Lecture 4

Dynamics and control of open quantum systems

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This lecture covers adiabatic elimination. Minimal suggested reading H. Carmichael, *Statistical Methods in Quantum Optics*, Volume 2, Chapter 9.

I. FACTS ABOUT SUPEROPERATORS

$$\left(a^{\dagger 2} b \cdot\right) \hat{O} \equiv a^{\dagger 2} b \hat{O}, \quad \left(a \cdot a^{\dagger}\right) \hat{O} \equiv a \hat{O} a^{\dagger}, \quad \left(\cdot b^{\dagger} b\right) \hat{O} \equiv \hat{O} b^{\dagger} b \tag{1}$$

$$\left(a^{\dagger 2} b \cdot\right) = \left(a^{\dagger 2} \cdot\right)(b \cdot), \quad \left(a \cdot a^{\dagger}\right) = (a \cdot)\left(\cdot a^{\dagger}\right), \quad \left(\cdot b^{\dagger} b\right) = (\cdot b)\left(\cdot b^{\dagger}\right) \tag{2}$$

$$\begin{aligned} \left(a \cdot a^{\dagger}\right)\left(b \cdot b^{\dagger}\right) &= \left(a b \cdot b^{\dagger} a^{\dagger}\right) \\ &= \left(b a \cdot a^{\dagger} b^{\dagger}\right) \\ &= \left(b \cdot b^{\dagger}\right)\left(a \cdot a^{\dagger}\right) \end{aligned} \tag{3}$$

Definition of conjugate superoperator

$$(\mathcal{S} \hat{O})^{\dagger} \equiv \mathcal{S}^{\dagger} \hat{O}^{\dagger} \tag{4}$$

$$\begin{aligned} \left[\left(a^{\dagger 2} b \cdot\right) \hat{O}\right]^{\dagger} &= \left(a^{\dagger 2} b \hat{O}\right)^{\dagger} \\ &= \left(\hat{O}^{\dagger} b^{\dagger} a^{2}\right) \\ &= \left(\cdot b^{\dagger} a^{2}\right) \hat{O}^{\dagger} \end{aligned} \tag{5}$$

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and therefore

$$\left(a^{\dagger 2}b\cdot\right)^{\dagger}=\left(\cdot b^{\dagger}a^{2}\right) \tag{6}$$

Conjugation of a product of superoperators does not reverse their order

$$\left(\mathcal{S}_1\mathcal{S}_2\right)^{\dagger}=\mathcal{S}_1^{\dagger}\mathcal{S}_2^{\dagger}. \tag{7}$$

For example

$$\begin{aligned}\left[\left(a\cdot a^{\dagger}\right)\left(\cdot a^{\dagger}a\right)\right]^{\dagger}&=\left(a\cdot a^{\dagger}aa^{\dagger}\right)^{\dagger}\\&=\left(aa^{\dagger}a\cdot a^{\dagger}\right)\\&=\left(a\cdot a^{\dagger}\right)\left(a^{\dagger}a\cdot\right)\\&=\left(a\cdot a^{\dagger}\right)^{\dagger}\left(\cdot a^{\dagger}a\right)^{\dagger}.\end{aligned} \tag{8}$$

Three examples of commutators of superoperators, which can be easily calculated once we know how to compose them

$$\begin{aligned}\left[\left(b\cdot b^{\dagger}\right),(b\cdot)\right]&=\left(b\cdot b^{\dagger}\right)(b\cdot)-(b\cdot)\left(b\cdot b^{\dagger}\right)=0\\\left[\left(b^{\dagger}b\cdot\right),(b\cdot)\right]&=\left(b^{\dagger}b\cdot\right)(b\cdot)-(b\cdot)\left(b^{\dagger}b\cdot\right)\\&=\left(b^{\dagger}b^{2}\cdot\right)-\left(bb^{\dagger}b\cdot\right)\\&=-(b\cdot)\\\left[\left(\cdot b^{\dagger}b\right),(b\cdot)\right]&=\left(\cdot b^{\dagger}b\right)(b\cdot)-(b\cdot)\left(\cdot b^{\dagger}b\right)\\&=0.\end{aligned} \tag{9}$$

