfrac{i}{\hbar}[H, \rho_t] + L\rho_t L^\dagger - \frac{1}{2}\left(L^\dagger L\rho_t + \rho_t L^\dagger L\right)\right) dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L + L^\dagger)\rho_t\right)\rho_t\right) dW_t$$

driven by the Wiener process dW_t (Gaussian law of mean 0 and variance dt).

With Itō rules, it can be written as the following "discrete-time" Markov model

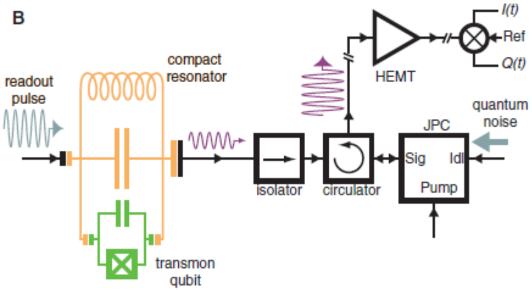
$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\text{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\tfrac{i}{\hbar}H - \tfrac{1}{2}\left(L^\dagger L\right)\right) dt + \sqrt{\eta}\, dy_t L$. The probability to detect dy_t is given by the following density

$$\mathbb{P}\left(dy_t \in [s, s+ds]\right) = \frac{\text{Tr}\left(M_s\rho_t M_s^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}{\sqrt{2\pi dt}} e^{-\frac{s^2}{2dt}}\, ds$$

close to a Gaussian law of variance dt and mean $\sqrt{\eta}\, \text{Tr}\left((L + L^\dagger)\,\rho_t\right) dt$.

A key physical example in circuit QED[8]



Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of both output field quadratures I_t and Q_t is described by a simple SME for the qubit density operator.

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right)dt$$
$$+ \sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)dW_t^I + \imath\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]dW_t^Q$$

with I_t and Q_t given by $dI_t = \sqrt{\eta\gamma/2}\,\mathrm{Tr}\left(2\boldsymbol{\sigma_z}\rho_t\right)dt + dW_t^I$ and $dQ_t = dW_t^Q$, where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0, 1]$ is the detection efficiency. u and v are the two control inputs.

[8] M. Hatridge et al. Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 2013, 339, 178-181.

With $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ satisfying $\imath\hbar\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$, the density operator corresponds to $\rho = |\psi\rangle\langle\psi|$. Then ρ is non negative Hermitian operator such that $\operatorname{Tr}(\rho) = 1$ and obeying to

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[\boldsymbol{H}, \rho].$$

For mixed states, $\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$ with

$$x = \operatorname{Tr}(\boldsymbol{\sigma_x}\rho),\ y = \operatorname{Tr}(\boldsymbol{\sigma_y}\rho) \text{ and } z = \operatorname{Tr}(\boldsymbol{\sigma_z}\rho).$$

Then $(x, y, z) \in \mathbb{R}^3$ are the cartesian coordinates of vector $\vec{M}$ inside Bloch sphere ($\operatorname{Tr}(\rho^2) = x^2 + y^2 + z^2 \leq 1$):

$$\frac{d}{dt}\vec{M} = (u\vec{\imath} + v\vec{\jmath}) \times \vec{M}.$$

is another formulation of $\frac{d}{dt}\rho = -\frac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho]$. Here u stands for the rotation speed around x-axis and v the rotation speed around y-axis.

QND measurement: asymptotic behavior in open-loop

Almost such convergence: Consider the SME

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right)dt$$
$$+ \sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)\boldsymbol{dW_t^I} + \imath\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}$$

with $u = v = 0$ and $\eta > 0$.

- For any initial state ρ_0, the solution ρ_t converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or $|e\rangle\langle e|$.
- The probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \mathrm{Tr}\left(|g\rangle\langle g|\rho_0\right)$ (respectively $\mathrm{Tr}\left(|e\rangle\langle e|\rho_0\right)$).

