Lectures 2-3

Dynamics and control of open quantum systems

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This lecture covers the derivation of the Lindblad master equation for a system coupled to an environment. Minimal suggested reading H. Carmichael, *Statistical Methods in Quantum Optics*, Volume 1, Chapter 1-2, and H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Chapter 3.

I. INTRODUCING FRICTION IN A QUANTUM SYSTEM

A. Classical approach does not apply

This lecture is on modeling friction in a quantum mechanical setting. In this subsection we show that in order to ensure consistency, additional degrees of freedom need to be introduced. To do so, we begin, following Carmichael, with a simple harmonic oscillator of mass m and frequency ω, whose Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2, \tag{1}$$

where p, q are the momentum and position of the particle, respectively. The Hamilton equations of motion, to which we *ad hoc* add a friction term proportional to the negative momentum, are

$$\dot{q} = p/m, \quad \dot{p} = -\gamma p - m\omega^2 q, \tag{2}$$

or, eliminating momentum,

$$\ddot{q} + \gamma \dot{q} + \omega^2 q = 0. \tag{3}$$

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In a quantum mechanical treatment, the coordinates obey the canonical commutation relation,

$$[\hat{q}, \hat{p}] = i\hbar, \tag{4}$$

and the equations of motion above, without the friction term, are exactly the same for operators in the Heisenberg picture (Heisenberg equations of motion). If we kept the friction term, we would obtain the following for the time derivative of the commutator of the coordinates

$$\begin{aligned} \frac{d}{dt}[\hat{q}, \hat{p}] &= \dot{\hat{q}}\hat{p} + \hat{q}\dot{\hat{p}} - \dot{\hat{p}}\hat{q} - \hat{p}\dot{\hat{q}}. \\ &= -\gamma[\hat{q}, \hat{p}] \end{aligned} \tag{5}$$

That is, the commutator decays exponentially in time,

$$[\hat{q}(t), \hat{p}(t)] = e^{-\gamma t}[\hat{q}(0), \hat{p}(0)] = e^{-\gamma t} i\hbar, \tag{6}$$

which would lead to the unexpected conclusion that the Heisenberg uncertainty constraint also decays exponentially

$$\Delta q \Delta p \geq \frac{1}{2}\hbar e^{-\gamma t}. \tag{7}$$

To remedy this difficulty, at least one additional degree of freedom needs to be added to model the physics of friction, as we explain below.

B. Additional degree of freedom

To see the source of the additional degree of freedom, assume that, in addition to the friction term appearing in the equations of motion, the particle is also subjected to a fluctuating random force $F(t)$. Then the second-order differential equation for position changes to

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = F(t)/m. \tag{8}$$

We may think of $F(t)$ as being exerted by a large collection of degrees of freedom (the environment), upon which the dynamics of the oscillator $q(t)$ has no backaction (that is, we are neglecting the effect of $q(t)$ on the dynamical equations for $F(t)$). These degrees of freedom may be a separate collection of harmonic oscillators coupled weakly to the harmonic degree of freedom described by the coordinates (q, p). These harmonic oscillators could be, for example, in thermal equilibrium at some temperature T.

To understand how the addition of degrees of freedom can remedy the problem we raised above, in the Hamiltonian operator the addition of a single new harmonic oscillator would amount to

$$H = \hbar\omega a^\dagger a + \hbar\omega b^\dagger b + \hbar\kappa \left(a^\dagger b + a b^\dagger\right), \tag{9}$$

where the canonical commutators hold

$$\left[a, a^\dagger\right] = 1, \quad \left[b, b^\dagger\right] = 1. \tag{10}$$

The Hamiltonian for a single oscillator

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2}\right) \tag{11}$$

is related to the one before via

$$\begin{aligned} a &\equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} + i\hat{p}), \\ a^\dagger &\equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{q} - i\hat{p}). \end{aligned} \tag{12}$$

