Dynamics and Control of Open Quantum Systems
Lectures 8 and 9 (November 2023):
Quantum measurement and continuous-time open systems

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1 Stochastic master equations

These models have their origins in the work of Davies [6], are related to quantum trajectories [3, 5] and to Belavkin quantum filters [2]. A modern and mathematical exposure of the diffusive models is given in [1]. For a tutorial passage of discret-time towards continuous-time formulation of stochastic master equations see [16]. These models are interpreted here as continuous-time versions of discrete-time (partial) Kraus maps. They are based on stochastic differential equations, also called Stochastic Master Equations (SME). They provide the evolution of the density operator $\boldsymbol{\rho}_t$ with respect to the time t. They are driven by a finite number of independent Wiener processes indexed by ν, $(W_{\nu,t})$, each of them being associated to a continuous classical and real signal, $y_{\nu,t}$, produced by detector ν. These SMEs admit the following form:

$$d\boldsymbol{\rho}_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \tfrac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu) \right) dt + \sum_\nu \sqrt{\eta_\nu} \left(\boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger) \boldsymbol{\rho}_t \right) \boldsymbol{\rho}_t \right) dW_{\nu,t} \quad (1)$$

where $\boldsymbol{H}$ is the Hamiltonian operator on the underlying Hilbert space $\mathcal{H}$ and $\boldsymbol{L}_\nu$ are arbitrary operators (not necessarily Hermitian) on $\mathcal{H}$. Each measured signal $y_{\nu,t}$ is related to $\boldsymbol{\rho}_t$ and $W_{\nu,t}$ by the following output relationship:

$$dy_{\nu,t} = dW_{\nu,t} + \sqrt{\eta_\nu} \operatorname{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger) \boldsymbol{\rho}_t \right) dt$$

where $\eta_\nu \in [0, 1]$ is the efficiency of detector ν.

For the case of a finite dimensional Hilbert space, it has been proven in [14, 1] that the above SME admits a unique strong solution in the space of well-defined density matrices

$$\mathcal{S} = \{ \boldsymbol{\rho} \mid \boldsymbol{\rho} = \boldsymbol{\rho}^\dagger, \boldsymbol{\rho} \geq 0, \operatorname{Tr}(\boldsymbol{\rho}) = 1 \}.$$

The ensemble average of $\boldsymbol{\rho}_t$ obeys thus to a linear differential equation, also called master or Lindblad-Kossakowski differential equation [9, 12]:

$$\frac{d}{dt}\boldsymbol{\rho} = -\tfrac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}] + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger - \tfrac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \boldsymbol{\rho}_t + \boldsymbol{\rho}_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu). \quad (2)$$

It is the continuous-time analogue of the Kraus map $\boldsymbol{K}$ associated to a discrete-time quantum Markov process.

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In fact (1) has the same structure. This becomes obvious if one remarks that, with standard Itō rules, (1) admits the following formulation

$$\boldsymbol{\rho}_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t}\boldsymbol{\rho}_t\boldsymbol{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt\right)}$$

with $\boldsymbol{M}_{dy_t} = \boldsymbol{I} + \left(-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\right)dt + \sum_\nu \sqrt{\eta_\nu}dy_{\nu t}\boldsymbol{L}_\nu$. Moreover the probability associated to the measurement outcome $dy = (dy_\nu)$, is given by the following density

$$p\left(dy \in \prod_\nu [\xi_\nu, \xi_\nu + d\xi_\nu] \;\Big/\; \boldsymbol{\rho}_t\right) = \mathrm{Tr}\left(\boldsymbol{M}_\xi\boldsymbol{\rho}_t\boldsymbol{M}_\xi^\dagger + \sum_\nu (1-\eta_\nu)\boldsymbol{L}_\nu\boldsymbol{\rho}_t\boldsymbol{L}_\nu^\dagger dt\right)\prod_\nu \tfrac{d\xi_\nu}{\sqrt{2\pi dt}} e^{-\xi_\nu^2/2dt}$$

where ξ stands for the vector (ξ_ν). With such a formulation, it becomes clear that (1) preserves the trace and the non-negativity of $\boldsymbol{\rho}$. This formulation provides also directly a time discretization numerical scheme preserving non-negativity of $\boldsymbol{\rho}$.

We recall here the basic rule of Itō differential calculus for the stochastic system of state $X \in \mathbb{R}^n$ and driven by m scalar Wiener independent processes $W_{v,t}$:

$$X_{t+dt} - X_t = dX_t = F(X_t,t)dt + \sum_\nu G_\nu(X_t,t)dW_{\nu,t}$$

where $F(X,t)$ and $(G_\nu(X,t))$ are smooth functions of X and piece-wise continuous functions of t. For any C^2 real function f of X, the computation of $df_t = f(X_{t+dt}) - f(X_t)$ is conducted up to including order one in dt with the following rules: $dW_{\nu,t} = O(\sqrt{dt})$, $(dW_{\nu,t})^2 = dt$, $dW_{\nu,t}DW_{\nu',t} = 0$ for $\nu \neq \nu'$ and any other products between the $dW_{\nu,t}$ being zero since of order greater than $(dt)^{3/2}$. This means that we have

$$\begin{aligned} df_t = f(X_{t+dt}) - f(X_t) &= f(X_t + dX_t) - f(X_t) \\ &= \left.\frac{\partial f}{\partial X}\right|_{X_t} dX_t + \tfrac{1}{2}\left.\frac{\partial^2 f}{\partial X^2}\right|_{X_t}(dX_t, dX_t) + \ldots \\ &= \left(\left.\frac{\partial f}{\partial X}\right|_{X_t} F(X_t,t) + \tfrac{1}{2}\sum_\nu \left.\frac{\partial^2 f}{\partial X^2}\right|_{X_t}(G_\nu(X_t,t), G_\nu(X_t,t))\right)dt \\ &\qquad + \sum_\nu \left.\frac{\partial f}{\partial X}\right|_{X_t} G_\nu(X_t,t)dW_{\nu,t}. \end{aligned}$$

Notice that we have removed terms with $dtdW_{\nu,t}$ since of order $dt^{3/2}$. For expectation values, all $dW_{\nu,t}$ are independent of X_t and $\mathbb{E}(dW_{\nu,t}) = 0$. Thus we have for any C^2 function f of X:

$$\mathbb{E}(df_t \mid X_t) = \left(\left.\frac{\partial f}{\partial X}\right|_{X_t} F(X_t,t) + \tfrac{1}{2}\sum_\nu \left.\frac{\partial^2 f}{\partial X^2}\right|_{X_t}(G_\nu(X_t,t), G_\nu(X_t,t))\right)dt.$$

2 QND measurement of a qubit and asymptotic behavior

In this section, we consider a continuous measurement protocol for a single qubit. The considered setup corresponds to the inverse of the photon box experiment. As illustrated in Figure 1, we consider the qubit to be fixed inside the cavity and interacting with the confined electromagnetic field. The cavity however is assumed to be not ideal and the confined field can leak out at a rate κ. This outgoing field is continuously measured through what is called a homodyne measurement process, corresponding to the measurement of a certain quadrature $\boldsymbol{X}_\lambda = (e^{i\lambda}\boldsymbol{a}^\dagger + e^{-i\lambda}\boldsymbol{a})/2$ as physical observable. Assuming a dispersive coupling between the qubit and the cavity and in the regime where the leakage rate κ is much stronger than the other dynamical time-scales, such as an eventual Rabi oscillation rate for the qubit, the cavity dynamics can be removed leading to a stochastic master equation for the qubit [7] (we will skip the details of this model reduction which includes some details that are out of the scope of these lectures).

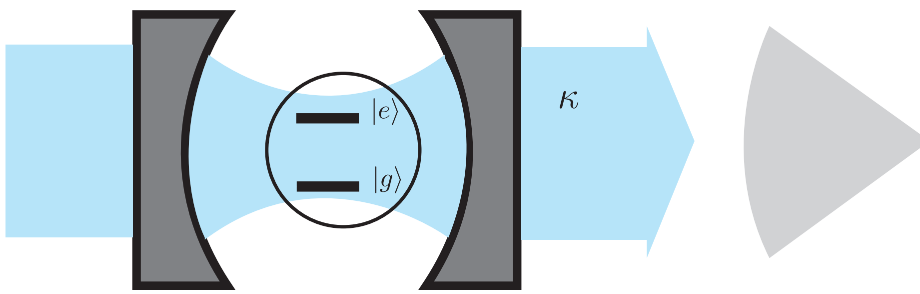

Figure 1: The cavity field interacts with the qubit and the cavity output gets measured providing information on the state of the qubit.

For a well-chosen measured quadrature $\boldsymbol{X}_\lambda$, this SME of the form (1) is given by

$$d\boldsymbol{\rho}_t = -\frac{i}{\hbar}[\boldsymbol{H}, \boldsymbol{\rho}_t]dt + \frac{\Gamma_m}{4}(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t\boldsymbol{\sigma_z} - \boldsymbol{\rho}_t)dt + \frac{\sqrt{\eta\Gamma_m}}{2}\left(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t + \boldsymbol{\rho}_t\boldsymbol{\sigma_z} - 2\operatorname{Tr}(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t)\right)dW_t, \tag{3}$$

where $\boldsymbol{H}$ is the qubit's Hamiltonian, the only Lindblad operator $\boldsymbol{L}_\nu$ is given by $\sqrt{\Gamma_m}\boldsymbol{\sigma_z}/2$, and $\eta \in [0, 1]$ represents the detector efficiency. The measured signal dy_t is given by

$$dy_t = dW_t + \sqrt{\eta\Gamma_m}\operatorname{Tr}(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t)\,dt. \tag{4}$$

Let us consider here the uncontrolled case where the Hamiltonian $\boldsymbol{H}/\hbar$ is simply given by $\omega_{\text{eg}}\boldsymbol{\sigma_z}/2$. Following the arguments of the previous section, the above SME correspond to a Markov process with the Kraus operators

$$\boldsymbol{M}_{dy_t} = \boldsymbol{I} - (i\frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \frac{\Gamma_m}{8}\boldsymbol{I})dt + \frac{\sqrt{\eta\Gamma_m}}{2}\boldsymbol{\sigma_z}dy_t \quad \text{and} \quad \sqrt{(1-\eta)dt}\boldsymbol{L} = \frac{\sqrt{(1-\eta)\Gamma_m dt}}{2}\boldsymbol{\sigma_z}.$$

Noting that the above operators commute with $\boldsymbol{\sigma_z}$. Thus we have a quantum non-demolition (QND) measurement of the observable $\boldsymbol{\sigma_z}$. We study here the asymptotic behavior of the open-loop system undergoing the above continuous measurement process.

Theorem 1. *Consider the SME (3) with $\boldsymbol{H}/\hbar = \omega_{\text{eg}}\boldsymbol{\sigma_z}/2$ and $\eta > 0$. For any initial density matrix $\boldsymbol{\rho}_0$, the solution $\boldsymbol{\rho}_t$ converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or*

$|e\rangle\langle e|$. *Furthermore the probability of convergence to* $|g\rangle\langle g|$ *(respectively* $|e\rangle\langle e|$*) is given by* $p_g = \mathrm{Tr}\left(|g\rangle\langle g|\,\boldsymbol{\rho}_0\right)$ *(respectively* $\mathrm{Tr}\left(|e\rangle\langle e|\,\boldsymbol{\rho}_0\right)$*).*

Proof. We consider the Lyapunov function

$$V(\boldsymbol{\rho}) = 1 - \mathrm{Tr}\left(\boldsymbol{\sigma_z}\boldsymbol{\rho}\right)^2.$$

Applying the Ito rules, we have

$$\frac{d}{dt}\mathbb{E}\left(V(\boldsymbol{\rho}_t)\right) = -\eta\Gamma_m\mathbb{E}\left(V^2(\boldsymbol{\rho}_t)\right) \leq 0,$$

and thus

$$\mathbb{E}\left(V(\boldsymbol{\rho}_t)\right) = V(\boldsymbol{\rho}_0) - \eta\Gamma_m\int_0^t \mathbb{E}\left(V^2(\boldsymbol{\rho}_s)\right)ds.$$

Noting that $V(\boldsymbol{\rho}) \geq 0$, we have

$$\eta\Gamma_m\int_0^t \mathbb{E}\left(V^2(\boldsymbol{\rho}_s)\right)ds = V(\boldsymbol{\rho}_0) - \mathbb{E}\left(V(\boldsymbol{\rho}_t)\right) \leq V(\boldsymbol{\rho}_0) < \infty.$$

Thus we have the monotone convergence

$$\mathbb{E}\left(\int_0^\infty V^2(\boldsymbol{\rho}_s)ds\right) < \infty \Rightarrow \int_0^\infty V^2(\boldsymbol{\rho}_s)ds < \infty \text{ almost surely.}$$

By Theorem 5 of Appendix B, the limit $V(\boldsymbol{\rho}_t)$ as $t \to \infty$ exists with probability one (as a supermartingale bounded from below) and hence, the above inequality implies that $V(\boldsymbol{\rho}_t) \to 0$ almost surely. But the only states $\boldsymbol{\rho}$ satisfying $V(\boldsymbol{\rho}) = 0$ are $\boldsymbol{\rho} = |g\rangle\langle g|$ or $\boldsymbol{\rho} = |e\rangle\langle e|$.

We can finish the proof by noting that $\mathrm{Tr}\left(\boldsymbol{\sigma_z}\boldsymbol{\rho}_t\right)$ is a martingale. Therefore the probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \mathrm{Tr}\left(|g\rangle\langle g|\,\boldsymbol{\rho}_0\right)$ (respectively $\mathrm{Tr}\left(|e\rangle\langle e|\,\boldsymbol{\rho}_0\right)$). □

The above theorem implies that the continuous QND measurement can be seen as a non-deterministic preparation protocol for the states $|g\rangle\langle g|$ and $|e\rangle\langle e|$. This preparation can be rendered deterministic by adding an appropriate feedback control. Indeed, it has been proven in [18, 14] that, a controlled Hamiltonian

$$\boldsymbol{H} = \frac{\omega_{\mathrm{eg}}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x},$$

with the feedback law

$$u(\boldsymbol{\rho}) = -\alpha\,\mathrm{Tr}\left(i[\boldsymbol{\sigma_x},\boldsymbol{\rho}]\boldsymbol{\rho}_{\mathrm{tag}}\right) + \beta(1 - \mathrm{Tr}\left(\boldsymbol{\rho}\boldsymbol{\rho}_{\mathrm{tag}}\right)), \qquad \alpha,\beta > 0 \quad \text{and} \quad \beta^2 < 8\alpha\eta,$$

globally stabilizes the target state $\boldsymbol{\rho}_{\mathrm{tag}} = |g\rangle\langle g|$ or $|e\rangle\langle e|$.

3 Lindblad master equation

The continuous-time analogue of the discrete-time quantum master equation (ensemble average dynamics) becomes a differential equation for the time-evolution of the density operator $t \mapsto \boldsymbol{\rho}(t)$:

$$\frac{d}{dt}\boldsymbol{\rho} = -\tfrac{i}{\hbar}[\boldsymbol{H},\boldsymbol{\rho}] + \sum_\nu \boldsymbol{L}_\nu\boldsymbol{\rho}\boldsymbol{L}_\nu^\dagger - \tfrac{1}{2}(\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{L}_\nu^\dagger\boldsymbol{L}_\nu) \tag{5}$$

where

- $\boldsymbol{H}$ is the Hamiltonian that could depend on t (Hermitian operator on the underlying Hilbert space $\mathcal{H}$)

- the L_ν's are operators on $\mathcal{H}$ that are not necessarily Hermitian.

The differential equation (5) preserves the positivity and the trace: if the initial condition $\boldsymbol{\rho}_0$ is Hermitian of trace one and non-negative, then its solution $\boldsymbol{\rho}(t)$ for $t \geq 0$ is also Hermitian, non-negative and of trace one. To avoid mathematical technicalities we consider in the theorem below that $\mathcal{H}$ is of finite dimension.

Theorem 2. *Assume that $\mathcal{H}$ is of finite dimension. Then for any Hermitian operator $t \mapsto \boldsymbol{H}(t)$ and any operators $\boldsymbol{L}_\nu(t)$ that are bounded and measurable functions of time, the solution of (5) with an initial condition $\boldsymbol{\rho}_0$ Hermitian, non-negative and of trace one, is defined for all $t > 0$, remains Hermitian, non-negative and of trace one.*

Proof. The existence and uniqueness of the solution for $t > 0$ is consequence of a standard result on linear ordinary differential systems of finite dimension and with bounded and time-measurable coefficients. The Hermiticity and trace conservation directly follows from the fact that the right-hand side of (5) is Hermitian as soon as $\boldsymbol{\rho}$ is Hermitian, and admits a zero trace. The positivity conservation is less simple. It can be seen from the following formulation of (5):

$$\frac{d}{dt}\boldsymbol{\rho} = \boldsymbol{A}\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{A}^\dagger + \sum_\nu \boldsymbol{L}_\nu \boldsymbol{\rho} \boldsymbol{L}_\nu^\dagger$$

with $\boldsymbol{A} = -\frac{i}{\hbar} H - \frac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu$. Consider the solution of the matrix equation $\frac{d}{dt}\boldsymbol{E} = \boldsymbol{A}\boldsymbol{E}$ with $\boldsymbol{E}_0 = \boldsymbol{I}$. Then $\boldsymbol{E}$ is always invertible and defines the following change of variables $\boldsymbol{\rho} = \boldsymbol{E}\boldsymbol{\xi}\boldsymbol{E}^\dagger$. We have then

$$\frac{d}{dt}\boldsymbol{\xi} = \sum_\nu \boldsymbol{M}_\nu \boldsymbol{\xi} \boldsymbol{M}_\nu^\dagger$$

with $\boldsymbol{M}_\nu = \boldsymbol{E}^{-1}\boldsymbol{L}_\nu \boldsymbol{E}$. The fact that $\boldsymbol{\xi}_0 = \boldsymbol{\rho}_0$ is Hermitian non-negative and that $\frac{d}{dt}\boldsymbol{\xi}$ is also Hermitian and non-negative, implies that $\boldsymbol{\xi}$ remains non-negative for all $t > 0$, and thus $\boldsymbol{\rho}$ remains also non-negative. □

The link between the discrete-time formulation and the continuous-time one (5), becomes clear if we consider the following identity for ϵ positive and small:

$$\boldsymbol{\rho} + \epsilon \frac{d}{dt}\boldsymbol{\rho} = \boldsymbol{M}_{\epsilon,0}\boldsymbol{\rho}\boldsymbol{M}_{\epsilon,0}^\dagger + \sum_\nu \boldsymbol{M}_{\epsilon,\nu}\boldsymbol{\rho}\boldsymbol{M}_{\epsilon,\nu}^\dagger + O(\epsilon^2)$$

where $\frac{d}{dt}\boldsymbol{\rho}$ is given by (5), $\boldsymbol{M}_{\epsilon,0} = \boldsymbol{I} - \epsilon\left(\frac{i}{\hbar}\boldsymbol{H} + \frac{1}{2}\sum_\nu \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu\right)$ and $\boldsymbol{M}_{\epsilon,\nu} = \sqrt{\epsilon}\boldsymbol{L}_\nu$. Since $\boldsymbol{\rho}(t+\epsilon) = \boldsymbol{\rho}(t) + \epsilon\frac{d}{dt}\boldsymbol{\rho}(t) + o(\epsilon)$ and $\boldsymbol{M}_{\epsilon,0}^\dagger \boldsymbol{M}_{\epsilon,0} + \sum_\nu \boldsymbol{M}_{\epsilon,\nu}^\dagger \boldsymbol{M}_{\epsilon,\nu} = \boldsymbol{I} + O(\epsilon^2)$, the continuous-time evolution (5) is attached to a discrete-time evolution with the following infinitesimal Kraus map

$$\boldsymbol{\rho}(t+dt) = \boldsymbol{M}_{dt,0}\boldsymbol{\rho}(t)\boldsymbol{M}_{dt,0}^\dagger + \sum_\nu \boldsymbol{M}_{dt,\nu}\boldsymbol{\rho}(t)\boldsymbol{M}_{dt,\nu}^\dagger \tag{6}$$

up to second order terms versus the time-step $dt > 0$. Such correspondence can be used to develop positivity preserving numerical schemes (see, e.g., [11, 15]).