From their commutation relations we may derive equations of motion for superoperators. For example, there is an equivalent to Heisenberg's equation of motion. If we define from some superoperator $\mathcal{S}$ another superoperator

$$\mathcal{S}'(t)\equiv e^{-\mathcal{L}t}\mathcal{S}e^{\mathcal{L}t}, \tag{10}$$

then the latter obeys the equation of motion

$$\frac{d\mathcal{S}'}{dt}=\left[\mathcal{S}',\mathcal{L}\right]. \tag{11}$$

For example, we may consider the one-photon dissipator $\mathcal{L}\equiv\kappa\left(2b\cdot b^{\dagger}-b^{\dagger}b\cdot-\cdot b^{\dagger}b\right)$ and for $\mathcal{S}=(b\cdot)$ we find

$$\frac{d(b\cdot)'}{dt}=\kappa\left[(b\cdot)',2\left(b\cdot b^{\dagger}\right)'-\left(b^{\dagger}b\cdot\right)'-\left(\cdot b^{\dagger}b\right)'\right]=-\kappa(b\cdot)', \tag{12}$$

which we may integrate to get

$$e^{-\mathcal{L}_b t}(b\cdot)e^{\mathcal{L}_b t} = e^{-\kappa_b t}(b\cdot). \tag{13}$$

Similarly, we can prove that (*exercise*) for $\mathcal{S}'(0) = \mathcal{S} = \left(b^\dagger\cdot\right)$

$$e^{-\mathcal{L}_b t}\left(b^\dagger\cdot\right)e^{\mathcal{L}_b t} = e^{\kappa_b t}\left(b^\dagger\cdot\right) + \left(e^{-\kappa_b t} - e^{\kappa_b t}\right)\left(\cdot b^\dagger\right) \tag{14}$$

Equation (13) and Eq. (14) will be useful in the next section where we discuss adiabatic elimination.

II. ADIABATIC ELIMINATION

Consider a system governed by the following Liouvillian (Lindblad master equation)

$$\dot{\rho} = \left(\mathcal{L}_a + \mathcal{L}_b + \mathcal{L}_{ab}\right)\rho \tag{15}$$

We assume that the first and last terms correspond to slow dynamics, whereas the middle term corresponds to fast dynamics. That is a is a slow mode, b is a fast mode, and the coupling between them is weak. Our task here is to eliminate the fast degree of freedom b in order to obtain an effective Liouvillian for the mode a.

For example, the three Liouvillians could correspond to

$$\begin{aligned}
\mathcal{L}_a &\equiv -i\left[H_a,\cdot\right] + \kappa\left(2a\cdot a^\dagger - a^\dagger a\cdot - \cdot a^\dagger a\right), \\
\mathcal{L}_b &\equiv \kappa_b\left(2b\cdot b^\dagger - b^\dagger b\cdot - \cdot b^\dagger b\right) \\
\mathcal{L}_{ab} &\equiv (g/2)\left[a^{\dagger 2}b - a^2 b^\dagger,\cdot\right].
\end{aligned} \tag{16}$$

In this lecture we will pursue the adiabatic elimination of b to show that this induces an effective two-photon dissipator on the mode a. The adiabatic elimination procedure can be done more formally, i.e. without regard to the microscopic details of the Liouvillians. We will however choose here the more explicit route in order to familiarize ourselves with superoperator algebra. Throughout this lecture we are following the approach given by Carmichael in Chapter 9 of his book (see abstract), with minor modifications for clarity.

The goal is to derive an equation for the reduced density matrix of subsystem a, under the *Born approximation* Ansatz (tensor product symbol omitted hereafter for brevity)

$$\rho(t) \approx \sigma(t)(|0\rangle\langle 0|)_b \tag{17}$$

Plugging the Ansatz Eq. (17) into Eq. (15) would give $\dot{\sigma} = \mathcal{L}_a\sigma$ which neglects the second-order contribution from the b subsystem. Instead we need to proceed as we did in the case of the derivation of the Lindblad master equation. The first step is, as we did before, to express the interaction Liouvillian in the interaction picture (for which we will use an overline)

$$\overline{\mathcal{L}}_{ab}(t) \equiv e^{-(\mathcal{L}_a+\mathcal{L}_b)t}\mathcal{L}_{ab}e^{(\mathcal{L}_a+\mathcal{L}_b)t}. \tag{18}$$

The interaction-picture density matrix

$$\bar{\rho}(t) \equiv e^{-(\mathcal{L}_a+\mathcal{L}_b)t}\rho(t) \tag{19}$$

then obeys

$$\dot{\bar{\rho}} = \overline{\mathcal{L}}_{ab}(t)\bar{\rho}. \tag{20}$$

To parallel our treatment for the derivation of the master equation, we integrate Eq. (20), we iterate it once, we differentiate with respect to time, and then we trace with respect to the b subsystem.

