Problem Set 2
(M2 Dynamics and control of open quantum systems 2023-2024)

This problem set is due on Monday, October 9th, 2023, at 23:59. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. SIMPLE HARMONIC OSCILLATOR COUPLED TO A BOSONIC BATH

Consider a harmonic oscillator coupled to a bosonic bath in thermal equilibrium at temperature T, as described by the Hamiltonian

$$\begin{aligned} H =& H_S + H_I + H_B \\ H_S =& \hbar\omega_0 a^\dagger a,\ H_B = \sum_l \hbar\omega_l b_l^\dagger b_l,\ H_I = \sum_l g_l (a + a^\dagger) \otimes (b_l + b_l^\dagger). \end{aligned} \tag{1}$$

a) Show that the Lindblad master equation for the reduced density matrix describing the simple harmonic oscillator mode annihilated by operator a can be written as

$$\begin{aligned} \dot{\rho}_S =& -i\left[\omega_0' a^\dagger a, \rho_S\right] + \frac{\gamma}{2}(\bar{n}+1)\left(2a\rho_S a^\dagger - a^\dagger a \rho_S - \rho_S a^\dagger a\right) \\ &+ \frac{\gamma}{2}\bar{n}\left(2a^\dagger \rho_S a - a a^\dagger \rho_S - \rho_S a a^\dagger\right). \end{aligned} \tag{2}$$

Do this without repeating derivations already in the lecture notes, and give expressions for γ, $\bar{n}$, and ω_0'.

b) Write down differential equations for the population of the n^{th} state of the simple harmonic oscillator, $p(n,t) \equiv \langle n|\rho_S(t)|n\rangle$. Suppose the system is in initial Fock state $|1\rangle$ at the beginning of the evolution under Eq. (2). Supposing the bath temperature is $T = 0$, what are $p(n,t)$? Let's keep $T \neq 0$ for the remainder of this problem.

c) Write down and solve the ordinary differential equation for $\langle a\rangle(t) = \mathrm{tr}_S\{\rho_S(t)a\}$.

d) Write down and solve the ordinary differential equation for $\langle n\rangle(t) = \mathrm{tr}_S\{\rho_S(t)n\}$, with $n = a^\dagger a$.

e) Show that

$$\rho_{\mathrm{eq}} = \frac{e^{-H_S/k_B T}}{\mathrm{tr}\left(e^{-H_S/k_B T}\right)} = \frac{e^{-\hbar\omega_0 a^\dagger a/k_B T}}{1 - e^{-\hbar\omega_0/k_B T}} \tag{3}$$

is a steady state of Eq. (2).

II. SPIN-1/2 COUPLED TO BOSONIC BATHS

Consider a spin-1/2 coupled to two bosonic baths, both at temperature T,

$$\begin{aligned} H_S =& \frac{1}{2}\hbar\omega_{01}\sigma_z, \\ H_I =& \sigma_x \otimes \sum_l g_{x,l}(b_{x,l} + b_{x,l}^\dagger) + \sigma_z \otimes \sum_l g_{z,l}(b_{z,l} + b_{z,l}^\dagger), \\ H_B =& \sum_{\alpha=x,z} \sum_l \hbar\omega_{\alpha,l} b_{\alpha,l}^\dagger b_{\alpha,l}, \end{aligned} \tag{4}$$

where the canonical commutators hold $[b_{\alpha,l}, b_{\beta,m}^\dagger] = \delta_{\alpha\beta}\delta_{lm}$.

a) Show that the Lindblad master equation for the dynamics of the reduced density matrix of the system is (express all quantities below in terms of two-point correlation functions of the baths at finite temperature):

$$\frac{d}{dt}\rho_S(t) = -i\left[\frac{1}{2}\omega'_{01}\sigma_z, \rho_S(t)\right] + \gamma_\downarrow \mathcal{D}\left[\sigma_-\right]\rho_S(t) + \gamma_\uparrow \mathcal{D}\left[\sigma_+\right]\rho_S(t) + \frac{1}{2}\gamma_\varphi \mathcal{D}\left[\sigma_z\right]\rho_S(t). \tag{5}$$

b) Find equations of motion for $\langle\sigma_{\pm,z}\rangle(t) = \mathrm{tr}_S\{\rho_S(t)\sigma_{\pm,z}\}$. Hint: one way to do this is to write down equations of motion first for the four entries of the reduced density matrix in the qubit Hilbert space, $\rho_S(t)$.

c) Show that the expectation value $\langle\sigma_z\rangle$ has an exponential decay with characteristic time T_1, i.e. $\langle\sigma_z\rangle(t) \propto e^{-t/T_1}$, and express T_1 in terms of the constants in Eq. (5). Show that $\langle\sigma_-\rangle(t)$ oscillates in time with an exponentially decaying envelope, with a characterstic timescale T_2, again to be expressed in terms of the constants in Eq. (5). Express $1/T_2$ in terms of $1/T_1$ and $1/T_\varphi \equiv \gamma_\varphi$.

d) Show that the density matrix ρ_S is pure if and only if $|\langle\vec{\sigma}\rangle| = 1$. Can you write down an equation of motion for the purity of the density matrix, $\mathrm{Tr}\{\rho^2\}$?

e) Assume that $T \to \infty$. What is the steady-state density matrix? Same question for $T \to 0$. What is $\langle\vec{\sigma}\rangle$ for each of these steady states?

f) How does the answer in part a) change if now you have $H_S = \hbar\omega_{01}\sigma_z + \hbar\omega_x\sigma_x$? What about the particular case when $\omega_{01} = 0$ and $\omega_x > 0$?