Proof based on the martingales $V_g(\rho) = \mathrm{Tr}\left(|g\rangle\langle g|\rho\right) = (1-z)/2$ and $V_e(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\rho\right) = (1+z)/2$, and on the sub-martingale $V(\rho) = \mathrm{Tr}^2\left(\boldsymbol{\sigma_z}\rho\right) = z^2$:

$$\mathbb{E}\left(dV_g|\rho_t\right) = \mathbb{E}\left(dV_e|\rho_t\right) = 0, \quad \mathbb{E}\left(dV|\rho_t\right) = 2\eta\gamma\left(1 - z^2\right)^2 dt \geq 0.$$

Confirmed by the quantum Monte Carlo simulations: `IdealModelQubit.m`

Adding decoherence due to finite life-time of $|e\rangle$

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right)dt$$
$$+\sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)\boldsymbol{dW_t^I} + \imath\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}$$
$$+\left(\boldsymbol{L_e}\rho_t\boldsymbol{L_e^\dagger} - \tfrac{1}{2}\left(\boldsymbol{L_e^\dagger}\boldsymbol{L_e}\rho_t + \rho_t\boldsymbol{L_e^\dagger}\boldsymbol{L_e}\right)\right)dt$$

where $\boldsymbol{L_e} = \sqrt{1/T_1}|g\rangle\langle e|$ and T_1 is the life time of the excited state $|e\rangle$.
For $u = v = 0$, convergence of all trajectories towards $|g\rangle$, the ground state. Proof based on the super-martingales $V_e(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\rho\right) = (1+z)/2$:

$$\mathbb{E}\left(dV_e|\rho_t\right) = -\frac{1}{T_1}V_e\,dt.$$

Confirmed by the quantum Monte Carlo simulations:
`RealisticModelQubit.m`

Feedback stabilization of the excited state

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right) dt$$
$$+ \sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right) \boldsymbol{dW_t^I} + \imath\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t] \boldsymbol{dW_t^Q}$$

With u and v arbitrary, we have for $V(\rho) = \mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho\right) = z$,

$$\mathbb{E}\left(dV_t|\rho_t\right) = u\,\mathrm{Tr}\left(\boldsymbol{\sigma_y}\rho_t\right) - v\,\mathrm{Tr}\left(\boldsymbol{\sigma_x}\rho_t\right) = uy - vx.$$

With the quantum state feedback

$$u = \frac{\mathrm{sign}(y)}{T}(1 - V_t), \quad v = -\frac{\mathrm{sign}(x)}{T}(1 - V_t)$$

we get in closed loop $\mathbb{E}\left(dV_t|\rho_t\right) = \frac{|x|+|y|}{T}(1 - V_t)$ and thus V tends to converge towards 1, i.e., z tends to converge towards 1. Confirmed by the closed-loop simulations: `IdealFeedbackQubit.m`
Robustness of such feedback illustrated by the more realistic simulations:
`RealisticFeedbackQubit.m`

With Poisson process $\boldsymbol{N(t)}$, $\langle \boldsymbol{dN(t)} \rangle = \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ and $\overline{\eta} \in [0,1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt + \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{0}}\rho_t M_{\boldsymbol{0}}^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt}{\mathrm{Tr}\left(M_{\boldsymbol{0}}\rho_t M_{\boldsymbol{0}}^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_{\boldsymbol{0}} = I + \left(-iH + \frac{1}{2}\left(\overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V\right)\right) dt$.

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L \rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V \rho_t + \rho_t V^\dagger V) \right) dt$$
$$+ \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\left((L+L^\dagger)\rho_t \right) \rho_t \right) \boldsymbol{dW_t}$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right)} - \rho_t \right) \left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger \right) \right) dt \right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt}{\operatorname{Tr}\left(M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt \right)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2}L^\dagger L + \frac{1}{2}\left(\overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V \right) \right) dt + \sqrt{\eta}\,\boldsymbol{dy_t} L$.

Continuous/discrete-time general diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu}\left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}$$
$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_\mu \rho_t V_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger\right)} - \rho_t\right)\left(dN_\mu(t) - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)\right) dt\right)$$

where $\eta_\nu \in [0,1], \overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $N_\mu(t+dt) - N_\mu(t) = 1$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'}\rho_t V_{\mu'}^\dagger\right)}$.

When $\forall \mu,\ dN_\mu(t) = 0$, we have

$$\rho_{t+dt} = \frac{M_{dy_t} \rho_t M_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\operatorname{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-iH - \frac{1}{2}\sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2}\sum_\mu \left(\overline{\eta}_\mu \operatorname{Tr}\left(V_\mu \rho_t V_\mu^\dagger\right) I - V_\mu^\dagger V_\mu\right)\right) dt + \sum_\nu \sqrt{\eta_\nu}\, dy_{\nu t} L_\nu$ and where $dy_{\nu,t} = \sqrt{\eta_\nu} \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right) dt + dW_{\nu,t}$.