Problem Set 1

(M2 Dynamics and control of open quantum systems 2023-2024)

This problem set is due on Friday, September 24th, 2023, at 5 PM. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. RABI-DRIVEN QUBIT

Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows:

$$V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}).$$

At $t = 0$, it is known that only the lower level is populated – that is, $c_1(0) = 1, c_2(0) = 0$.
a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by exactly solving the coupled differential equation

$$i\hbar \dot{c}_k = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n, \quad (k = 1, 2)$$

b) Do the same problem using time-dependent perturbation theory to lowest nonvanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21} and (ii) ω close to ω_{21}.

Hint: the answer for a) is Rabi's formula, which is so important that we reproduce it here

$$\begin{aligned} |c_2(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2 \left\{ \left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4} \right]^{1/2} t \right\}, \\ |c_1(t)|^2 &= 1 - |c_2(t)|^2 . \end{aligned} \tag{1}$$

SOLUTION

The Hamiltonian in the problem reads

$$H(t) = \begin{pmatrix} E_1 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & E_2 \end{pmatrix}. \tag{2}$$

Defining $\omega_{kn} = (E_k - E_n)/\hbar$, for $k, n = 1, 2$, we go to the interaction picture with respect to H_0 representing the diagonal part of the Hamiltonian, i.e. by applying $H'(t) - i\partial_t \equiv U^\dagger(t)\,[H(t) - i\partial_t]\,U(t)$ with $U(t) = e^{-iH_0 t}$. There is a partial derivative on the left hand side of the previous equation since we assume that both operators in that equation act on a test function from the left, and apply the chain rule.

With these notations, letting $\Omega = \omega + \omega_{12}$

$$H'(t) = \begin{pmatrix} 0 & \gamma e^{i\Omega t} \\ \gamma e^{-i\Omega t} & 0 \end{pmatrix}. \tag{3}$$

In this frame, the Schrödinger equation reads

$$i\partial_t \left|\psi'(t)\right\rangle = H'(t) \left|\psi'(t)\right\rangle, \tag{4}$$

which, upon using Eq. (3) and furthermore denoting the two components of $\left|\psi'(t)\right\rangle$ as $\left\langle 1\middle|\psi'(t)\right\rangle = c_1(t)$, and $\left\langle 2\middle|\psi'(t)\right\rangle = c_2(t)$, takes the following form

$$\begin{aligned} i\hbar\dot{c}_1(t) &= \gamma e^{i\Omega t} c_2(t), \\ i\hbar\dot{c}_2(t) &= \gamma e^{-i\Omega t} c_1(t), \end{aligned} \tag{5}$$

from which, by rearranging factors in the first equation, taking a time derivative, and inserting the second equation, we deduce (we set $\hbar = 1$ and will reinstate it at the end with dimensional analysis)

$$\begin{aligned} \gamma^{-1}\frac{d}{dt}\left[e^{-i\Omega t}\dot{c}_1(t)\right] &= -i\dot{c}_2(t) = -\gamma e^{-i\Omega t} c_1(t), \\ \frac{d}{dt}\left[e^{-i\Omega t}\dot{c}_1(t)\right] &= -\gamma^2 e^{-i\Omega t} c_1(t), \\ -i\Omega e^{-i\Omega t}\dot{c}_1(t) + e^{-i\Omega t}\ddot{c}_1(t) &= -\gamma^2 e^{-i\Omega t} c_1(t), \end{aligned} \tag{6}$$

or, after rearranging phase factors

$$\ddot{c}_1(t) - i\Omega\dot{c}_1(t) + \gamma^2 c_1(t) = 0. \tag{7}$$

If we look for solutions of the form $c_1(t) = e^{i\nu t}$, ν must obey the quadratic polynomial equation

$$-\nu^2 + \Omega\nu + \gamma^2 = 0, \tag{8}$$

with solutions

$$\nu_\pm = \frac{\Omega}{2} \pm \sqrt{\frac{\Omega^2}{4} + \gamma^2}, \tag{9}$$

with the general complex solution to the homogeneous differential equation

$$c_1(t) = A_+ e^{i\nu_+ t} + A_- e^{-i\nu_- t}, \tag{10}$$

with $A_\pm$ complex coefficients. Now we impose $c_1(0) = 1$, so

$$c_1(t) = Ae^{i\nu_+ t} + (1 - A)e^{i\nu_- t}, \tag{11}$$

for some complex-valued coefficient A, yet undetermined.