Since any Kraus map is a contraction for the trace-distance, we have the following theorem, the continuous-time counter part of discrete-time contraction properties.

Theorem 3. *Consider two solutions of* (5)*,* $\boldsymbol{\rho}$ *and* $\boldsymbol{\rho}'$*, starting form* $\boldsymbol{\rho}_0$ *and* $\boldsymbol{\rho}'_0$ *two Hermitian non negative operators of trace one. Assume that* $\mathcal{H}$ *is of finite dimension and the Hermitian operator* $\boldsymbol{H}(t)$ *and the operators* $\boldsymbol{L}_\nu(t)$ *are bounded and measurable functions of time. Then for any* $0 \leq t_1 \leq t_2$*,*

$$\text{Tr}\left(|\boldsymbol{\rho}(t_2), \boldsymbol{\rho}'(t_2)|\right) \leq \text{Tr}\left(|\boldsymbol{\rho}(t_1), \boldsymbol{\rho}'(t_1)|\right) \quad \textit{and} \quad F(\boldsymbol{\rho}(t_2), \boldsymbol{\rho}'(t_2)) \geq F(\boldsymbol{\rho}(t_1), \boldsymbol{\rho}'(t_1)).$$

The proof just consists in exploiting (6) with the discrete-time contraction properties.

4 Driven and damped quantum harmonic oscillator

4.1 Classical ordinary differential equations

Consider the following damped harmonic oscillator

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

where $\omega \gg \kappa, \sqrt{u_1^2 + u_2^2}$. Consider the following periodic change of variables $(x', p') \mapsto (x, p)$:

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Then, we have

$$\begin{aligned}
\cos(\omega t)\frac{d}{dt}x + \sin(\omega t)\frac{d}{dt}p &= 0 \\
-\sin(\omega t)\frac{d}{dt}x + \cos(\omega t)\frac{d}{dt}p &= -\kappa(-\sin(\omega t)x + \cos(\omega t)p) - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}x &= -\kappa \sin^2(\omega t)x + 2u_1 \sin^2(\omega t) + (\kappa p - 2u_2)\sin(\omega t)\cos(\omega t) \\
\frac{d}{dt}p &= -\kappa \cos^2(\omega t)p + 2u_2 \cos^2(\omega t) + (\kappa x - 2u_1)\sin(\omega t)\cos(\omega t).
\end{aligned}$$

Removing highly oscillating terms (rotating wave approximation), we get:

$$\frac{d}{dt}x = -\tfrac{\kappa}{2}x + u_1, \quad \frac{d}{dt}p = -\tfrac{\kappa}{2}p + u_2$$

that reads also with the complex variables $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}\alpha + u. \tag{7}$$

This yields to the following approximate model in the original frame (x', p'):

$$\frac{d}{dt}x' = -\tfrac{\kappa}{2}x' + \omega p + u_1 \cos(\omega t) + u_2 \sin(\omega t), \quad \frac{d}{dt}p' = -\omega x' - \tfrac{\kappa}{2}p' - u_1 \sin(\omega t) + u_2 \cos(\omega t)$$

or with complex variable $\alpha' = x' + ip' = e^{-i\omega t}\alpha$:

$$\frac{d}{dt}\alpha' = -(\tfrac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t} \tag{8}$$

4.2 Quantum master equation

We consider here the quantum model of the classical oscillator modeled by (7) and (8). It admits the infinite dimensional Hilbert-space $\mathcal{H}$ with $(|n\rangle)_{n\in\mathbb{N}}$ as orthonormal basis (Fock states) Its Hamiltonian with a resonant coherent drive of complex amplitude u ($|u| \ll \omega$) reads

$$\boldsymbol{H} = \hbar\left(\omega \boldsymbol{N} + i(ue^{-i\omega t}\boldsymbol{a}^\dagger - u^* e^{i\omega t}\boldsymbol{a})\right).$$

Consider the Lindblad master equation (5) with the above $\boldsymbol{H}$ and two operators $L_1 = \sqrt{(1+n_{\text{th}})\kappa}\ \boldsymbol{a}$ and $L_2 = \sqrt{n_{\text{th}}\kappa}\ \boldsymbol{a}^\dagger$ corresponding to decoherence via photon losses and thermal photon gains. We get the following master equation where $\boldsymbol{\rho}'$ is the density operator:

$$\frac{d}{dt}\boldsymbol{\rho}' = -\imath\omega[\boldsymbol{N},\boldsymbol{\rho}'] + [ue^{-i\omega t}\boldsymbol{a}^\dagger - u^* e^{i\omega t}\boldsymbol{a},\boldsymbol{\rho}'] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}'\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}' - \tfrac{1}{2}\boldsymbol{\rho}'\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}'\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho}' - \tfrac{1}{2}\boldsymbol{\rho}'\boldsymbol{a}\boldsymbol{a}^\dagger\right). \quad (9)$$

with parameter $\kappa > 0$ and $n_{\text{th}} \geq 0$. When $n_{\text{th}} = 0$, we recover (8) with $\alpha' = \text{Tr}\left(\boldsymbol{\rho}'\boldsymbol{a}\right)$.

Consider the change of frame $\boldsymbol{\rho}' = e^{-i\omega t\boldsymbol{N}}\boldsymbol{\rho}e^{i\omega t\boldsymbol{N}}$. Since $e^{i\omega t\boldsymbol{N}}\boldsymbol{a}e^{-i\omega t\boldsymbol{N}} = e^{-i\omega t}\boldsymbol{a}$, we get:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a},\boldsymbol{\rho}] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\right). \quad (10)$$

When $n_{\text{th}} = 0$, we recover with $\alpha = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{a}\right)$ the classical amplitude equation (7).

The above models (9) and (10) are valid only when $\omega \gg \kappa, |u|$: weak drive amplitude and high quality factor of the oscillator. With initial conditions $\boldsymbol{\rho}_0'$ and $\boldsymbol{\rho}_0$ being density operators (Hermitian non-negative trace-class operators on $\mathcal{H}$ of trace one, see appendix A), their solutions give the forward time evolution of $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}$. In the sequel, we focus on the dynamics of $\boldsymbol{\rho}$, i.e., on the dynamics in the frame rotating at the oscillator frequency ω.

4.3 Zero temperature case: $n_{\text{th}} = 0$

Assume that $n_{\text{th}} = 0$:

$$\frac{d}{dt}\boldsymbol{\rho} = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a},\boldsymbol{\rho}] + \kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

Set $\overline{\alpha} = \frac{2u}{\kappa}$. We recover the classical equation for the complex amplitude $\alpha = \text{Tr}\left(\boldsymbol{\rho}\boldsymbol{a}\right)$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}(\alpha - 2u/\kappa) = -\tfrac{\kappa}{2}(\alpha - \overline{\alpha}).$$

Consider the following change of frame

$$\boldsymbol{\rho} = e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}}\boldsymbol{\xi}e^{-\overline{\alpha}\boldsymbol{a}^\dagger + \overline{\alpha}^*\boldsymbol{a}}$$

corresponding to a displacement of amplitude $-\overline{\alpha}$ of $\boldsymbol{\rho}$. Since $e^{-\overline{\alpha}\boldsymbol{a}^\dagger + \overline{\alpha}^*\boldsymbol{a}}\boldsymbol{a}e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}} = \boldsymbol{a} + \overline{\alpha}$ and $e^{-\overline{\alpha}\boldsymbol{a}^\dagger + \overline{\alpha}^*\boldsymbol{a}}\boldsymbol{a}^\dagger e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}} = \boldsymbol{a} + \overline{\alpha}^*$ we have

$$\frac{d}{dt}\boldsymbol{\xi} = [u(\boldsymbol{a}^\dagger + \overline{\alpha}^*) - u^*(\boldsymbol{a} + \overline{\alpha}),\boldsymbol{\xi}] + \kappa\left((\boldsymbol{a}+\overline{\alpha})\boldsymbol{\xi}(\boldsymbol{a}^\dagger + \overline{\alpha}^*) - \tfrac{1}{2}(\boldsymbol{a}^\dagger + \overline{\alpha}^*)(\boldsymbol{a}+\overline{\alpha})\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}(\boldsymbol{a}^\dagger + \overline{\alpha}^*)(\boldsymbol{a}+\overline{\alpha})\right)$$
$$= \kappa\left(\boldsymbol{a}\boldsymbol{\xi}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

Consider $V(\boldsymbol{\xi}) = \text{Tr}\,(\boldsymbol{\xi}\boldsymbol{N})$ ($\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$). Since $\boldsymbol{\xi}$ is a density operator $V(\boldsymbol{\xi}) \geq 0$ and $V(\boldsymbol{\xi}) = 0$ if, and only if, $\boldsymbol{\xi} = |0\rangle\langle 0|$ (vacuum state). We have

$$\frac{d}{dt}V(\boldsymbol{\xi}) = -\kappa V(\boldsymbol{\xi}).$$

If the initial energy $V(\boldsymbol{\xi}_0) < +\infty$, $\boldsymbol{\xi}(t)$ remains of finite energy for all t and moreover, $V(\boldsymbol{\xi}(t)) = V(\boldsymbol{\xi}_0)e^{-\kappa t}$. Thus $V(\boldsymbol{\xi}(t))$ tends to 0 and thus $\boldsymbol{\xi}(t)$ converges towards $|0\rangle\langle 0|$. Since $\boldsymbol{\rho}$ is just $\boldsymbol{\xi}$ up to a coherent displacement $\overline{\alpha}$, this proves that $\boldsymbol{\rho}(t)$ converges towards $|\overline{\alpha}\rangle\langle\overline{\alpha}|$, the coherent and pure state of amplitude $\overline{\alpha}$.

The above arguments with the strict Lyapunov function V are not presented here above with all the necessarily mathematical rigour since $\mathcal{H}$ is an infinite dimensional Hilbert space. Nevertheless, they can be made rigorous to prove the following theorem

Theorem 4. *Consider (10) with $u \in \mathbb{C}$, $\kappa > 0$ and $n_{th} = 0$. Denote by $|\overline{\alpha}\rangle$ the coherent state of complex amplitude $\overline{\alpha} = \frac{2u}{\kappa}$. Assume that the initial state $\boldsymbol{\rho}_0$ is a density operator with finite energy $\text{Tr}\,(\boldsymbol{\rho}_0\boldsymbol{N}) < +\infty$. Then, there exists a unique solution to the Cauchy problem (10) initialized with $\boldsymbol{\rho}_0$ in the the Banach space $\mathcal{K}^1(\mathcal{H})$ (see appendix A). It is defined for all $t > 0$ with $t \mapsto \boldsymbol{\rho}(t)$ a density operator (Hermitian, non-negative and trace-class) that remains in the domain of the Lindblad super-operator*

$$\boldsymbol{\rho} \mapsto [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \boldsymbol{\rho}] + \kappa\left(\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho} - \tfrac{1}{2}\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

Thus $t \mapsto \boldsymbol{\rho}(t)$ is differentiable in the Banach space $\mathcal{K}^1(\mathcal{H})$. Moreover $\boldsymbol{\rho}(t)$ converges for the trace-norm towards $|\overline{\alpha}\rangle\langle\overline{\alpha}|$ when t tends to $+\infty$.

The following lemma gives the link with the classical damped oscillator.

Lemma 1. *Consider (10) with $u \in \mathbb{C}$, $\kappa > 0$ and $n_{th} = 0$.*

1. *for any initial density operator $\boldsymbol{\rho}_0$ with $\text{Tr}\,(\boldsymbol{\rho}_0\boldsymbol{N}) < +\infty$, we have $\frac{d}{dt}\alpha = -\frac{\kappa}{2}(\alpha - \overline{\alpha})$ where $\alpha = \text{Tr}\,(\rho\boldsymbol{a})$.*

2. *Assume that $\boldsymbol{\rho}_0 = |\beta_0\rangle\langle\beta_0|$ where β_0 is some complex amplitude. Then for all $t \geq 0$, $\boldsymbol{\rho}(t) = |\beta(t)\rangle\langle\beta(t)|$ remains a coherent and pure state of amplitude $\beta(t)$ solution of the following equation: $\frac{d}{dt}\beta = -\frac{\kappa}{2}(\beta - \overline{\alpha})$ with $\beta(0) = \beta_0$.*

Proof. Statement 1 follows from $\frac{d}{dt}\alpha = \text{Tr}\left(\boldsymbol{a}\frac{d}{dt}\boldsymbol{\rho}\right)$ with $\frac{d}{dt}\boldsymbol{\rho}$ given by (10). Statement 2 relies on the following relationships specific to coherent state:

$$\boldsymbol{a}|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = e^{-\frac{\beta\beta^*}{2}}e^{\beta\boldsymbol{a}^\dagger}|0\rangle \quad \text{and} \quad \frac{d}{dt}|\beta\rangle = \left(-\tfrac{1}{2}(\beta^*\dot{\beta} + \beta\dot{\beta}^*) + \dot{\beta}\boldsymbol{a}^\dagger\right)|\beta\rangle.$$

□

4.4 Wigner function and quantum Fokker-Planck equation

For a harmonic oscillator of space dimension 1, the phase space is the plane (x, p). To represent this quantum state and its link with classical statistical physics, it is useful to consider the Wigner function $\mathbb{R}^2 \ni (x, p) \mapsto W^{\{\boldsymbol{\rho}\}}(x, p) \in \mathbb{R}$ attached to the density operator $\boldsymbol{\rho}$. For a physical interpretation of $W^{\{\boldsymbol{\rho}\}}$ as a pseudo-probability density see appendix of [8] where the Wigner function is defined via the Fourier transform

$$W^{\{\boldsymbol{\rho}\}}(x,p) = \frac{1}{\pi^2} \iint_{\mathbb{R}^2} C_s^{\{\boldsymbol{\rho}\}}(\lambda_1 + i\lambda_2) e^{-2i(x\lambda_2 - p\lambda_1)}\, d\lambda_1\, d\lambda_2$$

of the symmetric characteristic function $C_s^{\{\boldsymbol{\rho}\}}$ attached to $\boldsymbol{\rho}$ (quantum probability):

$$\mathbb{C} \ni \lambda_1 + i\lambda_2 = \lambda \mapsto C_s^{\{\boldsymbol{\rho}\}}(\lambda) = \operatorname{Tr}\left(\boldsymbol{\rho} e^{\lambda \boldsymbol{a}^\dagger - \lambda^* \boldsymbol{a}}\right).$$

We will use here the following definition,

$$W^{\{\boldsymbol{\rho}\}}(x,p) = \tfrac{2}{\pi} \operatorname{Tr}\left(\boldsymbol{\rho} \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}\right) \quad \text{with} \quad \alpha = x + ip, \tag{11}$$

where $D_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a}}$ is the displacement of complex amplitude α. Consequently $W^{\{\boldsymbol{\rho}\}}(x,p)$ is real and well defined since $\boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}$ is a bounded, unitary and Hermitian operator (the dual of $\mathcal{K}^1(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$, see appendix A).

For a coherent state $\boldsymbol{\rho} = |\beta\rangle\langle\beta|$ with $\beta \in \mathbb{C}$ we have

$$W^{\{|\beta\rangle\langle\beta|\}}(x,p) = \tfrac{2}{\pi} \left\langle \beta \middle| \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha} \middle| \beta \right\rangle = \tfrac{2}{\pi} e^{-2|\beta-\alpha|^2}.$$

since $\langle\beta| \boldsymbol{D}_\alpha = \langle \beta - \alpha|$ with $\boldsymbol{D}_{-\alpha}|\beta\rangle = |\beta-\alpha\rangle$ and $e^{i\pi\boldsymbol{N}}|\beta - \alpha\rangle = |\alpha - \beta\rangle$. Thus $W^{\{|\beta\rangle\langle\beta|\}}$ is the usual Gaussian density function centered on β in the phase plane $\alpha = x + ip$ and of variance $1/2$ in all directions.