Without the b degree of freedom, the problem encountered before translates to

$$\left[a, a^\dagger\right] = e^{-\gamma t}. \tag{13}$$

However, in the presence of b, due to the linear coupling κ, and without the ad hoc addition of friction, the Heisenberg equations of motion yield the result

$$\begin{aligned} a(t) &= e^{-i\omega t}[a(0)\cos\kappa t - ib(0)\sin\kappa t] \\ b(t) &= e^{-i\omega t}[b(0)\cos\kappa t - ia(0)\sin\kappa t] \end{aligned} \tag{14}$$

which leads to a conservation of the commutator

$$\left[a(t), a^\dagger(t)\right] = \left[a(0), a^\dagger(0)\right]\cos^2\kappa t + \left[b(0), b^\dagger(0)\right]\sin^2\kappa t = 1. \tag{15}$$

In the following we will extrapolate this idea for an arbitrary system coupled to an environment. In Sec. II we will begin with a formal treatment of ‘bath degrees of freedom’ coupled to a system, but will soon give a microscopic model for the bath, which consists of a collection of harmonic oscillators.

II. MICROSCOPIC DERIVATION IN THE WEAK-COUPLING LIMIT

Following largely the notations of Breuer and Petruccione, we consider a system coupled to a bath via an interaction Hamiltonian,

$$H = H_S + H_B + H_I. \tag{16}$$

In the interaction picture with respect to $H_S + H_B$, the von Neumann equation for the full density matrix of the system is

$$\frac{d}{dt}\rho(t) = -i\left[H_I(t), \rho(t)\right]. \tag{17}$$

That equation can be integrated to lead to the integral form

$$\rho(t) = \rho(0) - i\int_0^t ds\left[H_I(s), \rho(s)\right]. \tag{18}$$

We may plug the equation Eq. (18) into itself once, and take the trace with respect to the bath degrees of freedom. Denoting $\rho_S(t) = \mathrm{tr}_B\,\rho(t)$, we have the exact equation

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds\,\mathrm{tr}_B\left[H_I(t), \left[H_I(s), \rho(s)\right]\right], \tag{19}$$

if we assume

$$\mathrm{tr}_B\left[H_I(t), \rho(0)\right] = 0. \tag{20}$$

The first approximation that one makes is the *Born approximation.* This assumes that the system and the bath are only weakly coupled, and therefore throughout the dynamics there is no significant back-action of the system onto the bath, and therefore the density matrix of the system+bath, $\rho(t)$, remains in a tensor-product form

$$\rho(t) \approx \rho_S(t) \otimes \rho_B, \tag{21}$$

and, as assumed, with the reduced density matrix of the bath ρ_B time-independent. After the *Born approximation*,

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds\,\mathrm{tr}_B\left[H_I(t), \left[H_I(s), \rho_S(s) \otimes \rho_B\right]\right] \tag{22}$$

Furthermore, under the Markov approximation, we replace under the integral $\rho_S(s)$ by $\rho_S(t)$

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds\,\mathrm{tr}_B\left[H_I(t), \left[H_I(s), \rho_S(t) \otimes \rho_B\right]\right], \tag{23}$$

which amounts to enforcing the (Markov) condition that the time-evolution of the reduced density matrix of the system $\rho_S(t)$ only depends on its value at time t, and not upon its past values at $s < t$. This is the *Redfield equation*.

To finish the Markov approximation, we replace $s \to t - s$ and change the integration limit from t to ∞. The first change is exact, since $\int_0^t ds f(s) = \int_0^t ds f(t-s)$, and the second change is valid insofar as the integrand decays sufficiently fast compared to bath correlation times, i.e. for $s \gg \tau_B$. The typical timescale of the integrand τ_R over which the state of the system varies appreciably must then be large compared to τ_B. Reservoir correlations have to decay fast. With these two steps done, we arrive at the Born-Markov master equation

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds\, \mathrm{tr}_B \left[H_I(t), \left[H_I(t-s), \rho_S(t) \otimes \rho_B\right]\right]. \tag{24}$$

In the remainder of this section, we will introduce microscopic details in order to bring this equation into Lindblad form.