We now write the expression for $c_2(t)$ using Eq. (5),

$$\begin{aligned} c_2(t) &= e^{-i\Omega t}\gamma^{-1} i\dot{c}_1'(t) = e^{-i\Omega t}\gamma^{-1} i \frac{d}{dt}\left\{Ae^{i\nu_+ t} + (1 - A)e^{i\nu_- t}\right\} \\ &= e^{-i\Omega t}\gamma^{-1} i \left\{Ai\nu_+ e^{i\nu_+ t} + i\nu_-(1 - A)e^{i\nu_- t}\right\} \\ &= -\gamma^{-1}\left\{A\nu_+ e^{i(\nu_+ - \Omega)t} + \nu_-(1 - A)e^{i(\nu_- - \Omega)t}\right\}. \end{aligned} \tag{12}$$

We now impose $c_2(t) = 0$, so that

$$A\nu_+ + \nu_-(1 - A) = 0, \text{ or } A = \frac{\nu_-}{\nu_- - \nu_+} = \frac{\sqrt{\frac{\Omega^2}{4} + \gamma^2} - \frac{\Omega}{2}}{2\sqrt{\frac{\Omega^2}{4} + \gamma^2}}. \tag{13}$$

Thus

$$\begin{aligned} c_2(t) &= -\gamma^{-1}\left\{\frac{\nu_-}{\nu_- - \nu_+}\nu_+ e^{i(\nu_+ - \Omega)t} + \nu_-(1 - \frac{\nu_-}{\nu_- - \nu_+})e^{i(\nu_- - \Omega)t}\right\} \\ &= -\gamma^{-1}\left\{\frac{\nu_-\nu_+}{\nu_- - \nu_+}e^{i(\nu_+ - \Omega)t} - \frac{\nu_-\nu_+}{\nu_- - \nu_+}e^{i(\nu_- - \Omega)t}\right\} \\ &= \frac{2i\gamma^{-1}\nu_-\nu_+}{\nu_+ - \nu_-}e^{-i\Omega t/2}\sin\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2}\right), \end{aligned} \tag{14}$$

with

$$|c_2(t)|^2 = \frac{\gamma^2}{\frac{\Omega^2}{4} + \gamma^2}\sin^2\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2}\right) \tag{15}$$

and

$$
\begin{aligned}
c_1(t) &= \frac{\nu_-}{\nu_- - \nu_+} e^{i\nu_+ t} - \frac{\nu_+}{\nu_- - \nu_+} e^{i\nu_- t} \\
&= \frac{e^{i\Omega t/2}}{\nu_- - \nu_+} \left[-2\sqrt{\frac{\Omega^2}{4} + \gamma^2} \cos\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) + i\Omega \sin\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) \right] \\
&= e^{i\Omega t/2} \left[\cos\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) + \frac{i\Omega}{\nu_- - \nu_+} \sin\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) \right],
\end{aligned}
\tag{16}
$$

and thus

$$
|c_1(t)|^2 = \cos^2\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) + \frac{\Omega^2/4}{\frac{\Omega^2}{4} + \gamma^2} \sin^2\left(t\sqrt{\frac{\Omega^2}{4} + \gamma^2} \right) \tag{17}
$$

Moreover, the sum of the populations of the two levels is one

$$
|c_1(t)|^2 = 1 - |c_2(t)|^2. \tag{18}
$$

Note that the result including $\hbar$ can be obtained by setting $\gamma \to \gamma/\hbar$ in Eq. (15) and Eq. (17).

b) Using Eq. (21) of the course notes,

$$
\begin{aligned}
c_n^{(0)}(t) &= \delta_{ni}, \\
c_n^{(1)}(t) &= -\frac{i}{\hbar} \int_{t_0}^{t} dt' e^{i\omega_{ni} t'} V_{ni}(t'),
\end{aligned}
\tag{19}
$$