In the sequel we will consider that $\boldsymbol{\rho}$ is in $\mathcal{K}^f(\mathcal{H})$ (support with a finite number of photons) and thus that the computations here below can be done without any divergence problem. Using $\boldsymbol{D}_\alpha = e^{\alpha\boldsymbol{a}^\dagger} e^{-\alpha^*\boldsymbol{a}} e^{-\alpha\alpha^*/2} = e^{-\alpha^*\boldsymbol{a}} e^{\alpha\boldsymbol{a}^\dagger} e^{\alpha\alpha^*/2}$ we have two equivalent formulations:

$$\tfrac{\pi}{2} W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*) = \operatorname{Tr}\left(\boldsymbol{\rho} e^{\alpha\boldsymbol{a}^\dagger} e^{-\alpha^*\boldsymbol{a}} e^{i\pi\boldsymbol{N}} e^{\alpha^*\boldsymbol{a}} e^{-\alpha\boldsymbol{a}^\dagger}\right) = \operatorname{Tr}\left(\boldsymbol{\rho} e^{-\alpha^*\boldsymbol{a}} e^{\alpha\boldsymbol{a}^\dagger} e^{i\pi\boldsymbol{N}} e^{-\alpha\boldsymbol{a}^\dagger} e^{\alpha^*\boldsymbol{a}}\right)$$

Here α and α^* are seen as independent variables. We have the following correspondence:

$$\frac{\partial}{\partial\alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial\alpha^*} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right)$$

We have

$$\tfrac{\pi}{2}\frac{\partial}{\partial\alpha} W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*) = \operatorname{Tr}\left((\boldsymbol{\rho}\boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{\rho}) e^{\alpha\boldsymbol{a}^\dagger} e^{-\alpha^*\boldsymbol{a}} e^{i\pi\boldsymbol{N}} e^{\alpha^*\boldsymbol{a}} e^{-\alpha\boldsymbol{a}^\dagger}\right) = \operatorname{Tr}\left((\boldsymbol{\rho}\boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{\rho}) \boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}} \boldsymbol{D}_{-\alpha}\right)$$

Since $\boldsymbol{a}^\dagger \boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}} \boldsymbol{D}_{-\alpha} = \boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}} \boldsymbol{D}_{-\alpha}(2\alpha^* - \boldsymbol{a}^\dagger)$, we have

$$\frac{\partial}{\partial\alpha} W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*) = 2\alpha^* W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*) - 2W^{\{\boldsymbol{a}^\dagger\boldsymbol{\rho}\}}(\alpha,\alpha^*).$$

Thus $W^{\{\boldsymbol{a}^\dagger\boldsymbol{\rho}\}}(\alpha,\alpha^*) = \alpha^* W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*) - \frac{1}{2}\frac{\partial}{\partial\alpha} W^{\{\boldsymbol{\rho}\}}(\alpha,\alpha^*)$.

Similar computations yield to the following correspondence rules:

$$W^{\{\boldsymbol{\rho a}\}} = \left(\alpha - \tfrac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W^{\{\rho\}}, \quad W^{\{\boldsymbol{a\rho}\}} = \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial\alpha^*}\right) W^{\{\boldsymbol{\rho}\}}$$
$$W^{\{\boldsymbol{\rho a}^\dagger\}} = \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial\alpha}\right) W^{\{\boldsymbol{\rho}\}}, \quad W^{\{\boldsymbol{a}^\dagger\boldsymbol{\rho}\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial\alpha}\right) W^{\{\boldsymbol{\rho}\}}.$$

With these rules the operator differential equation (10) for $\boldsymbol{\rho}$ becomes a partial differential equation for $W^{\{\boldsymbol{\rho}\}}(x,p)$. We have

$$W^{\{[u\boldsymbol{a}^\dagger - u^*\boldsymbol{a},\boldsymbol{\rho}]\}} = -\left(u\frac{\partial}{\partial\alpha} + u^*\frac{\partial}{\partial\alpha^*}\right)W^{\{\boldsymbol{\rho}\}}$$
$$W^{\{\boldsymbol{a}\boldsymbol{\rho}\boldsymbol{a}^\dagger - \frac{\boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{a}^\dagger\boldsymbol{a}}{2}\}} = \tfrac{1}{2}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)W^{\{\boldsymbol{\rho}\}}$$
$$W^{\{\boldsymbol{a}^\dagger\boldsymbol{\rho}\boldsymbol{a} - \frac{\boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger}{2}\}} = \tfrac{1}{2}\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*} - \frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)W^{\{\boldsymbol{\rho}\}}.$$

Consequently, the time-varying Wigner function $W^{\{\boldsymbol{\rho}\}}$ is governed by a partial differential equation

$$\frac{\partial}{\partial t}W^{\{\boldsymbol{\rho}\}} = \frac{\kappa}{2}\left(\frac{\partial}{\partial\alpha}(\alpha-\overline{\alpha}) + \frac{\partial}{\partial\alpha^*}(\alpha^*-\overline{\alpha}^*) + (1+2n_{\text{th}})\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)W^{\{\boldsymbol{\rho}\}}$$

with $\overline{\alpha} = 2u/\kappa$. Set $\overline{\alpha} = \overline{x} + i\overline{p}$. Using $\frac{\partial}{\partial\alpha}$ and $\frac{\partial}{\partial\alpha^*}$ as linear expressions in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial p}$, we get finally the following convection diffusion equation also called quantum Fokker-Planck equation:

$$\frac{\partial}{\partial t}W^{\{\boldsymbol{\rho}\}} = \frac{\kappa}{2}\left(\frac{\partial}{\partial x}\left((x-\overline{x})W^{\{\boldsymbol{\rho}\}}\right) + \frac{\partial}{\partial p}\left((p-\overline{p})W^{\{\boldsymbol{\rho}\}}\right) + \tfrac{1+2n_{\text{th}}}{4}\left(\frac{\partial^2 W^{\{\boldsymbol{\rho}\}}}{\partial x^2} + \frac{\partial^2 W^{\{\boldsymbol{\rho}\}}}{\partial p^2}\right)\right). \tag{12}$$

It can be also written in a more geometric form with $\nabla = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial p}\end{pmatrix}$:

$$\frac{\partial}{\partial t}W^{\{\boldsymbol{\rho}\}} = -\nabla\cdot\left(W^{\{\boldsymbol{\rho}\}}F\right) + \nabla\cdot\left(\sigma\nabla W^{\{\boldsymbol{\rho}\}}\right)$$

where $F = \frac{\kappa}{2}\begin{pmatrix}\overline{x}-x\\ \overline{y}-y\end{pmatrix}$ and $\sigma = \frac{\kappa(1+2n_{\text{th}})}{8}$.

The Green function $G(x,p,t,x_0,p_0)$ of (12), i.e., its solution with initial condition $W_0^{\{\boldsymbol{\rho}\}}(x,p) = \delta(x-x_0)\delta(p-p_0)$ where δ is the Dirac distribution, reads:

$$G(x,p,t,x_0,p_0) = \tfrac{1}{\pi(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}\exp\left(-\frac{\left(x-\overline{x}-(x_0-\overline{x})e^{-\frac{\kappa t}{2}}\right)^2 + \left(p-\overline{p}-(p_0-\overline{p})e^{-\frac{\kappa t}{2}}\right)^2}{(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}\right).$$

The general solution of (12) with an L^1 initial condition $W_0^{\{\boldsymbol{\rho}\}}(x,p)$ ($\iint_{\mathbb{R}^2}|W_0^{\{\boldsymbol{\rho}\}}(x,p)| < +\infty$), reads for $t>0$:

$$W_t^{\{\boldsymbol{\rho}\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\boldsymbol{\rho}\}}(x',p')G(x,p,t,x',p')\ dx'dp'.$$

For t large, $G(x,p,t,x',p')$ converges toward a Gaussian distribution independent of (x',p'). By application of the dominated convergence theorem we have:

$$\forall (x,p)\in\mathbb{R}^2,\quad \lim_{t\mapsto+\infty} W_t^{\{\boldsymbol{\rho}\}}(x,p) = \tfrac{\iint_{\mathbb{R}^2} W_0^{\{\boldsymbol{\rho}\}}}{\pi(n_{\text{th}}+\frac{1}{2})}\exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right).$$

Notice that Wigner functions associated to density operators satisfy $\iint_{\mathbb{R}^2} W^{\{\boldsymbol{\rho}\}} = 1$. Thus the steady state solution of (12) is a Gaussian probability density centered on $(\overline{x}, \overline{p})$ with variance $(n_{\text{th}} + \frac{1}{2})$ in all direction. Moreover any trajectory of (12) initialized with $W^{\{\boldsymbol{\rho}_0\}}$, $\boldsymbol{\rho}_0$ being a density operator, converges to this Gaussian function. When $n_{\text{th}} = 0$, we recover the Wigner function of the coherent state $\overline{\alpha}$.

Many other properties on Wigner and related functions can be founded in [8] and also in [4].

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A Operator spaces

This summary is strongly inspired from chapter 4 of [17] where detailed justifications can be found. $\mathcal{H}$ denotes a separable Hilbert space. We summarize the basic properties of the following spaces of linear operators on $\mathcal{H}$: finite rank operators $\mathcal{K}^f(\mathcal{H})$, trace-class operators $\mathcal{K}^1(\mathcal{H})$, Hilbert-Schmidt operators $\mathcal{K}^2(\mathcal{H})$, compact operators $\mathcal{K}^c(\mathcal{H})$ and bounded operators $\mathcal{B}(\mathcal{H})$. These operators spaces, $\mathcal{K}^f(\mathcal{H}) \subset \mathcal{K}^1(\mathcal{H}) \subset \mathcal{K}^2(\mathcal{H}) \subset \mathcal{K}^c(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$, are non-commutative analogue of the following usual spaces of complex-value series $(\lambda_k)_{k\geq 0}$:

- $\mathcal{K}^f(\mathcal{H})$ mimics series with a finite number of non zero terms.
- $\mathcal{K}^1(\mathcal{H})$ mimics absolutely converging series, $\sum_{k\geq 0} |\lambda_k| < +\infty$; the analogue of the l^1 norm is the trace-class norm.
- $\mathcal{K}^2(\mathcal{H})$ mimics l^2 series, $\sum_{k\geq 0} |\lambda_k|^2 < +\infty$; the analogue of the scalar product on l^2 is the Frobenius product.
- $\mathcal{K}^c(\mathcal{H})$ mimics series those general term converges to zero: $\lim_{k\mapsto +\infty} \lambda_k = 0$.
- $\mathcal{B}(\mathcal{H})$ mimics l^∞ series, i.e., bounded series; the analogue of the l^∞ norm becomes the sup norm on bounded operators.

Elements of $\mathcal{H}$ are vectors denoted usually with the Ket notation $|\psi\rangle \in \mathcal{H}$. The Hermitian product between two Kets $|\psi\rangle$ and $|\phi\rangle$ is denoted by $\langle\psi|\phi\rangle = \langle\psi|\,|\phi\rangle$ where $\langle\psi| = |\psi\rangle^\dagger$ is the Bra, the co-vector associated to $|\psi\rangle$, element of the dual $\mathcal{H}^*$ of $\mathcal{H}$, and defining a continuous linear map: $\mathcal{H} \ni |\phi\rangle \mapsto \langle\psi|\phi\rangle \in \mathbb{C}$. The length of $|\psi\rangle$ is denote by $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$.

$\mathcal{L}(\mathcal{H})$ denotes the vector space of linear operators from $\mathcal{H}$ to $\mathcal{H}$. For $\boldsymbol{A} \in \mathcal{L}(\mathcal{H})$, $A^\dagger$ denotes its Hermitian conjugate, another element of $\mathcal{L}(\mathcal{H})$ defined by $\forall\, |\psi\rangle, |\phi\rangle \in \mathcal{H}$, $\big\langle\psi\big|\big(\boldsymbol{A}\,|\phi\rangle\,\big)\big\rangle = \big\langle\big(\boldsymbol{A}^\dagger\,|\psi\rangle\,\big)\big|\phi\big\rangle$.

The set of bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. The vector space $\mathcal{B}(\mathcal{H})$ equipped with the following sup norm

$$\|\boldsymbol{A}\| = \underset{\substack{|\psi\rangle \in \mathcal{H} \\ \langle\psi|\psi\rangle = 1}}{\mathrm{Sup}} \sqrt{\langle\psi|\boldsymbol{A}^\dagger\boldsymbol{A}|\psi\rangle}$$

is a Banach space. Bounded operators of $\mathcal{L}(\mathcal{H})$ are continuous operators of $\mathcal{L}(\mathcal{H})$. An operator $\boldsymbol{U}$ of $\mathcal{L}(\mathcal{H})$ is called unitary, if it is invertible and if $\boldsymbol{U}^{-1} = \boldsymbol{U}^\dagger$. Any unitary operator $\boldsymbol{U}$ belongs to $\mathcal{B}(\mathcal{H})$.

Take two elements of $\mathcal{H}$, $|a\rangle$ and $|b\rangle$: they define a Ket-Bra operator $\boldsymbol{P}_{a,b} \in \mathcal{B}(\mathcal{H})$ via the following correspondence:

$$\forall \,|\psi\rangle \in \mathcal{H}, \boldsymbol{P}_{a,b}(|\psi\rangle) = \big(\langle b|\psi\rangle \big) \,|a\rangle \,.$$

Usual $\boldsymbol{P}_{a,b}$ is denoted by $|a\rangle \langle b|$ since $\boldsymbol{P}_{a,b}(|\psi\rangle) = |a\rangle \langle b| \,|\psi\rangle$.

Exercice 1. *Show that* $\|P_{a,b}\| = \frac{\sqrt{\langle a|a\rangle\langle b|b\rangle} + |\langle a|b\rangle|}{2}$

Let $|\psi\rangle$ be a unitary vector of $\mathcal{H}$ ($\langle\psi|\psi\rangle = 1$). The orthogonal projector on the line spanned by $|\psi\rangle$, $\{z\,|\psi\rangle \mid z \in \mathbb{C}\}$ is the Ket-Bra operator $P_{\psi,\psi} = |\psi\rangle \langle\psi|$. The orthogonal projector $\boldsymbol{P}_{\mathcal{H}_f}$ on a finite dimensional vector space $\mathcal{H}_f$ of $\mathcal{H}$ reads

$$\boldsymbol{P}_{\mathcal{H}_f} = \sum_{k=1}^{N} |a_k\rangle \langle a_k|$$

where $(|a_1\rangle, \ldots, |a_N\rangle)$ is any ortho-normal basis of $\mathcal{H}_f$.

An element $\boldsymbol{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be finite rank, if and only if, it can be expressed as a finite sum of length N of Ket-Bra operators:

$$\boldsymbol{A} = \sum_{k=1}^{N} |a_k\rangle \langle b_k|$$

where $|a_k\rangle$ and $|b_k\rangle$ belong to $\mathcal{H}$. The linear sub-space of $\mathcal{L}(\mathcal{H})$ of finite rank operators of $\mathcal{H}$ is noted by $\mathcal{K}^f(\mathcal{H})$. It is clear that $\mathcal{K}^f(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Moreover $\boldsymbol{A} \in \mathcal{L}(\mathcal{H})$ belongs to $\mathcal{K}^f(\mathcal{H})$ if and only if it range, the sub-vector space of $\mathcal{H}$ denoted by $R(\boldsymbol{A}) = \{\boldsymbol{A}\,|\psi\rangle \mid |\psi\rangle \in \mathcal{H}\}$, is finite dimensional. The rank of $\boldsymbol{A}$ is then the dimension of its range $R(\boldsymbol{A})$.

Exercice 2. *Show that for* $\boldsymbol{A} \in \mathcal{K}^f(\mathcal{H})$ *with* $\mathcal{H}$ *of infinite dimension, the kernel of* $\boldsymbol{A}$, $\ker(\boldsymbol{A}) = \{|\psi\rangle \in \mathcal{H} \mid \boldsymbol{A}\,|\psi\rangle = 0\}$ *is of infinite dimension.*

An element $\boldsymbol{A}$ of $\mathcal{L}(\mathcal{H})$ is said to be compact, if and only if, the image via $\boldsymbol{A}$ of any bounded sub-set of $\mathcal{H}$ admits a compact closure. The set of compact operators is denoted by $\mathcal{K}^c(\mathcal{H})$. Any compact operator is thus bounded, $\mathcal{K}^c(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$: it is a sub-vector space of $\mathcal{B}(\mathcal{H})$. The completion of $\mathcal{K}^f(\mathcal{H})$ with respect to the norm on $\mathcal{B}(\mathcal{H})$ is the set of compact operators $\mathcal{K}^c(\mathcal{H})$: by Hilbert theorem, any compact operator is the limit of finite rank operators for the sup norm on $\mathcal{B}(\mathcal{H})$. This implies that $\mathcal{K}^c(\mathcal{H})$ equipped with the sup norm inherited from $\mathcal{B}(\mathcal{H})$ is a Banach space.

Finally, any compact Hermitian operator $\boldsymbol{A}$ admits a discrete real spectrum $(\lambda_k)_{k\in\mathbb{N}}$ with $\lim_{k\mapsto+\infty} \lambda_k = 0$. To each λ_k we can associated a unitary Ket $|e_k\rangle$ such that $(|e_k\rangle)_{k\in\mathbb{N}}$ is an Hilbert basis of $\mathcal{H}$. Then we have

$$\boldsymbol{A} = \sum_{k\geq 0} \lambda_k \,|e_k\rangle \langle e_k| \,.$$

The above series is absolutely convergent in $\mathcal{B}(\mathcal{H})$ with the sup norm. In this decomposition, the λ_k's are countered with their possible multiplicities. Another equivalent and more intrinsic decomposition (unitary invariance) where each λ_k are different, is as follows

$$\boldsymbol{A} = \sum_k \lambda_k \boldsymbol{P}_k$$

where $\boldsymbol{P}_k$ is the orthogonal projector on the eigen-space associated to the eigenvalue λ_k.

Consider a non-negative Hermitian compact operator $\boldsymbol{A}$ with eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ counted with their multiplicities ($\boldsymbol{A} = \sum_{k\geq 0} \lambda_k \left|e_k\right\rangle \left\langle e_k\right|$). Then $\lambda_k \geq 0$. $\boldsymbol{A}$ is said trace class, if and only if , $\sum_{k\geq 0} \lambda_k < +\infty$. It is then simple to prove that $\sum_{k\geq 0} \lambda_k = \sum_{n\geq 0} \langle a_n|\boldsymbol{A}|a_n\rangle$ where $(|a_n\rangle)_{n\geq 0}$ is any ortho-normal basis of $\mathcal{H}$. Consequently, $\sum_{k\geq 0} \lambda_k$ is denote by $\mathrm{Tr}\left(\boldsymbol{A}\right)$.