Integrating and iterating, we get

$$\begin{aligned}\bar{\rho}(t) &= \bar{\rho}(0) + \int_0^t dt'\overline{\mathcal{L}}_{ab}(t')\bar{\rho}(t') \\ &= \bar{\rho}(0) + \int_0^t dt'\overline{\mathcal{L}}_{ab}(t')\left[\bar{\rho}(0) + \int_0^{t'} dt''\overline{\mathcal{L}}_{ab}(t'')\bar{\rho}(t'')\right] \\ &= \bar{\rho}(0) + \int_0^t dt'\overline{\mathcal{L}}_{ab}(t')\bar{\rho}(0) + \int_0^t dt'\overline{\mathcal{L}}_{ab}(t')\int_0^{t'} dt''\overline{\mathcal{L}}_{ab}(t'')\bar{\rho}(t'').\end{aligned} \tag{21}$$

We may now differentiate with respect to t, and take the trace with respect to subsystem b,

$$\dot{\bar{\sigma}} = \mathrm{tr}_b\left[\overline{\mathcal{L}}_{ab}(t)\rho(0)\right] + \int_0^t dt'\,\mathrm{tr}_b\left[\overline{\mathcal{L}}_{ab}(t)\overline{\mathcal{L}}_{ab}\left(t'\right)\bar{\rho}\left(t'\right)\right]. \tag{22}$$

Let us focus on the first term of Eq. (22) and show that it is vanishing.

$$\begin{aligned} &\mathrm{tr}_b\left[\overline{\mathcal{L}}_{ab}(t)\rho(0)\right] \\ &= \mathrm{tr}_b\left[e^{-(\mathcal{L}_a+\mathcal{L}_b)t}(g/2)\left[a^{\dagger 2}b - a^2b^\dagger, \cdot\right]e^{(\mathcal{L}_a+\mathcal{L}_b)t}\overline{\sigma}(0)\left|0\right\rangle_b\left\langle 0\right|_b\right] \\ &= \mathrm{tr}_b\left[e^{-(\mathcal{L}_a+\mathcal{L}_b)t}(g/2)\left[a^{\dagger 2}b - a^2b^\dagger, e^{(\mathcal{L}_a+\mathcal{L}_b)t}\overline{\sigma}(0)\left|0\right\rangle_b\left\langle 0\right|_b\right]\right] \\ &= \mathrm{tr}_b\left[e^{-\mathcal{L}_at}e^{-\mathcal{L}_bt}(g/2)(a^{\dagger 2}b - a^2b^\dagger)e^{\mathcal{L}_at}e^{\mathcal{L}_bt}\overline{\sigma}(0)\left|0\right\rangle_b\left\langle 0\right|_b\right] \\ &\quad - \mathrm{tr}_b\left[e^{-\mathcal{L}_at}e^{-\mathcal{L}_bt}(g/2)e^{\mathcal{L}_at}e^{\mathcal{L}_bt}\overline{\sigma}(0)\left|0\right\rangle_b\left\langle 0\right|_b(a^{\dagger 2}b - a^2b^\dagger)\right] \\ &= (g/2)\left[e^{-\mathcal{L}_at}{}_b\left\langle 0\right|\left(a^{\dagger 2}b - a^2b^\dagger\right)\left|0\right\rangle_b e^{\mathcal{L}_at}\sigma(0) - \sigma(0)_b\left\langle 0\right|\left(a^{\dagger 2}b - a^2b^\dagger\right)\left|0\right\rangle_b\right] = 0,\end{aligned} \tag{23}$$

where we have used in the last step the fact that $e^{\pm\mathcal{L}_b t}\left|0\right\rangle_b\left\langle 0\right|_b = \left|0\right\rangle_b\left\langle 0\right|_b$ (this is the steady-state of the dynamics of the fast-decaying subsystem b).

Returning to Eq. (22), we replace on the right-hand side the Born-approximation Ansatz Eq. (17), to finally obtain

$$\dot{\bar{\sigma}}(t) = \int_0^t dt'\,\mathrm{tr}_b\left[\overline{\mathcal{L}}_{ab}(t)\overline{\mathcal{L}}_{ab}\left(t'\right)\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right)_b\right]. \tag{24}$$

To complete the adiabatic approximation, we need to fill in the explicit form for the interaction-picture interaction $\overline{\mathcal{L}}_{ab}(t)$, and perform the time integral (which will involve the equivalent of a Markov approximation).

