Problem Set 1
(M2 Dynamics and control of open quantum systems 2023-2024)

This problem set is due on Friday, September 24th, 2023, at 5 PM. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. RABI-DRIVEN QUBIT

Consider a two-level system with $E_1 < E_2$. There is a time-dependent potential that connects the two levels as follows:

$$V_{11} = V_{22} = 0, \quad V_{12} = \gamma e^{i\omega t}, \quad V_{21} = \gamma e^{-i\omega t} \quad (\gamma \text{ real}).$$

At $t = 0$, it is known that only the lower level is populated – that is, $c_1(0) = 1, c_2(0) = 0$. a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by exactly solving the coupled differential equation

$$i\hbar \dot{c}_k = \sum_{n=1}^{2} V_{kn}(t) e^{i\omega_{kn} t} c_n, \quad (k = 1, 2)$$

b) Do the same problem using time-dependent perturbation theory to lowest nonvanishing order. Compare the two approaches for small values of γ. Treat the following two cases separately: (i) ω very different from ω_{21} and (ii) ω close to ω_{21}.

Hint: the answer for a) is Rabi's formula, which is so important that we reproduce it here

$$\begin{aligned} |c_2(t)|^2 &= \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2 \left\{ \left[\frac{\gamma^2}{\hbar^2} + \frac{(\omega - \omega_{21})^2}{4} \right]^{1/2} t \right\}, \\ |c_1(t)|^2 &= 1 - |c_2(t)|^2 . \end{aligned} \tag{1}$$

II. QUANTUM SPECTROMETER OF CLASSICAL NOISE

Consider a qubit described by the unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hbar\omega_{01}}{2} \hat{\sigma}_z. \tag{2}$$

Assume that at time $t = 0$ this qubit is coupled to a classical noise source

$$\hat{V} = AF(t)\hat{\sigma}_x, \tag{3}$$

where $F(t)$ is a noisy function with zero mean, $\overline{F(t)} = 0$, and time-translation invariant $\overline{F(t)F(t')} = \overline{F(t-t')F(0)}$. Moreover, assume that $\overline{F(t-t')F(0)}$ decays exponentially fast in $|t-t'|$ whenever $|t-t'| \gg \tau_c$, for some characteristic time τ_c. Furthermore, we define the noise spectral density

$$S_{FF}(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \overline{F(\tau)F(0)}. \tag{4}$$

You can assume that the system evolves according to the total Hamiltonian $\hat{H} + \hat{V}(t)$. We will use time-dependent perturbation theory to find how the relaxation and excitation rates of the qubit allow us to measure properties of the classical noise source $F(t)$.

a) Assume that the system starts in the ground state $|i\rangle = |0\rangle$ of H_0, i.e. $c_0(t) = 1$. Evaluate the time-dependent population of the excited state $|c_1(t)|^2$ to second order in perturbation theory in $\hat{V}$. You can leave your answer in terms of a double time-integral.

b) Ensemble average your result above over noise realizations, then perform the time integrals under the assumption that $t \gg \tau_c$, and using time-translation invariance. Hint: You should get a population that grows linearly on time: $|c_1(t)|^2 = t \cdot \# \cdot S_{FF}(\#)$, where $\#$ are constants that depend on $A, \hbar, \omega_{01}$ that you are to find.

c) Find the rate of excitation, or escape from the ground state due to the perturbation, $w_{0\to 1}$.

d) How would your results in b) and c) change if you now started in the excited state $|i\rangle = |0\rangle$ and were asked to give the population of the ground state, and the relaxation rate due to the classical noise source?