More generally a compact operator $\boldsymbol{A}$ is trace class, if and only if, $\mathrm{Tr}\left(\sqrt{\boldsymbol{A}^\dagger \boldsymbol{A}}\right) < +\infty$. Since $\boldsymbol{A}$ is compact, the non-negative Hermitian operator $\boldsymbol{A}^\dagger\boldsymbol{A}$ is also compact. Thus it admits a spectral decomposition $\boldsymbol{A}^\dagger\boldsymbol{A} = \sum_k \lambda_k \boldsymbol{P}_k$ where $\lambda_k \geq 0$. Then $\sqrt{\boldsymbol{A}^\dagger\boldsymbol{A}}$ is defined as $\sum_k \sqrt{\lambda_k}\boldsymbol{P}_k$: it is another non-negative Hermitian compact operator those square coincides with $\boldsymbol{A}^\dagger\boldsymbol{A}$.

Exercice 3. *Show that* $\boldsymbol{A} \in \mathcal{K}^c(\mathcal{H})$ *is trace-class if and only if* $\Re(\boldsymbol{A}) = (\boldsymbol{A} + \boldsymbol{A}^\dagger)/2$ *and* $\Im(\boldsymbol{A}) = (\boldsymbol{A} - \boldsymbol{A}^\dagger)/(2i)$ *are trace class. Show that for any trace class operator* $\boldsymbol{A}$ *and for any ortho-normal basis* $(|a_n\rangle)_{n\geq 0}$, $\sum_{n\geq 0} \langle a_n|\boldsymbol{A}|a_n\rangle$ *is an absolute convergent series. Show that its sum depends only on* $\boldsymbol{A}$*(this justifies the notation* $\mathrm{Tr}\left(\boldsymbol{A}\right)$*). When* $\boldsymbol{A}$ *is Hermitian and trace class, show that* $\mathrm{Tr}\left(\boldsymbol{A}\right)$ *coincides with the sum of its eigenvalues counted with their multiplicity.*

The set of trace class operators $\boldsymbol{A}$ is noted by $\mathcal{K}^1(\mathcal{H})$: it is equipped with the trace norm also called nuclear norm: $\|\boldsymbol{A}\|_1 = \mathrm{Tr}\left(\sqrt{\boldsymbol{A}^\dagger\boldsymbol{A}}\right)$. A finite rank operator is automatically trace class: $\mathcal{K}^f(\mathcal{H}) \subset \mathcal{K}^1(\mathcal{H})$. More-over the completion of $\mathcal{K}^f(\mathcal{H})$ for the trace-class norm is $\mathcal{K}^1(\mathcal{H})$: any element of $\mathcal{K}^1(\mathcal{H})$ can be approximated for the trace norm topology by a sequence of finite rank operators. For any trace-class operators $\boldsymbol{A}$, $\boldsymbol{B}$, we have :

- $\mathrm{Tr}\left(\boldsymbol{A}\right) \geq 0$ when $\boldsymbol{A}^\dagger = \boldsymbol{A} > 0$.
- $\mathrm{Tr}\left(\boldsymbol{A}\right)$ real when $\boldsymbol{A}^\dagger = \boldsymbol{A}$.
- $\mathrm{Tr}\left(\boldsymbol{A}^\dagger\right) = \left(\mathrm{Tr}\left(\boldsymbol{A}\right)\right)^\dagger$ where $^\dagger =^*$ stands for the conjugation of complex number.
- $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are also trace class and $\mathrm{Tr}\left(\boldsymbol{AB}\right) = \mathrm{Tr}\left(\boldsymbol{BA}\right)$.

For any trace class operator $\boldsymbol{A}$ and any bounded operator $\boldsymbol{M}$, the operators $\boldsymbol{AM}$ is also trace class: More over $|\,\mathrm{Tr}\left(\boldsymbol{AM}\right)| \leq \|\boldsymbol{M}\|\|\boldsymbol{A}\|_1$. Thus for any $\boldsymbol{M} \in \mathcal{B}(\mathcal{H})$, $\mathcal{K}^1(\mathcal{H}) \in \boldsymbol{A} \mapsto \mathrm{Tr}\left(\boldsymbol{AM}\right) \in \mathbb{C}$ is a continuous linear operator of the Banach space $\mathcal{K}^1(\mathcal{H})$ is equipped with the trace norm. Conversely, any linear map from $\mathcal{K}^1(\mathcal{H})$ to $\mathbb{C}$ that is continuous with the trace norm coincides with $\mathcal{K}^1(\mathcal{H}) \ni \boldsymbol{A} \mapsto \mathrm{Tr}\left(\boldsymbol{AM}\right)$ for some $\boldsymbol{M} \in \mathcal{B}(\mathcal{H})$. The dual of $\mathcal{K}^1(\mathcal{H})$ for the trace-class norm is $\mathcal{B}(\mathcal{H})$.

A compact operator $\boldsymbol{A}$ is an Hilbert-Schmidt operator if, and only if, $\mathrm{Tr}\left(\boldsymbol{A}^\dagger\boldsymbol{A}\right) < +\infty$. The set of Hilbert-Schmidt operators is denoted by $\mathcal{K}^2(\mathcal{H})$. Equipped with the Frobenius scalar product $\text{Tr}\left(\boldsymbol{A}\boldsymbol{B}^\dagger\right)$, this space admits an Hilbert-space: the Frobenius norm $\boldsymbol{A}$ is denoted by $\|\boldsymbol{A}\|_2 = \sqrt{\text{Tr}\left(\boldsymbol{A}^\dagger\boldsymbol{A}\right)}$. We have $\mathcal{K}^f(\mathcal{H}) \subset \mathcal{K}^1(\mathcal{H}) \subset \mathcal{K}^2(\mathcal{H})$. More-over, the closure of $\mathcal{K}^f(\mathcal{H})$ with the Frobenius norm coincides with $\mathcal{K}^2(\mathcal{H})$.

We have the following list of properties:

1. For any $\boldsymbol{A} \in \mathcal{K}^1(\mathcal{H}) \subset \mathcal{K}^2(\mathcal{H})$:
$$\|\boldsymbol{A}\|_2 \leq \|\boldsymbol{A}\|_1, \quad |\,\text{Tr}\,(\boldsymbol{A})\,| \leq \|\boldsymbol{A}\|_1, \quad \|\boldsymbol{A}^\dagger\|_1 = \|\boldsymbol{A}\|_1.$$
2. if $\boldsymbol{A} \in \mathcal{K}^1(\mathcal{H})$ and $\boldsymbol{B} \in \mathcal{B}(\mathcal{H})$, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are in $\mathcal{K}^1(\mathcal{H})$ and
$$\|\boldsymbol{AB}\|_1 = \|\boldsymbol{BA}\|_1 \leq \|\boldsymbol{A}\|_1\|\boldsymbol{B}\|.$$
3. if $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to $\mathcal{K}^2(\mathcal{H})$, then $\boldsymbol{AB}$ belongs to $\mathcal{K}^1(\mathcal{H})$ and
$$\|\boldsymbol{AB}\|_1 = \|\boldsymbol{BA}\|_1 \leq \|\boldsymbol{A}\|_2\|\boldsymbol{B}\|_2.$$
4. if $\boldsymbol{A} \in \mathcal{K}^2(\mathcal{H})$ and $\boldsymbol{B} \in \mathcal{B}(\mathcal{H})$, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are in $\mathcal{K}^2(\mathcal{H})$.

An operator $\boldsymbol{\rho} \in \mathcal{K}^1(\mathcal{H})$ that is additionally Hermitian, non negative and of trace one is called a density operator. The set of density operators is a closed convex subset of the Banach space $\mathcal{K}^1(\mathcal{H})$ equipped with the trace norm.

B Markov chains, martingales and convergence theorems

This Appendix has for aim to give a very brief overview of some definitions and some theorems in the theory of random processes. The stability Theorems 5, 6 and 7 can be seen as stochastic analogues of deterministic Lyapunov function techniques.

We start the appendix by defining three types of convergence for random processes:

Definition 1. *Consider (X_n) a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, p)$ and taking values in a metric space $\mathcal{X}$. The random process X_n is said to,*

- *converge in probability towards the random variable X if for all $\epsilon > 0$,*
$$\lim_{n\to\infty} p\left(|X_n - X| > \epsilon\right) = \lim_{n\to\infty} p\left(\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \epsilon\right) = 0;$$
- *converge almost surely towards the random variable X if*
$$p\left(\lim_{n\to\infty} X_n = X\right) = p\left(\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1;$$
- *converge in mean towards the random variable X if*
$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|\right) = 0.$$

We can prove that the almost sure convergence and the convergence in mean imply the convergence in probability. However no such relation can be proved between the convergence in mean and the almost sure convergence in general.

Before defining the Markov processes, martingales, and discussing their convergence theorems, we provide two useful results from probability theory that are used for the proof of convergence of QND measurement process. The first result is the Markov's inequality

Lemma 2 (Markov's inequality)**.** *If $X \geq 0$ is a random variable and $\epsilon > 0$, we have*

$$\mathbb{P}\left(X \geq \epsilon\right) \leq \frac{\mathbb{E}\left(X\right)}{\epsilon}.$$

The second result is the Borel-Cantelli lemma about sequences of events in the σ-algebra $\mathcal{F}$.

Lemma 3 (Borel-Cantelli lemma)**.** *Let $E_k \in \mathcal{F}$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, p)$. Assuming*

$$\sum_{n=1}^{\infty} p(E_n) < \infty,$$

we have

$$p\left(\limsup_{n\to\infty} E_n\right) = p\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k\right) = 0.$$

Let $(\Omega, \mathcal{F}, p)$ be a probability space, and let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ be a nondecreasing family of sub-σ-algebras. We have the following definitions

Definition 2. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called a Markov process with respect to $F = (\mathcal{F}_n)_{n=1}^{\infty}$, if for $n' > n$ and any measurable function $f(x)$ with $\sup_x |f(x)| < \infty$,*

$$\mathbb{E}\left(f(X_{n'}) \mid \mathcal{F}_n\right) = \mathbb{E}\left(f(X_{n'}) \mid X_n\right).$$

Definition 3. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called respectively a supermartingale, a submartingale or a martingale, if $\mathbb{E}\left(|X_n|\right) < \infty$ for $n = 1, 2, \cdots,$ and*

$$\mathbb{E}\left(X_n \mid \mathcal{F}_m\right) \leq X_m \qquad (p \textit{ almost surely}), \qquad n \geq m,$$

or

$$\mathbb{E}\left(X_n \mid \mathcal{F}_m\right) \geq X_m \qquad (p \textit{ almost surely}), \qquad n \geq m,$$

or finally,

$$\mathbb{E}\left(X_n \mid \mathcal{F}_m\right) = X_m \qquad (p \textit{ almost surely}), \qquad n \geq m.$$

Remark 1. *A time-continuous version of the above definitions can also be considered for $(X_t, \mathcal{F}_t)_{t\geq 0}$, where $F = (\mathcal{F}_t)_{t\geq 0}$, is non decreasing family of sub-σ-alegbras of $\mathcal{F}$.*

The following theorem characterizes the convergence of bounded martingales:

Theorem 5 (Doob's first martingale convergence theorem)**.** *Let $(X_n, \mathcal{F}_n)_{n<\infty}$ be a submartingale such that (x^+ is the positive part of x)*

$$\sup_n \mathbb{E}\left(X_n^+\right) < \infty.$$

Then $\lim_n X_n$ $(= X_\infty)$ exists with probability 1, and $\mathbb{E}\left(X_\infty^+\right) < \infty$.

For a proof we refer to [13, Chapter 2, Page 43].

Here, we recall two results that are often referred as the stochastic versions of the Lyapunov stability theory and the LaSalle's invariance principle. For detailed discussions and proofs we refer to [10, Sections 8.4 and 8.5]. The first theorem is the following:

Theorem 6 (Doob's Inequality)**.** *Let $\{X_n\}$ be a Markov chain on state space $\mathcal{X}$. Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_1) \mid X_0 = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then*

$$p\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

Corollary 1. *Consider the same assumptions as in Theorem 6. Assume moreover that there exists $\bar{x} \in \mathcal{X}$ such that $V(\bar{x}) = 0$ and that $V(x) \neq 0$ for all x different from $\bar{x}$. Then the Theorem 6 implies that the Markov process X_n is* ***stable in probability*** *around $\bar{x}$, i.e.*

$$\lim_{x \to \bar{x}} p\left(\sup_n \|X_n - \bar{x}\| \geq \epsilon \mid X_0 = x\right) = 0, \qquad \forall \epsilon > 0.$$

Theorem 7. *Let $\{X_n\}$ be a Markov chain on the compact state space S. Suppose that there exists a non-negative function $V(x)$ satisfying $\mathbb{E}\left(V(X_{n+1}) \mid X_n = x\right) - V(x) = -k(x)$, where $k(x) \geq 0$ is a positive continuous function of x. Then the ω-limit set (in the sense of almost sure convergence) of X_n is included in the following set*

$$I = \{X \mid k(X) = 0\}.$$

Trivially, the same result holds true for the case where $\mathbb{E}\left(V(X_{n+1}) \mid X_n = x\right) - V(x) = k(x)$ ($V(X_n)$ is a submartingale and not a supermartingale), with $k(x) \geq 0$ and $V(x)$ bounded from above.

The proof is just an application of the Theorem 1 in [10, Ch. 8], which shows that $k(X_n)$ converges to zero for almost all paths. It is clear that the continuity of $k(x)$ with respect to x and the compactness of S implies that the ω-limit set of X_n is necessarily included in the set I.

C A tutorial introduction to stochastc master equation

Next pages are the arXiv preprint of the article entitled *A tutorial introduction to quantum stochastic master equations based on the qubit/photon system* published in Annual Reviews in Control, Vol. 54, p. 252-261, 2022.

A tutorial introduction to quantum stochastic master equations based on the qubit/photon system

Pierre Rouchon*

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Abstract

From the key composite quantum system made of a two-level system (qubit) and a harmonic oscillator (photon) with resonant or dispersive interactions, one derives the corresponding quantum Stochastic Master Equations (SME) when either the qubits or the photons are measured. Starting with an elementary discrete-time formulation based on explicit formulae for the interaction propagators, one shows how to include measurement imperfections and decoherence. This qubit/photon quantum system illustrates the Kraus-map structure of general discrete-time SME governing the dynamics of an open quantum system subject to measurement back-action and decoherence induced by the environment. Then, on the qubit/photon system, one explains the passage to a continuous-time mathematical model where the measurement signal is either a continuous real-value signal (typically homodyne or heterodyne signal) or a discontinuous and integer-value signal obtained from a counter. During this derivation, the Kraus map formulation is preserved in an infinitesimal way. Such a derivation provides also an equivalent Kraus-map formulation to the continuous-time SME usually expressed as stochastic differential equations driven either by Wiener or Poisson processes. From such Kraus-map formulation, simple linear numerical integration schemes are derived that preserve the positivity and the trace of the density operator, i.e. of the quantum state.

Keywords: Open quantum systems, decoherence, quantum stochastic master equation, Lindblad master equation, Kraus-map, quantum channel, quantum filtering, Wiener process, Poisson process, qubit/photon composite system. Positivity and trace preserving numerical scheme.

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1 Intoduction

An increasing number of experiments controlling quantum states are conducted with various physical supports such as spins, atoms, trapped ions, photons, superconducting circuits, electro-mechanical circuits, optomechanical cavities (see, e.g., [25, 19, 7, 20, 13]). As illustrated on Fig. 1, the quantum dynamics of these experiments can be precisely described by well structured stochastic differential equations, called Stochastic Master Equations (SME). They govern the relationships between the input u corresponding to the classical parameters manipulated by the experimentalists and the classical output y corresponding to the observed measurements. These SME are expressed with operators for which non-commutative calculus and commutation relationships play a fundamental role. These SME are the quantum analogue of the classical Kalman state-space descriptions, $\frac{d}{dt}x = Ax + Bu + w$ and $y = Cx + v$, with noise (w, v) [29].

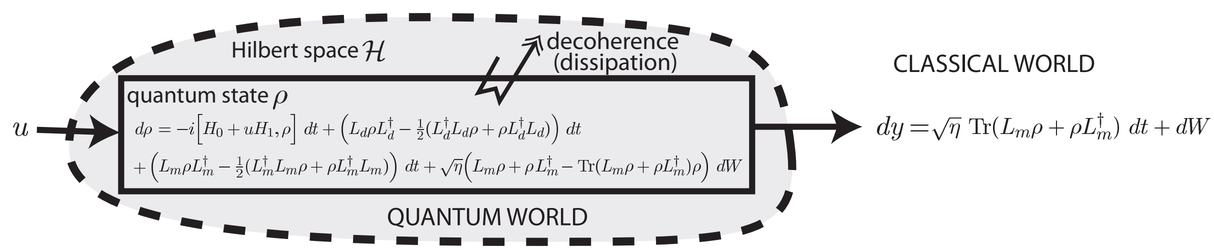

Figure 1: Classical Kalman state-space descriptions are replaced, for quantum systems, by Stochastic Master Equation (SME) descriptions where the measurement back-action is revealed here by the same Wiener process W shared by the quantum state ρ and by the output y.