we have, taking $t_0 = 0$,

$$
\begin{aligned}
c_1(t) &= 1 - \frac{i}{\hbar} \int_0^t dt' e^{i\omega_{11} t'} V_{11}(t') = 1 + O(V^2), \\
c_2(t) &= 0 - \frac{i}{\hbar} \int_0^t dt' e^{i\omega_{21} t'} V_{21}(t') = -\frac{i}{\hbar} \int_0^t dt' e^{i\omega_{21} t'} \gamma e^{-i\omega t'} = \frac{\gamma}{\hbar} \frac{e^{i(\omega_{21} - \omega)t} - 1}{\omega - \omega_{21}} \\
&= \frac{\gamma}{\hbar(\omega - \omega_{21})} [e^{i(\omega_{21} - \omega)t} - 1],
\end{aligned}
\tag{20}
$$

so that the population of state 2 is

$$
\begin{aligned}
|c_2(t)|^2 &= \frac{\gamma^2}{\hbar^2(\omega - \omega_{21})^2} |e^{i(\omega_{21} - \omega)t} - 1|^2 = \frac{\gamma^2}{\hbar^2(\omega - \omega_{21})^2} \left(e^{i(\omega_{21} - \omega)t} - 1 \right) \left(e^{-i(\omega_{21} - \omega)t} - 1 \right) \\
&= \frac{\gamma^2}{\hbar^2(\omega - \omega_{21})^2} \left\{ 2 - 2\cos\left[(\omega - \omega_{21}) t \right] \right\} = \frac{4\gamma^2}{\hbar^2(\omega - \omega_{21})^2} \sin^2\left(\frac{\omega - \omega_{21}}{2} t \right).
\end{aligned}
\tag{21}
$$

This expression should match, to lowest order, the exact result Eq. (15). To see this, we take that equation and Taylor expand in γ

$$\begin{aligned}|c_2(t)|^2 &= \frac{\gamma^2}{\frac{\Omega^2}{4}+\gamma^2}\sin^2\left(t\sqrt{\frac{\Omega^2}{4}+\gamma^2}\right) = \frac{4\gamma^2}{\Omega^2}\left(1-\frac{4\gamma^2}{\Omega^2}+\dots\right)\sin^2\left[\frac{\Omega}{2}\left(1+\frac{2\gamma^2}{\Omega^2}+\dots\right)t\right] \\ &= \frac{4\gamma^2}{\Omega^2}\sin^2\left[\frac{\Omega}{2}t\right]+O(\gamma^4),\end{aligned} \tag{22}$$

the same as Eq. (21) after recalling our expression for Ω. Same as before, the result for $\hbar \neq 1$ is obtained by passing $\gamma \to \gamma/\hbar$. If the drive is nearly resonant, then $\Omega \approx 0$, and the perturbative result gives

$$|c_2(t)|^2 \approx \frac{\gamma^2}{\hbar^2}t^2, \tag{23}$$

consistent with the exact result which gives as $\Omega \to 0$

$$\begin{aligned} c_1(t) &= \cos\left(\frac{\gamma}{\hbar}t\right), \\ c_2(t) &= i\sin\left(\frac{\gamma}{\hbar}t\right). \end{aligned} \tag{24}$$

These are the Rabi oscillations with Rabi frequency $\gamma/\hbar$ that allows us to perform single-qubit gates on the qubit, such as the π pulse and the $\pi/2$ pulse.

II. QUANTUM SPECTROMETER OF CLASSICAL NOISE

Consider a qubit described by the unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hbar\omega_{01}}{2}\hat{\sigma}_z. \tag{25}$$

Assume that at time $t = 0$ this qubit is coupled to a classical noise source

$$\hat{V} = AF(t)\hat{\sigma}_x, \tag{26}$$

where $F(t)$ is a noisy function with zero mean, $\overline{F(t)} = 0$, and time-translation invariant $\overline{F(t)F(t')} = \overline{F(t-t')F(0)}$. Moreover, assume that $\overline{F(t-t')F(0)}$ decays exponentially fast in $|t-t'|$ whenever $|t-t'| \gg \tau_c$, for some characteristic time τ_c. Furthermore, we define the noise spectral density

$$S_{FF}(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau}\overline{F(\tau)F(0)}. \tag{27}$$

You can assume that the system evolves according to the total Hamiltonian $\hat{H} + \hat{V}(t)$. We will use time-dependent perturbation theory to find how the relaxation and excitation rates of the qubit allow us to measure properties of the classical noise source $F(t)$.