To this end, we may rewrite $\overline{\mathcal{L}}_{ab}(t)$ as follows

$$\begin{aligned}
\overline{\mathcal{L}}_{ab}(t) =& (g/2)e^{-(\mathcal{L}_a+\mathcal{L}_b)t}\left[\left(a^{\dagger 2}b\cdot\right) - \left(a^2 b^\dagger\cdot\right) - \left(\cdot a^{\dagger 2}b\right) + \left(\cdot a^2 b^\dagger\right)\right]e^{(\mathcal{L}_a+\mathcal{L}_b)t}\\
=& (g/2)e^{-(\mathcal{L}_a+\mathcal{L}_b)t}\left[\left(a^{\dagger 2}\cdot\right)(b\cdot) - \left(a^2\cdot\right)\left(b^\dagger\cdot\right) - \left(a^2\cdot\right)^\dagger\left(b^\dagger\cdot\right)^\dagger + \left(a^{\dagger 2}\cdot\right)^\dagger(b\cdot)^\dagger\right]e^{(\mathcal{L}_a+\mathcal{L}_b)t}\\
=& (g/2)\left[\overline{\mathcal{A}}_1(t)\overline{\mathcal{B}}_1(t) - \overline{\mathcal{A}}_2(t)\overline{\mathcal{B}}_2(t) + \overline{\mathcal{A}}_1^\dagger(t)\overline{\mathcal{B}}_1^\dagger(t) - \overline{\mathcal{A}}_2^\dagger(t)\overline{\mathcal{B}}_2^\dagger(t)\right],
\end{aligned} \tag{25}$$

where we introduce two superoperators acting on the slow subspace and two superoperators acting on the fast subspace

$$\begin{aligned}
\overline{\mathcal{A}}_1(t) &\equiv e^{-\mathcal{L}_a t}\left(a^{\dagger 2}\cdot\right)e^{\mathcal{L}_a t},\\
\overline{\mathcal{A}}_2(t) &\equiv e^{-\mathcal{L}_a t}\left(a^2\cdot\right)e^{\mathcal{L}_a t},\\
\overline{\mathcal{B}}_1(t) &\equiv e^{-\mathcal{L}_b t}(b\cdot)e^{\mathcal{L}_b t},\\
\overline{\mathcal{B}}_2(t) &\equiv e^{-\mathcal{L}_b t}\left(b^\dagger\cdot\right)e^{\mathcal{L}_b t}.
\end{aligned} \tag{26}$$

The explicit time dependences of $\overline{\mathcal{B}}_1(t)$ and $\overline{\mathcal{B}}_2(t)$ are given by Eq. (13) and Eq. (14), and their conjugates are obtained from Eq. (4). Then

$$\begin{aligned}
\overline{\mathcal{B}}_1(t) &= e^{-\kappa_b t}(b\cdot)\\
\overline{\mathcal{B}}_1^\dagger(t) &= e^{-\kappa_b t}\left(\cdot b^\dagger\right)\\
\overline{\mathcal{B}}_2(t) &= e^{\kappa_b t}\left(b^\dagger\cdot\right) + \left(e^{-\kappa_b t} - e^{\kappa_b t}\right)\left(\cdot b^\dagger\right)\\
\overline{\mathcal{B}}_2^\dagger(t) &= e^{\kappa_b t}(\cdot b) + \left(e^{-\kappa_b t} - e^{\kappa_b t}\right)(b\cdot).
\end{aligned} \tag{27}$$

Plugging Eq. (27) into Eq. (25), the expression for $\overline{\mathcal{L}}_{ab}(t)\overline{\mathcal{L}}_{ab}(t')$ appearing in Eq. (24) will contain 36 contributions, coming from six terms for each factor.

Many contributions vanish. Our strategy is to make $\overline{\mathcal{B}}_{1,2}(t)$ explicit, but leave $\overline{\mathcal{A}}_{1,2}$ in place. This is because, of the six terms of $\overline{\mathcal{L}}_{ab}(t')$, when evaluating $\overline{\mathcal{L}}_{ab}(t')\left|0\right\rangle_b \left\langle 0\right|_b$ four will vanish since either b will hit $\left|0\right\rangle_b$ or $\left\langle 0\right|_b$ will hit $b^\dagger$. Then

$$\overline{\mathcal{L}}_{ab}\left(t'\right)\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right)_b = -(g/2)e^{\kappa_b t'}\left[\overline{\mathcal{A}}_2\left(t'\right)\left(b^\dagger\cdot\right) + \overline{\mathcal{A}}_2^\dagger\left(t'\right)\left(\cdot b\right)\right]\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right)_b. \tag{28}$$