For Fig. 1, with classical input u and classical output y, this SME reads (diffusive case with Itō formulation [8], $L_\nu^\dagger$ stands for Hermitian conjugate of operator L_ν):

$$d\rho_t = -i\Big[H_0 + uH_1, \rho_t\Big]\ dt + \left(\sum_{\nu=d,m} L_\nu\rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu\rho_t + \rho_t L_\nu^\dagger L_\nu)\right)\ dt + \dots$$
$$\dots + \sqrt{\eta}\Big(L_m\rho_t + \rho_t L_m^\dagger - \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\rho_t\Big)\ dW_t, \quad (1)$$

where the same Wiener process W_t is shared by the state dynamics and the output map

$$dy_t = \sqrt{\eta}\ \mathrm{Tr}(L_m\rho_t + \rho_t L_m^\dagger)\ dt + dW_t. \quad (2)$$

The state ρ is a density operator (a self-adjoint, non-negative and trace-one operator) on a Hilbert space $\mathcal{H}$. Its dynamics (1) are parameterized here via two self-adjoint operators (Hamiltonians) H_0 and H_1 ($[\cdot,\cdot]$ stands for the commutator) and two Lindblad operators, L_d describing a decoherence channel and L_m a measurement channel of efficiency $\eta \in [0, 1]$. When $\eta = 0$, ρ follows a deterministic linear master equation, called Lindblad master equation with two decoherence channels described by L_d and L_m,

$$\frac{d}{dt}\rho = -i\Big[H_0 + uH_1, \rho\Big] + \sum_{\nu=d,m} L_\nu \rho L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho + \rho L_\nu^\dagger L_\nu) \tag{3}$$

and the measurement $y_t = W_t$ boils down to a Wiener process without any relations with u and ρ and thus can be discarded. Notice that (3) corresponds also to the ensemble average dynamics of the SME (1). Notice also that the initial value problems (Cauchy problems) attached to (1) or to (3) are non trivial mathematical problems when the dimension of the underlying Hilbert space $\mathcal{H}$ is infinite and the Hamiltonian or Lindblad operators are unbounded (see e.g [2, 45]).

These SME rely on the well developed theory of open quantum systems combining irreversibility due to decoherence (quantum dissipation) [18, 14] and stochasticity due to measurement back-action [49, 16, 17, 27, 26]. More general SME than the one depicted on Fig. 1, with several Lindblad operators and/or driven by Poisson processes (counting measurement), admit similar structures [39]. Even if the initial system is known to be non Markovian, it is always possible in general to adjunct a dynamical model of the environment and to recover a Markovian model with an SME structure but of larger dimension [14, part IV]. For composite systems made of several interacting quantum sub-systems such SME models are also derived from a quantum network theory [24, 23] gathering in a concise way, quantum stochastic calculus [35], Heisenberg description of input/output theory [50, 22] and quantum filtering [12].

The goal of this paper is to provide an introduction to the structure of these quantum SME illustrated via a composite system made of two key quantum sub-systems (qubits and photons, see A) and based on three fundamental quantum rules (unitary evolution derived from Schrödinger equation, measurement back-action with the collapse of the wave-packet, composite systems relying on tensor products, see B).

Section 2 is devoted to discrete-time formulation of SME. One starts with the Markov chain modelling the LKB[1] photon box of Fig. 2 [25] where photons are measured by probe atoms described by two-level systems, i.e. qubits. Two kinds of interactions between the photons and the atoms are considered (see A for operator notations) :

- dispersive interaction leading to Quantum Non Demolition (QND) measurement of photons; the qubit/photon interaction is dispersive where $\mathbf{H}_{int} = -\chi\big(|e\rangle\langle e| - |g\rangle\langle g|\big) \otimes \mathbf{n}$ (with χ a constant parameter) yields $\mathbf{U}_{\theta=\chi T} = e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given by the explicit formula:

$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes e^{-i\theta \mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta \mathbf{n}}, \quad \theta = \chi T \tag{4}$$

  where $\theta = \chi T$.

- resonant interaction stabilizing then the photons in vacuum state; the qubit/photon interaction is here resonant where $\mathbf{H}_{int} = i\frac{\omega}{2}\Big(|g\rangle\langle e| \otimes \mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\Big)$ (with ω a

[1]LKB for Laboratoire Kastler Brossel.

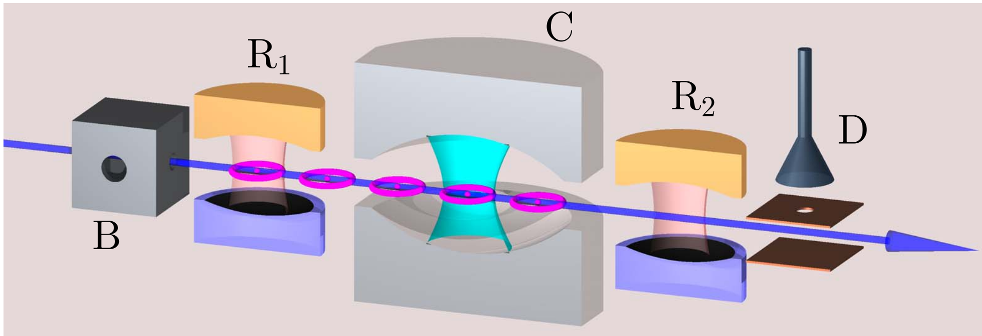

Figure 2: Scheme of the LBK experiment where photons are observed via probe atoms. The photons in blue are trapped between the two mirrors of the cavity C. They are probed by two-level atoms (the small pink torus) flying out the preparation box B, passing through the cavity C and measured in D. Each atom is manipulated before and after C in Ramsey cavities R_1 and R_2, respectively. It is finally detected in D either in ground state $|g\rangle$ or in excited state $|e\rangle$.

constant parameter) yields $\mathbf{U}_{\theta=\omega T} = e^{-iT\mathbf{H}_{int}}$, the Schrödinger propagator during the time $T > 0$, given by the explicit formula:

$$\mathbf{U}_\theta = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\mathbf{n}+\mathbf{I}}) + |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g| \otimes \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}} \quad (5)$$

where $\theta = \omega T$.

One explains on this key system how to take into account measurement errors and why the density operator as quantum state is then crucial. One concludes section 2 with the general structure of discrete-time SME governing the stochastic dynamics of open quantum systems subject to unperfect measurement and decoherence.

Section 3 is devoted to continuous-time SME. One considers here the reverse situation where the qubit is measured by probe photons. Dispersive interaction, measurement of one quadrature $\mathbf{Q}$ of the photons (an observable with a continuous spectrum) and $\theta = \chi T$ scaling as $\sqrt{dt}$ yields a continuous-time SME driven by a Wiener process of form (1) with $L_d = 0$, $L_m \propto \sigma_z$ and $\eta = 1$. One shows how measurement errors tend to decrease η towards 0. Then the general structure of diffusive SME is presented with an equivalent Kraus-map formulation yielding alinear time-integration numerical scheme preserving the positivity and the trace. Resonant interaction, measurement of the photon-number operator $\mathbf{n}$ and $\theta = \omega T$ scaling as dt yield a continuous-time SME driven by a Poisson process associated to the measurement counter. One shows how to include measurement imperfections and gives the structure of continuous-time SME driven by Poisson processes. Last subsection of section 3 provides a very general structure of continuous-time SME driven by Wiener and Poisson processes governing the stochastic dynamics of open quantum systems subject to diffusive/counting unperfect measurements and decoherence.

The conclusion section 4 provides some comments and references related to feedback, parameter estimation and filtering issues. These comments and references are far from being exhaustive.

2 Discrete-time formulation

2.1 Photons measured by qubits (dispersive interaction)

The wave function of the photon is denoted here by $|\psi\rangle$. From the scheme of Fig. 2, the qubit produced by the B in state $|g\rangle$ is subject in R_1 to a rotation of $\pi/4$ in the plane span$\{|g\rangle, |e\rangle\}$, then interacts during T dispersively with the photons in C, is subject to the reverse rotation of $-\pi/4$ in R_2 and finally is measured in D according to σ_z. The Schrödinger evolution of the qubit/photon wave function $|\Psi\rangle$ between B and just before D is given by the following unitary evolution $\mathbf{U}$:

$$\mathbf{U} = \left(\left(\left(\tfrac{|g\rangle-|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\tfrac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)\otimes \mathbf{I}\right) \left(|g\rangle\langle g| \otimes e^{-i\theta\mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta\mathbf{n}}\right) \left(\left(\left(\tfrac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\tfrac{-|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)\otimes \mathbf{I}\right).$$

Applied to the value of $|\Psi\rangle = |g\rangle \otimes |\psi\rangle$ when the qubit leaves B, one gets[2]

$$\mathbf{U}\ (|g\rangle\,|\psi\rangle) = |g\rangle\ \ \cos(\theta\mathbf{n})\,|\psi\rangle + |e\rangle\ \ i\sin(\theta\mathbf{n})\,|\psi\rangle\,.$$

Measuring in D, the observable σ_z yields the collapse of $|\Psi\rangle$ into a separable state $|g\rangle\ \cos(\theta\mathbf{n})\,|\psi\rangle$ or $|e\rangle\ i\sin(\theta\mathbf{n})\,|\psi\rangle$, eigen-vectors of $\sigma_z \otimes \mathbf{I} \equiv \sigma_z$ with eigenvalues -1 or 1, respectively. Numbering the qubit by the integer k and removing the qubit state, one gets the following Markov process induced by the passage of qubit number k:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta\mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta\mathbf{n})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta\mathbf{n})\,|\psi_k\rangle\ ; \\ \frac{i\sin(\theta\mathbf{n})|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta\mathbf{n})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta\mathbf{n})\,|\psi_k\rangle\ ; \end{cases}$$

where $y_k \in \{g, e\}$ is the classical signal produced by the quantum measurement of qubit k. The density operator formulation of this Markov process reads ($\rho \equiv |\psi\rangle\langle\psi|$):

$$\rho_{k+1} = \begin{cases} \frac{\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger}{\mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right)\ ; \\ \frac{\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger}{\mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)\ ; \end{cases}$$

[2] Tensor sign $\otimes$ and tensor product with identity operator $\mathbf{I}$ are not explicitly recalled in the formula as it is usually done when there is no ambiguity: $|g\rangle \otimes |\psi\rangle$ and $|g\rangle\langle g| \otimes e^{-i\theta\mathbf{n}} + |e\rangle\langle e| \otimes e^{i\theta\mathbf{n}}$ read then $|g\rangle\,|\psi\rangle$ and $|g\rangle\langle g|\, e^{-i\theta\mathbf{n}} + |e\rangle\langle e|\, e^{i\theta\mathbf{n}}$; similarly $\left(\left(\frac{|g\rangle-|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)\otimes\mathbf{I}$ just becomes $\left(\left(\frac{|g\rangle-|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)$.

with measurement Kraus operators $\mathbf{M}_g = \cos(\theta\mathbf{n})$ and $\mathbf{M}_e = \sin(\theta\mathbf{n})$. Notice that $\mathbf{M}_g^\dagger\mathbf{M}_g + \mathbf{M}_e^\dagger\mathbf{M}_e = \mathbf{I}$.

When θ/π is irrational, each realization of this Markov process, starting from ρ_0 satisfying $\rho_0 |n\rangle = 0$ for n large enough, converges almost surely towards a Fock state $|\bar{n}\rangle\langle\bar{n}|$ for some $\bar{n}$. More precisely, the probability that a realisation converges towards $|\bar{n}\rangle\langle\bar{n}|$ is given by the initial population $\langle\bar{n}|\rho_0|\bar{n}\rangle$ (see, e.g.,[40, 11, 4]). This almost sure convergence can be seen from the following Lyapunov function (super-martingale)

$$V(\rho) = \sum_{0\leq n_1<n_2} \sqrt{\langle n_1|\rho|n_1\rangle\langle n_2|\rho|n_2\rangle}$$

that converges in average towards 0 since its expectation value from step to step satisfies :

$$\mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) \leq \left(\max_{0\leq n_1<n_2} |\cos(\theta(n_1 \pm n_2)|\right) V(\rho_k).$$

2.2 Photons measured by qubits (resonant interaction)

The photon wave function is still denoted by $|\psi\rangle$. The qubit coming from box B of Fig. 2 is in $|g\rangle$. The Ramsey zones R_1 and R_2 are inactive. The resonant interaction during the passage of the qubit in C yields the propagator (5). Thus, the wave function $|\Psi\rangle$ of the composite qubit/photon system, just before the qubit measurement in D, is as follows:

$$\Big(|g\rangle\langle g|\cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e|\cos(\theta\sqrt{\mathbf{n}+\mathbf{I}}) + |g\rangle\langle e|\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\mathbf{a}^\dagger - |e\rangle\langle g|\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}\Big)|g\rangle|\psi\rangle = |g\rangle\;\cos(\theta\sqrt{\mathbf{n}})|\psi\rangle - |e\rangle\;\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi\rangle$$

Therefore, the resulting Markov process associated to the measurement of the observable $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ with classical signal $y \in \{g, e\}$ is as follows:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\cos(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}{\sqrt{\langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = g \text{ with probability } \langle\psi_k|\cos^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \\ -\frac{\mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}|\psi_k\rangle}{\sqrt{\langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle}} & \text{if } y_k = e \text{ with probability } \langle\psi_k|\sin^2(\theta\sqrt{\mathbf{n}})|\psi_k\rangle\ ; \end{cases}$$

The corresponding density operator formulation is then

$$\rho_{k+1} = \begin{cases} \frac{\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger}{\mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right)\ ; \\ \frac{\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger}{\mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)\ ; \end{cases}$$

with measurement Kraus operators $\mathbf{M}_g = \cos(\theta\sqrt{\mathbf{n}})$ and $\mathbf{M}_e = \mathbf{a}\frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}}$. Notice that, once again, $\mathbf{M}_g^\dagger\mathbf{M}_g + \mathbf{M}_e^\dagger\mathbf{M}_e = \mathbf{I}$.

When $\theta\sqrt{n}/\pi$ is irrational for all positive integer n, this Markov process converges almost surely towards vacuum state $|0\rangle\langle 0|$ when $\rho_0|n\rangle = 0$ for n large enough. This results from the following the Lyapunov function (super-martingale)

$$V(\rho) = \mathrm{Tr}\left(\mathbf{n}\rho\right)$$

since

$$\mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) = V(\rho_k) - \mathrm{Tr}\left(\sin^2(\theta\sqrt{\mathbf{n}})\rho_k\right).$$

2.3 Measurement errors

In presence of measurement imperfections and errors, one has to update the quantum ρ according to Bayes rule by taking as quantum state, the expectation value of ρ_{k+1} given by

$$\rho_{k+1} = \begin{cases} \frac{\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger}{\mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right)} & \text{if } y_k = g \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger\right); \\ \frac{\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger}{\mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)} & \text{if } y_k = e \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right); \end{cases}$$

knowing ρ_k and the information provides by the imperfect measurement outcome. Assume firstly, that the detector D is broken. Then, we get the following linear, trace-preserving and completely positive map

$$\rho_{k+1} = \mathbb{K}(\rho_k) \triangleq \mathbb{E}\left(\rho_{k+1} \;\middle|\; \rho_k\right) = \mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + \mathbf{M}_e\rho_k\mathbf{M}_e^\dagger$$

called in quantum information a quantum channel (see [34]).

When the qubit detector D, producing the classical measurement signal $y_k \in \{g, e\}$, has symmetric errors characterized by the single error rate $\eta \in (0, 1)$, the probability of detector outcome g (resp. e) knowing that the perfect outcome is e (resp. g), Bayes law gives directly

$$\rho_{k+1} = \begin{cases} \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = g, \rho_k\right) = \frac{(1-\eta)\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + \eta\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta)\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + \eta\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = g|\rho_k) = \mathrm{Tr}\left((1-\eta)\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + \eta\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right), \\ \mathbb{E}\left(\rho_{k+1} \;\middle|\; y_k = e, \rho_k\right) = \frac{\eta\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + (1-\eta)\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger}{\mathrm{Tr}\left(\eta\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + (1-\eta)\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right)} \\ \qquad \text{with probability } \mathbb{P}(y_k = e|\rho_k) = \mathrm{Tr}\left(\eta\mathbf{M}_g\rho_k\mathbf{M}_g^\dagger + (1-\eta)\mathbf{M}_e\rho_k\mathbf{M}_e^\dagger\right) \end{cases} \tag{6}$$

Notice that a broken detector corresponds to $\eta = 1/2$ and one recovers the above quantum channel.

2.4 Stochastic Master Equation (SME) in discrete-time

In fact, the general structure of discrete-time SME can always be constructed from the knowledge of a quantum channel (trace preserving completely positive map) having the following Kraus decomposition (which is not unique)

$$\mathbb{K}(\rho) = \sum_{\mu} \mathbf{M}_\mu \rho \mathbf{M}_\mu^\dagger \quad \text{where} \sum_{\mu} \mathbf{M}_\mu^\dagger \mathbf{M}_\mu = \mathbf{I}$$

and a left stochastic matrix $(\eta_{y,\mu})$ where y corresponds to the different imperfect measurement outcomes. Set $\mathbb{K}_y(\rho) = \sum_\mu \eta_{y,\mu} \mathbf{M}_\mu \rho \mathbf{M}_\mu^\dagger$. The SME associated to $\mathbb{K}$ and η reads

$$\rho_{k+1} = \frac{\mathbb{K}_{y_k}(\rho_k)}{\mathrm{Tr}\left(\mathbb{K}_{y_k}(\rho_k)\right)} \quad \text{where } y_k = y \text{ with probability } \mathrm{Tr}\left(\mathbb{K}_y(\rho_k)\right)$$

Notice that $\mathbb{K} = \sum_y \mathbb{K}_y$ since η is left stochastic. Here the Hilbert space $\mathcal{H}$ is arbitrary and can be of infinite dimension, the Kraus operator $\mathbf{M}_\mu$ are bounded operator on $\mathcal{H}$ and ρ is a density operator on $\mathcal{H}$ (Hermitian, trace-class with trace one, non-negative).

To recover the previous discrete-time SME (6), use the above general formulae with the quantum channel $\mathbb{K}(\rho) = \mathbf{M}_g \rho \mathbf{M}_g^\dagger + \mathbf{M}_e \rho \mathbf{M}_e^\dagger$ and the left stochastic matrix

$$\begin{pmatrix} \eta_{g,g} = 1 - \eta_e & \eta_{g,e} = \eta_g \\ \eta_{e,g} = \eta_e & \eta_{e,e} = 1 - \eta_g \end{pmatrix}$$

where η_g (resp. η_e) is the error probability associated to outcome g (resp. e). Notice that in (6) the error model is symmetric with $\eta_g = \eta_e$ corresponding to $\eta \in [0, 1]$.

3 Continuous-time formulation

Contrarily to section 2, photons measure here a qubit.