a) Assume that the system starts in the ground state $|i\rangle = |0\rangle$ of H_0, i.e. $c_0(t) = 1$. Evaluate the time-dependent population of the excited state $|c_1(t)|^2$ to second order in perturbation theory in $\hat{V}$. You can leave your answer in terms of a double time-integral.

b) Ensemble average your result above over noise realizations, then perform the time integrals under the assumption that $t \gg \tau_c$, and using time-translation invariance. Hint: You should get a population that grows linearly on time: $|c_1(t)|^2 = t \cdot \# \cdot S_{FF}(\#)$, where $\#$ are constants that depend on $A, \hbar, \omega_{01}$ that you are to find.

c) Find the rate of excitation, or escape from the ground state due to the perturbation, $w_{0\to 1}$.

d) How would your results in b) and c) change if you now started in the excited state $|i\rangle = |0\rangle$ and were asked to give the population of the ground state, and the relaxation rate due to the classical noise source?

SOLUTION

a) Note that in this problem we use $\omega_{01} > 0$ for the qubit frequency. In our conventions for Eq. (21), we have to flip the sign of the frequency to accommodate the notation in this problem

$$\begin{aligned} c_1^{(0)}(t) &= 0 \\ c_1^{(1)}(t) &= -\frac{i}{\hbar}\int_0^t dt' e^{i\omega_{10}t'} V_{10}(t') \\ &= -\frac{iA}{\hbar}\int_0^t dt' e^{i\omega_{01}t'} F(t'). \end{aligned} \tag{28}$$

and therefore to second order in the amplitude of the perturbation A,

$$|c_1(t)|^2 = -\frac{A^2}{\hbar^2}\int_0^t dt' \int_0^t dt'' e^{i\omega_{01}(t'-t'')} F(t')F(t''). \tag{29}$$

b) Ensemble averaging over noise realizations gives

$$\begin{aligned}
\overline{|c_1(t)|^2} &= \frac{A^2}{\hbar^2}\int_0^t dt' \int_0^t dt'' e^{i\omega_{01}(t'-t'')}\overline{F(t')F(t'')} \\
&= \frac{A^2}{\hbar^2}\int_0^t dt' \int_0^t dt'' e^{i\omega_{01}(t'-t'')}\overline{F(t'-t'')F(0)} \\
&= \frac{A^2}{\hbar^2}\int_0^t d\left(\frac{t'+t''}{2}\right) \int_{-t}^t d(t'-t'') e^{i\omega_{01}(t'-t'')}\overline{F(t'-t'')F(0)} \\
&= \frac{A^2}{\hbar^2} t \int_{-t}^t d\tau e^{i\omega_{01}\tau}\overline{F(\tau)F(0)},
\end{aligned} \tag{30}$$

where we have used the time-translation invariance property first, then made a change of variable which allowed us to perform one of the time integrals. Finally, assuming $t \ll \tau_c$, we can change the limits of integration from $-t, t$ to $-\infty, \infty$, since the integrand is non-negligible on a small interval of size $2\tau_c \ll t$. So

$$\begin{aligned}
\overline{|c_1(t)|^2} &= \frac{A^2}{\hbar^2} t \int_{-\infty}^{\infty} d\tau e^{i\omega_{01}\tau}\overline{F(\tau)F(0)} \\
&= \frac{A^2}{\hbar^2} t S_{FF}(\omega_{01}).
\end{aligned} \tag{31}$$

c) The rate of populating the 1 state is the time derivative of the population calculated above

$$\gamma_\uparrow \equiv \frac{d}{dt}\overline{|c_1(t)|^2} = \frac{A^2}{\hbar^2} S_{FF}(\omega_{01}). \tag{32}$$

d) If the two states are reversed, we would have the following changes. Eq. (30) changes to

$$\overline{|c_0(t)|^2} = \frac{A^2}{\hbar^2} t \int_{-t}^t d\tau e^{-i\omega_{01}\tau}\overline{F(\tau)F(0)}, \tag{33}$$

from which the population of the ground state evaluates to

$$\overline{|c_0(t)|^2} = \frac{A^2}{\hbar^2} t S_{FF}(-\omega_{01}), \tag{34}$$

and therefore the relaxation rate is

$$\gamma_\downarrow \equiv \frac{d}{dt}\overline{|c_0(t)|^2} = \frac{A^2}{\hbar^2} S_{FF}(-\omega_{01}). \tag{35}$$