Then there are only twelve contributions in $\overline{\mathcal{L}}_{ab}(t)\overline{\mathcal{L}}_{ab}(t')\left|0\right\rangle_b \left\langle 0\right|_b$

$$\begin{aligned}
&\overline{\mathcal{L}}_{ab}(t)\overline{\mathcal{L}}_{ab}\left(t'\right)\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right)_b \\
&= -(g/2)^2 e^{\kappa_b t'}\Big\{\Big[e^{-\kappa_b t}\overline{\mathcal{A}}_1(t)(b\cdot) - e^{\kappa_b t}\overline{\mathcal{A}}_2(t)\left(b^\dagger\cdot\right) - e^{\kappa_b t}\overline{\mathcal{A}}_2^\dagger(t)(\cdot b) - \left(e^{-\kappa_b t} - e^{-\kappa_b t}\right)\overline{\mathcal{A}}_2^\dagger(t)(b\cdot)\Big]\overline{\mathcal{A}}_2\left(t'\right)\left(b^\dagger\cdot\right) \\
&+ \Big[-e^{\kappa_b t}\overline{\mathcal{A}}_2(t)\left(b^\dagger\cdot\right) - \left(e^{-\kappa_b t} - e^{\kappa_b t}\right)\overline{\mathcal{A}}_2(t)\left(\cdot b^\dagger\right) + e^{-\kappa_b t}\overline{\mathcal{A}}_1^\dagger(t)\left(\cdot b^\dagger\right) - e^{\kappa_b t}\overline{\mathcal{A}}_2^\dagger(t)(\cdot b)\Big]\overline{\mathcal{A}}_2^\dagger\left(t'\right)\left(\cdot b\right)\Big\}\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right) \\
&= -(g/2)^2\Big\{e^{-\kappa_b(t-t')}\Big[\Big(\overline{\mathcal{A}}_1(t)\overline{\mathcal{A}}_2\left(t'\right) - \overline{\mathcal{A}}_2^\dagger(t)\overline{\mathcal{A}}_2\left(t'\right)\Big)\left(bb^\dagger\cdot\right) + \text{ s.c. }\Big] \\
&-e^{\kappa_b(t+t')}\Big[\overline{\mathcal{A}}_2(t)\overline{\mathcal{A}}_2\left(t'\right)\left(b^{\dagger 2}\cdot\right) + \overline{\mathcal{A}}_2^\dagger(t)\overline{\mathcal{A}}_2\left(t'\right)\left(\left(b^\dagger\cdot b\right) - \left(bb^\dagger\cdot\right)\right) + \text{ s.c. }\Big]\Big\}\bar{\sigma}\left(t'\right)\left(|0\rangle\langle 0|\right)_b,
\end{aligned} \tag{29}$$

where s.c. denotes superoperator conjugation according to Eq. (4). There are two types of terms that survive in Eq. (29). However, the ones that diverge as a function of time $\propto e^{\kappa_b(t+t')}$ vanish under tr_b.

For the remaining terms, the ones that decay in time $\propto e^{-\kappa_b(t+t')}$, we perform an analogue of the Markov approximation, assuming $1/\kappa_b$ is a timescale much faster than the timescale of system evolution in $\mathcal{A}_2(t')$, which justifies replacing

$$e^{-\kappa_b(t-t')}\overline{\mathcal{A}}_2\left(t'\right)\bar{\sigma}\left(t'\right) \to 2\kappa_b^{-1}\delta\left(t-t'\right)\overline{\mathcal{A}}_2\left(t'\right)\bar{\sigma}\left(t'\right). \tag{30}$$

Upon doing this the time integral becomes trivial, and we find from Eq. (24)

$$\dot{\bar{\sigma}} = \frac{g^2}{4\kappa_b}e^{-\mathcal{L}_a t}\left[2\left(a^2\cdot a^{\dagger 2}\right) - \left(a^{\dagger 2}a^2\cdot\right) - \left(\cdot a^{\dagger 2}a^2\right)\right]e^{\mathcal{L}_a t}\bar{\sigma}. \tag{31}$$

We can undo the interaction picture by writing the equation of motion for $\sigma(t) = e^{\mathcal{L}_a t}\overline{\sigma}(t)$, which yields

$$\dot{\sigma} = \mathcal{L}_a\sigma + \frac{g^2}{4\kappa_b}\left[2\left(a^2\cdot a^{\dagger 2}\right) - \left(a^{\dagger 2}a^2\cdot\right) - \left(\cdot a^{\dagger 2}a^2\right)\right]\sigma, \tag{32}$$

where we now appropriately see the two-photon dissipator appear.