3.1 Qubits measured by photons: dispersive interaction and discrete-time

The qubit wave function is denoted by $|\psi\rangle$. The photons, before interacting with the qubit, are in the coherent state $\left|i\frac{\alpha}{\sqrt{2}}\right\rangle$ with α real and strictly positive. The interaction is dispersive according to (4). After the interaction and just before the measurement performed on the photons, the composite qubit/photon wave function $|\Psi\rangle$ reads:

$$\Big(|g\rangle\langle g|\, e^{-i\theta\mathbf{n}} + |e\rangle\langle e|\, e^{i\theta\mathbf{n}}\Big) |\psi\rangle \left|i\tfrac{\alpha}{\sqrt{2}}\right\rangle = \langle g|\psi\rangle\, |g\rangle \quad \left|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\right\rangle + \langle e|\psi\rangle\, |e\rangle \quad \left|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\right\rangle$$

since for any coherent state $|\beta\rangle$ of complex amplitude β, $e^{i\theta\mathbf{n}}|\beta\rangle$ is also a coherent state of complex amplitude $e^{i\theta}\beta$ ($e^{i\theta\mathbf{n}}|\beta\rangle = \left|e^{i\theta}\beta\right\rangle$).

Assume that the perfect measurement outcome y belongs to $\mathbb{R}$ and corresponds to the phase-plane observable $\mathbf{Q} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{\sqrt{2}}$ having the entire real line as spectrum. Its spectral decomposition reads formally $\mathbf{Q} = \int_{-\infty}^{+\infty} q \left|q\right\rangle\!\left\langle q\right| dq$ where $\left|q\right\rangle$ is the wave function associated to the eigen-value q. $\left|q\right\rangle$ is not a usual wave function, i.e., in $L^2(\mathbb{R},\mathbb{C})$, but one has formally $\left\langle q | q'\right\rangle = \delta(q-q')$ (see, e.g., [9]). Since

$$\left|ie^{\pm i\theta}\tfrac{\alpha}{\sqrt{2}}\right\rangle = \tfrac{1}{\pi^{1/4}} \int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta} e^{-\frac{(q\pm\alpha\sin\theta)^2}{2}} \left|q\right\rangle dq,$$

we have

$$\begin{aligned}\left\langle g|\psi\right\rangle \left|g\right\rangle & \left|ie^{-i\theta}\tfrac{\alpha}{\sqrt{2}}\right\rangle + \left\langle e|\psi\right\rangle \left|e\right\rangle \left|ie^{i\theta}\tfrac{\alpha}{\sqrt{2}}\right\rangle \\ &= \tfrac{1}{\pi^{1/4}} \int_{-\infty}^{+\infty} e^{iq\alpha\cos\theta} \left(e^{-\frac{(q-\alpha\sin\theta)^2}{2}} \left\langle g|\psi\right\rangle \left|g\right\rangle + e^{-\frac{(q+\alpha\sin\theta)^2}{2}} \left\langle e|\psi\right\rangle \left|e\right\rangle \right) \left|q\right\rangle dq.\end{aligned}$$

Thus

$$\left|\psi_{k+1}\right\rangle = e^{iy_k\alpha\cos\theta} \frac{e^{-\frac{(y_k-\alpha\sin\theta)^2}{2}} \left\langle g|\psi_k\right\rangle \left|g\right\rangle + e^{-\frac{(y_k+\alpha\sin\theta)^2}{2}} \left\langle e|\psi_k\right\rangle \left|e\right\rangle}{\sqrt{e^{-(y_k-\alpha\sin\theta)^2} |\left\langle g|\psi_k\right\rangle|^2 + e^{-(y_k+\alpha\sin\theta)^2} |\left\langle e|\psi_k\right\rangle|^2}}$$

where $y_k \in [y, y+dy]$ with probability $\frac{e^{-(y-\alpha\sin\theta)^2}|\langle g|\psi_k\rangle|^2 + e^{-(y+\alpha\sin\theta)^2}|\langle e|\psi_k\rangle|^2}{\sqrt{\pi}} dy$.

The density operator formulation reads then

$$\rho_{k+1} = \frac{\mathbf{M}_{y_k}\rho_k\mathbf{M}_{y_k}^\dagger}{\operatorname{Tr}\left(\mathbf{M}_{y_k}\rho_k\mathbf{M}_{y_k}^\dagger\right)} \quad \text{where } y_k \in [y, y+dy] \text{ with probability } \operatorname{Tr}\left(\mathbf{M}_y\rho_k\mathbf{M}_y^\dagger\right) dy$$

and measurement Kraus operators

$$\mathbf{M}_y = \tfrac{1}{\pi^{1/4}} e^{-\frac{(y-\alpha\sin\theta)^2}{2}} \left|g\right\rangle\!\left\langle g\right| + \tfrac{1}{\pi^{1/4}} e^{-\frac{(y+\alpha\sin\theta)^2}{2}} \left|e\right\rangle\!\left\langle e\right|.$$

Notice that

$$\operatorname{Tr}\left(\mathbf{M}_y\rho\mathbf{M}_y^\dagger\right) = \frac{1}{\sqrt{\pi}} e^{-(y-\alpha\sin\theta)^2} \left\langle g\right|\rho\left|g\right\rangle + \frac{1}{\sqrt{\pi}} e^{-(y+\alpha\sin\theta)} \left\langle e\right|\rho\left|e\right\rangle \tag{7}$$

and $\int_{-\infty}^{+\infty} \mathbf{M}_y^\dagger\mathbf{M}_y \, dy = \left|g\right\rangle\!\left\langle g\right| + \left|e\right\rangle\!\left\langle e\right| = \mathbf{I}$.

For $\alpha \neq 0$, one has almost sure convergence towards $\left|g\right\rangle$ or $\left|e\right\rangle$ deduced from the following Lyapunov function

$$V(\rho) = \sqrt{\left\langle g\right|\rho\left|g\right\rangle \left\langle e\right|\rho\left|e\right\rangle}$$

and

$$\mathbb{E}\left(V(\rho_{k+1}) \;\middle|\; \rho_k\right) = e^{-\alpha^2\sin^2\theta}\, V(\rho_k).$$

Assume that the detection of y is not perfect. The probability density of y knowing that the perfect detection is q is also a Gaussian given by $\frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}}$ for some error parameter $\sigma > 0$. Then the above Markov process becomes

$$\rho_{k+1} = \frac{\mathbb{K}_{y_k}(\rho_k)}{\operatorname{Tr}\left(\mathbb{K}_{y_k}(\rho_k)\right)}$$

where

$$\mathbb{K}_y(\rho) = \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{\pi\sigma}} e^{-\frac{(y-q)^2}{\sigma}} \mathbf{M}_q \rho \mathbf{M}_q^\dagger \, dq$$

Standard computations show that

$$\mathbb{K}_y(\rho) = \tfrac{1}{\sqrt{\pi(1+\sigma)}} \Bigg(e^{-\frac{(y-\alpha\sin\theta)^2}{1+\sigma}} \langle g|\rho|g\rangle \, |g\rangle\langle g| + e^{-\frac{(y+\alpha\sin\theta)^2}{1+\sigma}} \langle e|\rho|e\rangle \, |e\rangle\langle e| \\ + e^{-\frac{y^2}{1+\sigma} - (\alpha\sin\theta)^2} \big(\langle e|\rho|g\rangle \, |e\rangle\langle g| + \langle g|\rho|e\rangle \, |g\rangle\langle e| \, \big) \Bigg).$$

3.2 Dispersive interaction and continuous-time diffusive limit

Consider the above Markov process with perfect detection $y \in \mathbb{R}$. From (7), one gets

$$\mathbb{E}\Big(y_k \;\Big|\; \rho_k = \rho\Big) \triangleq \overline{y} = -\alpha \sin\theta \ \mathrm{Tr}\left(\sigma_z\rho\right), \ \mathbb{E}\Big(y_k^2 \;\Big|\; \rho_k = \rho\Big) \triangleq \overline{y^2} = 1/2 + (\alpha\sin\theta)^2.$$

When $0 < \alpha\sin\theta = \epsilon \ll 1$, we have up-to third order terms versus ϵy,

$$\frac{\mathbf{M}_y\rho\mathbf{M}_y^\dagger}{\mathrm{Tr}\left(\mathbf{M}_y\rho\mathbf{M}_y^\dagger\right)} = \frac{(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)\rho(\cosh(\epsilon y) - \sinh(\epsilon y)\sigma_z)}{\cosh(2\epsilon y) - \sinh(2\epsilon y)\mathrm{Tr}\left(\sigma_z\rho\right)}$$

$$\approx \frac{\rho - \epsilon y(\sigma_z\rho + \rho\sigma_z) + (\epsilon y)^2(\rho + \sigma_z\rho\sigma_z)}{1 - 2\epsilon y \mathrm{Tr}\left(\sigma_z\rho\right) + 2(\epsilon y)^2}$$

$$= \rho + (\epsilon y)^2\Big(\sigma_z\rho\sigma_z - \rho\Big) + \Big(\sigma_z\rho + \rho\sigma_z - 2\mathrm{Tr}\left(\sigma_z\rho\right)\rho\Big)\Big(- \epsilon y - 2(\epsilon y)^2\mathrm{Tr}\left(\sigma_z\rho\right)\Big).$$

Replacing $\epsilon^2 y^2$ by its expectation value independent of ρ one gets, up to third order terms versus ϵy and ϵ:

$$\frac{\mathbf{M}_y\rho\mathbf{M}_y^\dagger}{\mathrm{Tr}\left(\mathbf{M}_y\rho\mathbf{M}_y^\dagger\right)} \approx \rho + \tfrac{\epsilon^2}{2}\Big(\sigma_z\rho\sigma_z - \rho\Big) + \Big(\sigma_z\rho + \rho\sigma_z - 2\mathrm{Tr}\left(\sigma_z\rho\right)\rho\Big)\Big(- \epsilon y - \epsilon^2\mathrm{Tr}\left(\sigma_z\rho\right)\Big).$$

Set $\epsilon^2 = 2dt$ and $\epsilon y = -2\mathrm{Tr}\left(\sigma_z\rho\right) dt - dW$. Since by construction

$$\mathbb{E}\Big(\epsilon y_k \;\Big|\; \rho_k = \rho\Big) = -\epsilon^2\mathrm{Tr}\left(\sigma_z\rho\right) \ \text{ and } \mathbb{E}\Big((\epsilon y_k)^2 \;\Big|\; \rho_k = \rho\Big) = \epsilon^2 + \epsilon^4$$

one has $\mathbb{E}\Big(dW \;\Big|\; \rho\Big) = 0$ and $\mathbb{E}\Big(dW^2 \;\Big|\; \rho\Big) = dt$ up to order 4 versus ϵ. Thus for dt very small, we recover the following diffusive SME

$$\rho_{t+dt} = \rho_t + dt\Big(\sigma_z\rho_t\sigma_z - \rho\Big) + \Big(\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}\left(\sigma_z\rho_t\right)\rho\Big)\Big(dy_t - 2\mathrm{Tr}\left(\sigma_z\rho_t\right) dt\Big)$$

with $dy_t = 2\mathrm{Tr}\left(\sigma_z\rho_t\right) dt + dW_t$ replacing $-\epsilon y$ and $dy_t^2 = dW_t^2 = dt$ according to Ito rules.

With measurement errors parameterized by $\sigma > 0$, the partial Kraus map

$$\mathbb{K}_y(\rho) = \tfrac{1}{\sqrt{\pi(1+\sigma)}} \Bigg(e^{-\frac{(y-\epsilon)^2}{1+\sigma}} \langle g|\rho|g\rangle \, |g\rangle\langle g| + e^{-\frac{(y+\epsilon)^2}{1+\sigma}} \langle e|\rho|e\rangle \, |e\rangle\langle e| \\ + e^{-\frac{y^2}{1+\sigma} - \epsilon^2} \big(\langle e|\rho|g\rangle \, |e\rangle\langle g| + \langle g|\rho|e\rangle \, |g\rangle\langle e| \big) \Bigg)$$

yields

$$\mathbb{E}\left(y_k \;\middle|\; \rho_k\right) \triangleq \overline{y} = -\epsilon \mathrm{Tr}\left(\sigma_z \rho\right), \; \mathbb{E}\left(y_k^2 \;\middle|\; \rho_k\right) \triangleq \overline{y^2} = (1+\sigma)/2 + \epsilon^2.$$

From

$$\mathbb{K}_y(\rho) \\ = \frac{e^{-\frac{y^2+\epsilon^2}{1+\sigma}}}{2\sqrt{\pi(1+\sigma)}} \Big(\cosh\left(\tfrac{2y\epsilon}{1+\sigma}\right) \big(\rho + \sigma_z \rho \sigma_z\big) - \sinh\left(\tfrac{2y\epsilon}{1+\sigma}\right) \big(\sigma_z \rho + \rho \sigma_z\big) + e^{-\frac{\sigma \epsilon^2}{1+\sigma}} \big(\rho - \sigma_z \rho \sigma_z\big) \Big).$$

and

$$\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right) = \frac{e^{-\frac{y^2+\epsilon^2}{1+\sigma}}}{2\sqrt{\pi(1+\sigma)}} \left(\cosh\left(\tfrac{2y\epsilon}{1+\sigma}\right) - \sinh\left(\tfrac{2y\epsilon}{1+\sigma}\right) \mathrm{Tr}\left(\sigma_z \rho\right) \right)$$

one gets

$$\frac{\mathbb{K}_y(\rho)}{\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right)} = \frac{\left(1 + \frac{e^{-\frac{\sigma\epsilon^2}{1+\sigma}}}{\cosh\left(\frac{2y\epsilon}{1+\sigma}\right)}\right) \rho + \left(1 - \frac{e^{-\frac{\sigma\epsilon^2}{1+\sigma}}}{\cosh\left(\frac{2y\epsilon}{1+\sigma}\right)}\right) \sigma_z \rho \sigma_z - \tanh\left(\frac{2y\epsilon}{1+\sigma}\right) (\sigma_z \rho + \rho \sigma_z)}{2\left(1 - \tanh\left(\frac{2y\epsilon}{1+\sigma}\right) \mathrm{Tr}\left(\sigma_z \rho\right)\right)}.$$

Up to third order terms versus ϵy, one has then

$$\frac{\mathbb{K}_y(\rho)}{\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right)} = \rho + \left(\tfrac{(\epsilon y)^2}{(1+\sigma)^2} + \tfrac{\sigma \epsilon^2}{2(1+\sigma)} \right) (\sigma_z \rho \sigma_z - \rho) - \left(\tfrac{\epsilon y}{1+\sigma} + \tfrac{2(\epsilon y)^2}{(1+\sigma)^2} \mathrm{Tr}\left(\sigma_z \rho\right) \right) \big(\sigma_z \rho + \rho \sigma_z - 2\mathrm{Tr}\left(\sigma_z \rho\right) \rho \big).$$

Replacing $\epsilon^2 y^2$ by its average independent of ρ one gets

$$\frac{\mathbb{K}_y(\rho)}{\mathrm{Tr}\left(\mathbb{K}_y(\rho)\right)} = \rho + \tfrac{\epsilon^2}{2} (\sigma_z \rho \sigma_z - \rho) - \left(\tfrac{\epsilon y}{1+\sigma} + \tfrac{\epsilon^2}{1+\sigma} \mathrm{Tr}\left(\sigma_z \rho\right) \right) \big(\sigma_z \rho + \rho \sigma_z - 2\mathrm{Tr}\left(\sigma_z \rho\right) \rho \big).$$

Set $\epsilon^2 = 2dt$ and $\epsilon y = -2\mathrm{Tr}\left(\sigma_z \rho\right) dt - \sqrt{1+\sigma}\; dW$. Since by construction

$$\mathbb{E}\left(\epsilon y \;\middle|\; \rho\right) = -\epsilon^2 \mathrm{Tr}\left(\sigma_z \rho\right) \text{ and } \mathbb{E}\left((\epsilon y)^2 \;\middle|\; \rho\right) = \epsilon^2/(1+\sigma) + \epsilon^4$$

one has $\mathbb{E}\left(dW \;\middle|\; \rho\right) = 0$ and $\mathbb{E}\left(dW^2 \;\middle|\; \rho\right) = dt$. Thus for dt very small, we recover the following SME with detection efficiency $\eta = \frac{1}{1+\sigma}$

$$\rho_{t+dt} = \rho_t + dt \Big(\sigma_z \rho_t \sigma_z - \rho \Big) + \sqrt{\eta} \Big(\sigma_z \rho_t + \rho_t \sigma_z - 2\mathrm{Tr}\left(\sigma_z \rho_t\right) \rho \Big) dW_t$$

with $dy_t = \sqrt{\eta}\mathrm{Tr}\left(\sigma_z\rho_t + \rho_t\sigma_z\right) + dW_t$ corresponding to $-\epsilon y/\sqrt{1+\sigma}$ and dW_t a Wiener process satisfying Ito rules $dW_t^2 = dt$.

Convergence towards either $|g\rangle$ or $|e\rangle$ is based on the following Lyapunov fonction $V(\rho) = \sqrt{1 - \mathrm{Tr}\left(\sigma_z\rho\right)^2}$. According to Ito rules, one has

$$dV = -\frac{zdz}{\sqrt{1-z^2}} - \frac{dz^2}{2(1-z^2)^{3/2}} = -\frac{zdz}{\sqrt{1-z^2}} - 2\eta^2 V dt$$

where $z = \mathrm{Tr}\left(\sigma_z\rho\right)$, $dz = 2\eta(1-z^2)dW$ and $dz^2 = 4\eta^2(1-z^2)^2 dt$. Since $\mathbb{E}\left(dz \;\middle|\; z\right) = 0$, $\bar{V}_t = \mathbb{E}\left(V(z_t) \;\middle|\; z_0\right)$ converges exponentially to 0 since governed by the linear differential equation

$$\frac{d}{dt}\bar{V}_t = -2\eta^2\bar{V}_t, \quad \bar{V}_0 = V(z_0).$$

For more general and precise results on diffusive SME corresponding to QND measurements and measurement-based feedback issues see [10, 32, 15].

3.3 Diffusive SME

As studied in [8], the general form of diffusive SME admits the following Ito formulation:

$$\begin{aligned} d\rho_t &= \left(-i[\mathbf{H}, \rho_t] + \sum_\nu \mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger - \frac{1}{2}(\mathbf{L}_\nu^\dagger\mathbf{L}_\nu\rho_t + \rho_t\mathbf{L}_\nu^\dagger\mathbf{L}_\nu)\right) dt \\ &\quad + \sum_\nu \sqrt{\eta_\nu}\left(\mathbf{L}_\nu\rho_t + \rho_t\mathbf{L}_\nu^\dagger - \mathrm{Tr}\left((\mathbf{L}_\nu + \mathbf{L}_\nu^\dagger)\rho_t\right)\rho_t\right) dW_{\nu,t}, \\ dy_{\nu,t} &= \sqrt{\eta_\nu}\mathrm{Tr}\left(\mathbf{L}_\nu\rho_t + \rho_t\mathbf{L}_\nu^\dagger\right) dt + dW_{\nu,t} \end{aligned}$$

with efficiencies $\eta_\nu \in [0,1]$ and $dW_{\nu,t}$ being independent Wiener processes. Here the Hilbert space $\mathcal{H}$ is arbitrary, $\mathbf{H}$ is Hermitian and $\mathbf{L}_\nu$ are arbitrary operators of $\mathcal{H}$ not necessarily Hermitian. Each label μ such that $\eta_\mu = 0$ corresponds here to a decoherence channel that can be seen as an unread measurement performed by a sub-system belonging to the environment, see [25, chapter 4].

With Ito rules, this SME admits also the following equivalent formulation:

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger dt}{\mathrm{Tr}\left(\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu(1-\eta_\nu)\mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger dt\right)}$$

with

$$\mathbf{M}_{dy_t} = \mathbf{I} + \left(-i\mathbf{H} - \frac{1}{2}\sum_\nu \mathbf{L}_\nu^\dagger\mathbf{L}_\nu\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t}\mathbf{L}_\nu.$$

Moreover $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ follows the following probability density knowing ρ_t:

$$\mathbb{P}\Big(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \;\mid \rho_t\Big) = \mathrm{Tr}\left(\mathbf{M}_{s\sqrt{dt}}\, \rho_t \mathbf{M}^\dagger_{s\sqrt{dt}} + \sum_\nu (1-\eta_\nu)\mathbf{L}_\nu \rho_t \mathbf{L}_\nu^\dagger dt \right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}$$

that remains a linear function of ρ, as imposed by the quantum measurement law.

In finite dimension N , this formulation implies directly that any diffusive SME admits a unique solution remaining for all $t \geq 0$ in $\{\rho \in \mathbb{C}^{N\times N} \;:\; \rho = \rho^\dagger,\ \rho \geq 0,\ \mathrm{Tr}\,(\rho) = 1\}$.

3.4 Kraus maps and numerical schemes for diffusive SME

From the above formulation, one can construct a linear, positivity and trace preserving numerical integration scheme for such diffusive SME (see [28, appendix B]):

$$\begin{aligned} d\rho_t &= \left(-i[\mathbf{H}, \rho_t] + \sum_\nu \mathbf{L}_\nu \rho_t \mathbf{L}_\nu^\dagger - \frac{1}{2}(\mathbf{L}_\nu^\dagger \mathbf{L}_\nu \rho_t + \rho_t \mathbf{L}_\nu^\dagger \mathbf{L}_\nu) \right) dt \\ &\quad + \sum_\nu \sqrt{\eta_\nu}\left(\mathbf{L}_\nu \rho_t + \rho_t \mathbf{L}_\nu^\dagger - \mathrm{Tr}\left((\mathbf{L}_\nu + \mathbf{L}_\nu^\dagger)\rho_t \right) \rho_t \right) dW_{\nu,t}, \\ dy_{\nu,t} &= \sqrt{\eta_\nu}\mathrm{Tr}\left(\mathbf{L}_\nu \rho_t + \rho_t \mathbf{L}_\nu^\dagger \right) dt + dW_{\nu,t} \end{aligned}$$

With

$$\mathbf{M}_0 = \mathbf{I} + \big(-i\mathbf{H} - \tfrac{1}{2}\sum_\nu \mathbf{L}_\nu^\dagger \mathbf{L}_\nu \big) dt, \quad \mathbf{S} = \mathbf{M}_0^\dagger \mathbf{M}_0 + \left(\sum_\nu \mathbf{L}_\nu^\dagger \mathbf{L}_\nu \right) dt$$

set

$$\widetilde{\mathbf{M}}_0 = \mathbf{M}_0 \mathbf{S}^{-1/2}, \quad \widetilde{\mathbf{L}}_\nu = \mathbf{L}_\nu \mathbf{S}^{-1/2}$$

Sampling of $dy_{\nu,t} = s_{\nu,t}\sqrt{dt}$ according to the following probability law:

$$\mathbb{P}\Big(\left(s_{\nu,t} \in [s_\nu, s_\nu + ds_\nu]\right)_\nu \;\mid \rho_t\Big) = \mathrm{Tr}\left(\widetilde{\mathbf{M}}_{s\sqrt{dt}} \rho_t \widetilde{\mathbf{M}}^\dagger_{s\sqrt{dt}} + \sum_\nu (1-\eta_\nu)\widetilde{\mathbf{L}}_\nu \rho_t \widetilde{\mathbf{L}}_\nu^\dagger dt \right) \prod_\nu \frac{e^{-\frac{s_\nu^2}{2}} ds_\nu}{\sqrt{2\pi}}.$$

where

$$\widetilde{\mathbf{M}}_{dy_t} = \widetilde{\mathbf{M}}_0 + \sum_\nu \sqrt{\eta_\nu} dy_{\nu,t} \widetilde{\mathbf{L}}_\nu.$$

The update ρ_{t+dt} is then given by the following exact Kraus-map formulation:

$$\rho_{t+dt} = \frac{\widetilde{\mathbf{M}}_{dy_t} \rho_t \widetilde{\mathbf{M}}^\dagger_{dy_t} + \sum_\nu (1-\eta_\nu)\widetilde{\mathbf{L}}_\nu \rho_t \widetilde{\mathbf{L}}_\nu^\dagger dt}{\mathrm{Tr}\left(\widetilde{\mathbf{M}}_{dy_t} \rho_t \widetilde{\mathbf{M}}^\dagger_{dy_t} + \sum_\nu (1-\eta_\nu)\widetilde{\mathbf{L}}_\nu \rho_t \widetilde{\mathbf{L}}_\nu^\dagger dt \right)}.$$

Notice that the operators $\widetilde{\mathbf{M}}_{dy_t}$ and $\widetilde{\mathbf{L}}_\nu$ are bounded operators even if $\mathbf{H}$ and $\mathbf{L}_\nu$ are unbounded.

One can also use the following splitting scheme when the unitary operator $e^{-\frac{idt}{2}\mathbf{H}}$ is numerically available and where in the above calculations $\mathbf{M}_0$ is reduced to $\mathbf{I}-\frac{dt}{2}\sum_\nu \mathbf{L}_\nu^\dagger \mathbf{L}_\nu$:

$$\rho_{t+dt} = e^{-\frac{idt}{2}\mathbf{H}} \frac{\widetilde{\mathbf{M}}_{dy_t} e^{-\frac{idt}{2}\mathbf{H}} \rho_t e^{\frac{idt}{2}\mathbf{H}} \widetilde{\mathbf{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathbf{L}}_\nu e^{-\frac{idt}{2}\mathbf{H}} \rho_t e^{\frac{idt}{2}\mathbf{H}} \widetilde{\mathbf{L}}_\nu^\dagger dt}{\mathrm{Tr}\left(\widetilde{\mathbf{M}}_{dy_t} e^{-\frac{idt}{2}\mathbf{H}} \rho_t e^{\frac{idt}{2}\mathbf{H}} \widetilde{\mathbf{M}}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu) \widetilde{\mathbf{L}}_\nu e^{-\frac{idt}{2}\mathbf{H}} \rho_t e^{\frac{idt}{2}\mathbf{H}} \widetilde{\mathbf{L}}_\nu^\dagger dt \right)} e^{\frac{idt}{2}\mathbf{H}}.$$

3.5 Qubits measured by photons (resonant interaction)

The qubit wave function is denoted by $|\psi\rangle$. The photons, before interacting with the qubit, are in the vacuum state $|0\rangle$. The interaction is resonant according to (5). After the interaction and just before the measurement performed on the photons, the composite qubit/photon wave function $|\Psi\rangle$ reads:

$$\Big(|g\rangle\langle g| \cos(\theta\sqrt{\mathbf{n}}) + |e\rangle\langle e| \cos(\theta\sqrt{\mathbf{n}+\mathbf{I}}) \\ + |g\rangle\langle e| \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}} \mathbf{a}^\dagger - |e\rangle\langle g| \mathbf{a} \frac{\sin(\theta\sqrt{\mathbf{n}})}{\sqrt{\mathbf{n}}} \Big) |\psi\rangle\, |0\rangle \\ = \big(\langle g|\psi\rangle\, |g\rangle + \cos\theta\, \langle e|\psi\rangle\, |e\rangle \big)\, |0\rangle + \sin\theta\, \langle e|\psi\rangle\, |g\rangle\, |1\rangle\, .$$

The Markov process with measurement observable $\mathbf{n} = \sum_{n\geq 0} n\, |n\rangle\langle n|$ and outcome $y \in \{0,1\}$ reads (density operator formulation)

$$\rho_{k+1} = \begin{cases} \frac{\mathbf{M}_0 \rho_k \mathbf{M}_0^\dagger}{\mathrm{Tr}\left(\mathbf{M}_0 \rho_k \mathbf{M}_0^\dagger\right)} & \text{if } y_k = 0 \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_0 \rho_k \mathbf{M}_0^\dagger\right)\,; \\ \frac{\mathbf{M}_1 \rho_k \mathbf{M}_1^\dagger}{\mathrm{Tr}\left(\mathbf{M}_1 \rho_k \mathbf{M}_1^\dagger\right)} & \text{if } y_k = 1 \text{ with probability } \mathrm{Tr}\left(\mathbf{M}_1 \rho_k \mathbf{M}_1^\dagger\right)\,; \end{cases}$$

with measurement Kraus operators $\mathbf{M}_0 = |g\rangle\langle g| + \cos\theta\, |e\rangle\langle e|$ and $\mathbf{M}_1 = \sin\theta\, |g\rangle\langle e|$. Notice that $\mathbf{M}_0^\dagger \mathbf{M}_0 + \mathbf{M}_1^\dagger \mathbf{M}_1 = \mathbf{I}$.

Almost convergence analysis when $\cos^2(\theta) < 1$ towards $|g\rangle$ can be seen via the Lyapunov function (super martingale)

$$V(\rho) = \mathrm{Tr}\left(|e\rangle\langle e|\, \rho \right)$$

since

$$\mathbb{E}\left(V(\rho_{k+1}) \;\Big|\; \rho_k \right) = \cos^2\theta\ V(\rho_k).$$

3.6 Towards jump SME

Since in the above Markov process $\mathrm{Tr}\left(\mathbf{M}_0 \rho \mathbf{M}_0^\dagger\right) = 1 - \sin^2\theta \mathrm{Tr}\left(\sigma_- \rho \sigma_+\right)$ and $\mathrm{Tr}\left(\mathbf{M}_1 \rho \mathbf{M}_1^\dagger\right) = \sin^2\theta \mathrm{Tr}\left(\sigma_- \rho \sigma_+\right)$, one gets with $\sin^2\theta = dt$ and y being denoted by dN, an SME driven by

Poisson process $dN_t \in \{0,1\}$ of expectation value $\text{Tr}\left(\sigma_-\rho_t\sigma_+\right) dt$ knowing ρ_t:

$$d\rho_t = \left(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_-)\right)\, dt \\ + \left(\frac{\sigma_-\rho_t\sigma_+}{\text{Tr}\left(\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\text{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt\right).$$

At each time-step, one has the following choice:

- with probabilty $1 - \text{Tr}\left(\sigma_-\rho_t\sigma_+\right)\, dt$, $dN_t = N_{t+dt} - N_t = 0$ and
$$\rho_{t+dt} = \frac{\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger}{\text{Tr}\left(\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger\right)}$$
with $\mathbf{M}_0 = \mathbf{I} - \frac{dt}{2}\sigma_+\sigma_-$.
- with probability $\text{Tr}\left(\sigma_-\rho_t\sigma_+\right)\, dt$, $dN_t = N_{t+dt} - N_t = 1$ and
$$\rho_{t+dt} = \frac{\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger}{\text{Tr}\left(\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)}$$
with $\mathbf{M}_1 = \sqrt{dt}\;\sigma_-$.

To take into account shot noise of rate $\bar{\theta} \geq 0$ and detection efficiency $\bar{\eta} \in [0,1]$, consider the following left stochastic matrix

$$\begin{pmatrix} 1 - \bar{\theta}dt & 1 - \bar{\eta} \\ \bar{\theta}dt & \bar{\eta} \end{pmatrix}$$

where $\bar{\theta}dt$ is the probability to detect $y = 1$, knowing that the true outcome is 0 (fault detection associated to shot noise) and where $\bar{\eta}$ is the probability to detect $y = 1$ knowing that the true outcome is 1 (detection efficiency). Then the above stochastic master equation becomes

$$d\rho_t = \left(\sigma_-\rho_t\sigma_+ - \tfrac{1}{2}(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_-)\right)\, dt \\ + \left(\frac{\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+}{\text{Tr}\left(\bar{\theta}\rho_t + \bar{\eta}\sigma_-\rho_t\sigma_+\right)} - \rho_t\right)\left(dN_t - \left(\bar{\theta} + \bar{\eta}\text{Tr}\left(\sigma_-\rho_t\sigma_+\right)\right) dt\right).$$

At each time-step, one has the following recipe

- $dN_t = N_{t+dt} - N_t = 0$ and
$$\rho_{t+dt} = \frac{(1-\bar{\theta}dt)\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger + (1-\bar{\eta})\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger}{\text{Tr}\left((1-\bar{\theta}dt)\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger + (1-\bar{\eta})\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)} \\ = \frac{\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger + (1-\bar{\eta})\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger}{\text{Tr}\left(\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger + (1-\bar{\eta})\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)} + O(dt^2).$$

with probability

$$1-\Big(\bar\theta+\bar\eta\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\Big)dt=\mathrm{Tr}\left((1-\bar\theta dt)\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+(1-\bar\eta)\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)+O(dt^2)$$

and where $\mathbf{M}_0=\mathbf{I}-\frac{dt}{2}\sigma_+\sigma_-$ and $\mathbf{M}_1=\sqrt{dt}\ \sigma_-$.

- $dN_t=N_{t+dt}-N_t=1$ and

$$\rho_{t+dt}\quad=\quad\frac{\bar\theta\,dt\,\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+\bar\eta\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger}{\mathrm{Tr}\left(\bar\theta\,dt\,\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+\bar\eta\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)}\quad=\quad\frac{\bar\theta\rho_t+\bar\eta\sigma_-\rho_t\sigma_+}{\bar\theta+\bar\eta\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)}\;+\;O(dt)$$

with probability

$$(\bar\theta+\bar\eta\mathrm{Tr}\left(\sigma_-\rho_t\sigma_+\right)\Big)dt=\mathrm{Tr}\left(\bar\theta\,dt\,\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+\bar\eta\mathbf{M}_1\rho_t\mathbf{M}_1^\dagger\right)+O(dt^2)$$

3.7 Jump SME in continuous-time

The above computations with dt very small emphasize the following general structure of a Jump SME in continuous time. With the counting process N_t having increment expectation value knowing ρ_t given by $\langle dN_t\rangle=\Big(\bar\theta+\bar\eta\mathrm{Tr}\left(V\rho_tV^\dagger\right)\Big)\,dt$, and detection imperfections modeled by $\bar\theta\geq 0$ (shot-noise rate) and $\bar\eta\in[0,1]$ (detection efficiency), the quantum state ρ_t is usually mixed and obeys to

$$\begin{aligned}d\rho_t=&\left(-i[\mathbf{H},\rho_t]+\mathbf{V}\rho_t\mathbf{V}^\dagger-\tfrac{1}{2}(\mathbf{V}^\dagger\mathbf{V}\rho_t+\rho_t\mathbf{V}^\dagger\mathbf{V})\right)\,dt\\&+\left(\frac{\bar\theta\rho_t+\bar\eta\mathbf{V}\rho_t\mathbf{V}^\dagger}{\bar\theta+\bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)}-\rho_t\right)\left(dN_t-\Big(\bar\theta+\bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\,dt\right).\end{aligned}$$

Here $\mathbf{H}$ and $\mathbf{V}$ are operators on an underlying Hilbert space $\mathcal{H}$, $\mathbf{H}$ being Hermitian. At each time-step between t and $t+dt$, one has

- $dN_t=0$ with probability $1-\Big(\bar\theta+\bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\,dt$

$$\rho_{t+dt}=\frac{\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+(1-\bar\eta)\mathbf{V}\rho_t\mathbf{V}^\dagger dt}{\mathrm{Tr}\left(\mathbf{M}_0\rho_t\mathbf{M}_0^\dagger+(1-\bar\eta)\mathbf{V}\rho_t\mathbf{V}^\dagger dt\right)}$$

where $\mathbf{M}_0=I-\left(iH+\frac{1}{2}\mathbf{V}^\dagger\mathbf{V}\right)dt$.

- $dN_t=1$ with probability $\Big(\bar\theta+\bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\,dt$,

$$\rho_{t+dt}=\frac{\bar\theta\rho_t+\bar\eta\mathbf{V}\rho_t\mathbf{V}^\dagger}{\bar\theta+\bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)}.$$

These SME have been introduced in the Physics literature in [17, 21].

3.8 General mixed diffusive/jump SME

One can combine in a single SME Wiener and Poisson noises induced by diffusive and counting measurements. The quantum state ρ_t, usually mixed, obeys to

$$
\begin{aligned}
d\rho_t = \Big(-i[\mathbf{H},\rho_t] + \mathbf{L}\rho_t\mathbf{L}^\dagger - \tfrac{1}{2}(\mathbf{L}^\dagger\mathbf{L}\rho_t + \rho_t\mathbf{L}^\dagger\mathbf{L}) + \mathbf{V}\rho_t\mathbf{V}^\dagger - \tfrac{1}{2}(\mathbf{V}^\dagger\mathbf{V}\rho_t + \rho_t\mathbf{V}^\dagger\mathbf{V})\Big)\, dt \\
+ \sqrt{\eta}\bigg(\mathbf{L}\rho_t + \rho_t\mathbf{L}^\dagger - \mathrm{Tr}\left((\mathbf{L}+\mathbf{L}^\dagger)\rho_t\right)\rho_t\bigg)dW_t \\
+ \left(\frac{\bar\theta\rho_t + \bar\eta\mathbf{V}\rho_t\mathbf{V}^\dagger}{\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)} - \rho_t\right)\left(dN_t - \Big(\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\, dt\right)
\end{aligned}
$$

With $dy_t = \sqrt{\eta}\mathrm{Tr}\left((\mathbf{L}+\mathbf{L}^\dagger)\,\rho_t\right)\, dt + dW_t$ and $dN_t = 0$ with probability $1 - \Big(\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\, dt$. The Kraus-map equivalent formulation reads:

- for $dN_t = 0$ of probability $1 - \Big(\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\, dt$

$$
\rho_{t+dt} = \frac{\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + (1-\eta)\mathbf{L}\rho_t\mathbf{L}^\dagger dt + (1-\bar\eta)\mathbf{V}\rho_t\mathbf{V}^\dagger dt}{\mathrm{Tr}\left(\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + (1-\eta)\mathbf{L}\rho_t\mathbf{L}^\dagger dt + (1-\bar\eta)\mathbf{V}\rho_t\mathbf{V}^\dagger dt\right)}
$$

  with $\mathbf{M}_{dy_t} = I - \left(i\mathbf{H} + \frac{1}{2}\mathbf{L}^\dagger\mathbf{L} + \frac{1}{2}\mathbf{V}^\dagger\mathbf{V}\right)dt + \sqrt{\eta}dy_t\mathbf{L}$.

- for $dN_t = 1$ of probability $\Big(\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)\Big)\, dt$:

$$
\rho_{t+dt} = \frac{\mathbf{M}_{dy_t}\tilde\rho_t\mathbf{M}_{dy_t}^\dagger + (1-\eta)\mathbf{L}\tilde\rho_t\mathbf{L}^\dagger dt + (1-\bar\eta)\mathbf{V}\tilde\rho_t\mathbf{V}^\dagger dt}{\mathrm{Tr}\left(\mathbf{M}_{dy_t}\tilde\rho_t\mathbf{M}_{dy_t}^\dagger + (1-\eta)\mathbf{L}\tilde\rho_t\mathbf{L}^\dagger dt + (1-\bar\eta)\mathbf{V}\tilde\rho_t\mathbf{V}^\dagger dt\right)} \text{ with } \tilde\rho_t = \frac{\bar\theta\rho_t + \bar\eta\mathbf{V}\rho_t\mathbf{V}^\dagger}{\bar\theta + \bar\eta\mathrm{Tr}\left(\mathbf{V}\rho_t\mathbf{V}^\dagger\right)}
$$

More generally, one can consider several independent Wiener and Poisson processes. The corresponding SME reads then

$$
\begin{aligned}
d\rho_t = \left(-i[\mathbf{H},\rho_t] + \sum_\nu \mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger - \tfrac{1}{2}(\mathbf{L}_\nu^\dagger\mathbf{L}_\nu\rho_t + \rho_t\mathbf{L}_\nu^\dagger\mathbf{L}_\nu) + \sum_\mu \mathbf{V}_\mu\rho_t\mathbf{V}_\mu^\dagger - \tfrac{1}{2}(\mathbf{V}_\mu^\dagger\mathbf{V}_\mu\rho_t + \rho_t\mathbf{V}_\mu^\dagger\mathbf{V}_\mu)\right) dt \\
+ \sum_\nu \sqrt{\eta_\nu}\left(\mathbf{L}_\nu\rho_t + \rho_t\mathbf{L}_\nu^\dagger - \mathrm{Tr}\left((\mathbf{L}_\nu+\mathbf{L}_\nu^\dagger)\rho_t\right)\rho_t\right)dW_{\nu,t} \\
+ \sum_\mu\left(\frac{\bar\theta_\mu\rho_t + \sum_{\mu'}\bar\eta_{\mu,\mu'}\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger}{\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\mathrm{Tr}\left(\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger\right)} - \rho_t\right)\left(dN_{\mu,t} - \Big(\bar\theta_\mu + \sum_{\mu'}\bar\eta_{\mu,\mu'}\mathrm{Tr}\left(\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger\right)\Big)\, dt\right)
\end{aligned}
$$

where $\eta_\nu \in [0,1]$, $\bar\theta_\mu, \bar\eta_{\mu,\mu'} \geq 0$ with $\bar\eta_{\mu'} = \sum_\mu \bar\eta_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

The equivalent Kraus-map formulation is the following

- When $\forall\mu,\ dN_{\mu,t} = 0$ (probability $1 - \sum_\mu \left(\bar{\theta}_\mu + \bar{\eta}_\mu \text{Tr}\left(\mathbf{V}_\mu \rho_t \mathbf{V}_\mu^\dagger\right)\right) dt$) we have

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger dt + \sum_\mu (1-\bar{\eta}_\mu)\mathbf{V}_\mu\rho_t\mathbf{V}_\mu^\dagger dt}{\text{Tr}\left(\mathbf{M}_{dy_t}\rho_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathbf{L}_\nu\rho_t\mathbf{L}_\nu^\dagger dt + \sum_\mu (1-\bar{\eta}_\mu)\mathbf{V}_\mu\rho_t\mathbf{V}_\mu^\dagger dt\right)}$$

  with $\mathbf{M}_{dy_t} = I - \left(iH + \frac{1}{2}\sum_\nu \mathbf{L}_\nu^\dagger\mathbf{L}_\nu + \frac{1}{2}\sum_\mu \ \mathbf{V}_\mu^\dagger\mathbf{V}_\mu\right) dt + \sum_\nu \sqrt{\eta_\nu} dy_{\nu t}\mathbf{L}_\nu$ and where $dy_{\nu,t} = \sqrt{\eta_\nu}\text{Tr}\left((\mathbf{L}_\nu + \mathbf{L}_\nu^\dagger)\rho_t\right) dt + dW_{\nu,t}$.

- If, for some μ, $dN_{\mu,t} = 1$ (probability $\left(\bar{\theta}_\mu + \sum_{\mu'} \bar{\eta}_{\mu,\mu'}\text{Tr}\left(\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger\right)\right) dt$) we have a similar transition rule

$$\rho_{t+dt} = \frac{\mathbf{M}_{dy_t}\tilde{\rho}_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathbf{L}_\nu\tilde{\rho}_t\mathbf{L}_\nu^\dagger dt + \sum_{\mu'} (1-\bar{\eta}_{\mu'})\mathbf{V}_{\mu'}\tilde{\rho}_t\mathbf{V}_{\mu'}^\dagger dt}{\text{Tr}\left(\mathbf{M}_{dy_t}\tilde{\rho}_t\mathbf{M}_{dy_t}^\dagger + \sum_\nu (1-\eta_\nu)\mathbf{L}_\nu\tilde{\rho}_t\mathbf{L}_\nu^\dagger dt + \sum_{\mu'} (1-\bar{\eta}_{\mu'})\mathbf{V}_{\mu'}\tilde{\rho}_t\mathbf{V}_{\mu'}^\dagger dt\right)}$$

  with $\tilde{\rho}_t = \frac{\bar{\theta}_\mu\rho_t + \sum_{\mu'}\bar{\eta}_{\mu,\mu'}\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger}{\bar{\theta}_\mu + \sum_{\mu'}\bar{\eta}_{\mu,\mu'}\text{Tr}\left(\mathbf{V}_{\mu'}\rho_t\mathbf{V}_{\mu'}^\dagger\right)}$.

4 Conclusion

These SME driven by diffusive measurements or counting measurements are now the object of numerous control-theoretical and mathematical investigations that can be divided into two main issues. The first issues are related to feedback stabilization of a target quantum state (quantum state preparation) or of quantum subspace as in quantum error correction. One can distinguish several kinds of quantum feedback:

- Markovian feedback [49] which is in fact a static output feedback usually used in discrete-time quantum error correction. Its main interest relies on the closed-loop ensemble average dynamics which is a linear quantum channel for which several stability properties are available (see, e.g.,[34, 36, 43]).

- measurement-based feedback where the control-loop is still achieved by a classical controller taking into account the past measurement outcomes. Quantum-state feedback is such typical feedback where the quantum state is estimated via a quantum filter (see, e.g.,[1, 42, 4, 32, 15]).

- Coherent feedback where the controller is a quantum dissipative system [24]. It has its origin in optical pumping and coherent population trapping [30, 6]. Such feedback structures are now the object of active researches in the context of autonomous quantum error correction (see, e.g., [48, 41, 31, 33, 37]).

The second issues are related to filtering and estimation. They are closely related to quantum-state or quantum-process tomography:

- Quantum filtering that can be seen as the quantum analogue of state asymptotic observers. It has its origin in the seminal work of [12]. It can be shown that quantum filtering is always a stable process in average (see [38, 3]). Characterization of asymptotic almost-sure convergence is an open-problem with recent progresses (see, e.g., [47, 5]).

- Estimation of a quantum state or classical parameters estimation based on repeated measurements including imperfection and decoherence during the measurement process relies on quantum SME (see, e.g., [44, 46])

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A Notations used for qubits and photons

1. The qubit with a two-dimensional Hilbert space:

   - Hilbert space: $\mathcal{H} = \mathbb{C}^2 = \Big\{\psi_g \left|g\right\rangle + \psi_e \left|e\right\rangle,\ \psi_g, \psi_e \in \mathbb{C}\Big\}$ with ortho-normal basis $\left|g\right\rangle$ and $\left|e\right\rangle$ (Dirac notations).
   - Quantum state space: $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
   - Pauli operators and commutations:
     $\sigma_- = \left|g\right\rangle\left\langle e\right|$, $\sigma_+ = \sigma_-^\dagger = \left|e\right\rangle\left\langle g\right|$
     $\sigma_x = \sigma_- + \sigma_+ = \left|g\right\rangle\left\langle e\right| + \left|e\right\rangle\left\langle g\right|$;
     $\sigma_y = i\sigma_- - i\sigma_+ = i\left|g\right\rangle\left\langle e\right| - i\left|e\right\rangle\left\langle g\right|$;
     $\sigma_z = \sigma_+\sigma_- - \sigma_-\sigma_+ = \left|e\right\rangle\left\langle e\right| - \left|g\right\rangle\left\langle g\right|$;
     $\sigma_x^2 = \mathbf{I}$, $\sigma_x\sigma_y = i\sigma_z$, $[\sigma_x, \sigma_y] = 2i\sigma_z$, ...
   - Hamiltonian: $\mathbf{H} = \omega_q\sigma_z/2 + u_q\sigma_x$.
   - Bloch sphere representation:
     $\mathcal{D} = \Big\{\frac{1}{2}\big(\mathbf{I} + x\sigma_x + y\sigma_y + z\sigma_z\big)\ \big|\ (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1\Big\}$

2. The photons of the quantum harmonic oscillator with an infinite dimensional Hilbert:

   - Hilbert space: $\mathcal{H} = \Big\{\sum_{n\geq 0}\psi_n\left|n\right\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C})\Big\} \equiv L^2(\mathbb{R}, \mathbb{C})$ with the infinite dimensional orthonormal basis $\big(\left|n\right\rangle\big)_{n=0,1,2,...}$.
   - Quantum state space: $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H})$ trace class, $\rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$ corresponding to trace-class Hermitian operators on $\mathcal{H}$ with unit trace.
   - Operators and commutations:
     Annihilation and creation operator: $\mathbf{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$, $\mathbf{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$;
     Number operator: $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$, $\mathbf{n}\left|n\right\rangle = n\left|n\right\rangle$;
     $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{I}$, $\mathbf{a}f(\mathbf{n}) = f(\mathbf{n} + \mathbf{I})\mathbf{a}$ for any function f;
     Coherent displacement unitary operator $\mathbf{D}_\alpha = e^{\alpha\mathbf{a}^\dagger - \alpha^\dagger\mathbf{a}}$.
     Position $\mathbf{Q}$ and momentum operators $\mathbf{P}$:
     $\mathbf{a} = \frac{\mathbf{Q}+i\mathbf{P}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right)$, $[\mathbf{Q}, \mathbf{P}] = \imath\mathbf{I}$.
   - Hamiltonian: $\mathbf{H} = \omega_c\mathbf{a}^\dagger\mathbf{a} + u_c(\mathbf{a} + \mathbf{a}^\dagger)$.
     (associated classical dynamics: $\frac{dq}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c q - \sqrt{2}u_c$).
   - Quasi-classical pure state $\equiv$ coherent state $\left|\alpha\right\rangle$ $\alpha \in \mathbb{C}$: $\left|\alpha\right\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)\left|n\right\rangle$;
     $\left|\alpha\right\rangle \equiv \frac{1}{\pi^{1/4}}e^{\imath\sqrt{2}q\Im\alpha}e^{-\frac{(q-\sqrt{2}\Re\alpha)^2}{2}}$; $\mathbf{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$, $\mathbf{D}_\alpha\left|0\right\rangle = \left|\alpha\right\rangle$.

B Three quantum rules

This appendix is borrowed from [39]

1. *The state of a quantum system is described either by the wave function* $|\psi\rangle$ *a vector of length one belonging to some separable Hilbert space* $\mathcal{H}$ *of finite or infinite dimension, or, more generally, by the density operator* ρ *that is a non-negative Hermitian operator on* $\mathcal{H}$ *with trace one. When the system can be described by a wave function* $|\psi\rangle$ *(pure state), the density operator* ρ *coincides with the orthogonal projector on the line spanned by* $|\psi\rangle$ *and* $\rho = |\psi\rangle\langle\psi|$ *with usual Dirac notations. In general the rank of* ρ *exceeds one, the state is then mixed and cannot be described by a wave function. When the system is closed, the time evolution of* $|\psi\rangle$ *is governed by the Schrödinger equation (here* $\hbar \equiv 1$*)*
$$\frac{d}{dt}|\psi\rangle = -i\mathbf{H}|\psi\rangle \tag{8}$$
where $\mathbf{H}$ *is the system Hamiltonian, an Hermitian operator on* $\mathcal{H}$ *that could possibly depend on time* t *via some time-varying parameters (classical control inputs). When the system is closed, the evolution of* ρ *is governed by the Liouville/von-Neumann equation*
$$\frac{d}{dt}\rho = -i\big[\mathbf{H}, \rho\big] = -i\big(\mathbf{H}\rho - \rho\mathbf{H}\big). \tag{9}$$

2. *Dissipation and irreversibility has its origin in the "collapse of the wave packet" induced by the measurement. A measurement on the quantum system of state* $|\psi\rangle$ *or* ρ *is associated to an observable* $\mathbf{O}$*, an Hermitian operator on* $\mathcal{H}$*, with spectral decomposition* $\sum_\mu \lambda_\mu \mathbf{P}_\mu$*:* $\mathbf{P}_\mu$ *is the orthogonal projector on the eigen-space associated to the eigen-value* λ_μ *(*$\lambda_\mu \neq \lambda_{\mu'}$ *for* $\mu \neq \mu'$*). The measurement process attached to* $\mathbf{O}$ *is assumed to be instantaneous and obeys to the following rules:*

   - *the measurement outcome* μ *is obtained with probability* $\mathbb{P}_\mu = \langle\psi|\mathbf{P}_\mu|\psi\rangle$ *or* $\mathbb{P}_\mu = \mathrm{Tr}(\rho\mathbf{P}_\mu)$*, depending on the state* $|\psi\rangle$ *or* ρ *just before the measurement;*
   - *just after the measurement process, the quantum state is changed to* $|\psi\rangle_+$ *or* ρ_+ *according to the mappings*
   $$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\mathbf{P}_\mu|\psi\rangle}{\sqrt{\langle\psi|\mathbf{P}_\mu|\psi\rangle}} \quad \text{or} \quad \rho \mapsto \rho_+ = \frac{\mathbf{P}_\mu\rho\mathbf{P}_\mu}{\mathrm{Tr}(\rho\mathbf{P}_\mu)}$$
   where μ *is the observed measurement outcome. These mappings describe the measurement back-action and have no classical counterpart.*

3. *Most systems are composite systems built with several sub-systems. The quantum states of such composite systems live in the tensor product of the Hilbert spaces of each sub-system. This is a crucial difference with classical composite systems where the state space is built with Cartesian products. Such tensor products have important implications such as entanglement with existence of non separable states. Consider a bi-partite system made of two sub-systems: the sub-system of interest* S *with Hilbert space* $\mathcal{H}_S$ *and the measured sub-system* M *with Hilbert space* $\mathcal{H}_M$*. The quantum state of this bi-partite system* (S, M) *lives in* $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$*. Its Hamiltonian* $\mathbf{H}$ *is constructed with the Hamiltonians of the sub-systems,* $\mathbf{H}_S$ *and* $\mathbf{H}_M$*, and an interaction*

Hamiltonian $\mathbf{H}_{int}$ *made of a sum of tensor products of operators (not necessarily Hermitian) on* S *and* M*:*

$$\mathbf{H} = \mathbf{H}_S \otimes \mathbf{I}_M + \mathbf{H}_{int} + \mathbf{I}_S \otimes \mathbf{H}_M$$

with $\mathbf{I}_S$ *and* $\mathbf{I}_M$ *identity operators on* $\mathcal{H}_S$ *and* $\mathcal{H}_M$*, respectively. The measurement operator* $\mathbf{O} = \mathbf{I}_S \otimes \boldsymbol{O}_M$ *is here a simple tensor product of identity on* $\mathcal{H}_S$ *and the Hermitian operator* $\mathbf{O}_M$ *on* $\mathcal{H}_M$*, since only system* M *is directly measured. Its spectrum is degenerate: the multiplicities of the eigenvalues are necessarily greater or equal to the dimension of* $\mathcal{H}_S$*.*