We assume that the system-bath interaction takes the form

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha, \tag{25}$$

with the system operators $A_\alpha^\dagger = A_\alpha$, and bath operators $B_\alpha^\dagger = B_\alpha$ being Hermitian. Note that the bath operators can act on multiple bath Hilbert spaces $\alpha = 1, 2, 3, \ldots$, or on the same Hilbert space. The former case is suited when we model multiple sources of noise, such as charge noise and flux noise in a superconducting qubit. The latter case covers the situation where the coupling between the system and some bath cannot be expressed as a single tensor product of two operators. For concrete examples, see below in Sec. V the treatment of a spin-1/2 and of a harmonic oscillator.

We next define the 'collapse operator' at frequency ω by

$$A_\alpha(\omega) \equiv \sum_{\varepsilon' - \varepsilon = \omega} \Pi(\varepsilon) A_\alpha \Pi\left(\varepsilon'\right). \tag{26}$$

The following hold

$$\begin{aligned} \left[H_S, A_\alpha(\omega)\right] &= -\omega A_\alpha(\omega), \\ \left[H_S, A_\alpha^\dagger(\omega)\right] &= +\omega A_\alpha^\dagger(\omega), \end{aligned} \tag{27}$$

which gives in the interaction picture chosen at the beginning of this section

$$\begin{aligned} e^{iH_S t} A_\alpha(\omega) e^{-iH_S t} &= e^{-i\omega t} A_\alpha(\omega), \\ e^{iH_S t} A_\alpha^\dagger(\omega) e^{-iH_S t} &= e^{+i\omega t} A_\alpha^\dagger(\omega). \end{aligned} \tag{28}$$

Moreover,

$$\left[H_S, A_\alpha^\dagger(\omega) A_\beta(\omega)\right] = 0, \tag{29}$$

and negative frequency collapse operators are merely the Hermitian conjugates of their positive frequency counterparts

$$A_\alpha^\dagger(\omega) = A_\alpha(-\omega). \tag{30}$$

Resolution of identity $\sum_\epsilon \Pi(\epsilon) = I$ leads to

$$\sum_\omega A_\alpha(\omega) = \sum_\omega A_\alpha^\dagger(\omega) = A_\alpha, \tag{31}$$

and, moreover, the Schrödinger picture interaction Hamiltonian is

$$H_I = \sum_{\alpha,\omega} A_\alpha(\omega) \otimes B_\alpha = \sum_{\alpha,\omega} A_\alpha^\dagger(\omega) \otimes B_\alpha^\dagger. \tag{32}$$

Putting these all together, the interaction-picture system-bath interaction Hamiltonian appearing in the Born-Markov master equation Eq. (24) is

$$H_I(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t) = \sum_{\alpha,\omega} e^{+i\omega t} A_\alpha^\dagger(\omega) \otimes B_\alpha^\dagger(t), \tag{33}$$

where

$$B_\alpha(t) = e^{iH_B t} B_\alpha e^{-iH_B t}. \tag{34}$$

We assume that the bath operators have zero expectation value in the state ρ_B

$$\langle B_\alpha(t)\rangle \equiv \mathrm{tr}\left\{B_\alpha(t)\rho_B\right\} = 0. \tag{35}$$

With these assumptions, we can insert the microscopic form of the interaction Hamiltonian Eq. (33) into the Born-Markov master equation Eq. (24) to get

$$\begin{aligned} \frac{d}{dt}\rho_S(t) &= \int_0^\infty ds\, \mathrm{tr}_B \left\{H_I(t-s)\rho_S(t)\rho_B H_I(t) - H_I(t) H_I(t-s)\rho_S(t)\rho_B\right\} + \text{ h.c.} \\ &= \sum_{\omega,\omega'} \sum_{\alpha,\beta} e^{i(\omega'-\omega)t} \Gamma_{\alpha\beta}(\omega) \left(A_\beta(\omega)\rho_S(t) A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega') A_\beta(\omega)\rho_S(t)\right) \\ &\quad + \text{ h.c.} \end{aligned} \tag{36}$$

Above, we have defined

$$\Gamma_{\alpha\beta}(\omega) \equiv \int_0^{\infty} ds e^{i\omega s} \left\langle B_\alpha^\dagger(t) B_\beta(t-s) \right\rangle , \tag{37}$$

and we need to introduce the two-time correlation function of the bath

$$\left\langle B_\alpha^\dagger(t) B_\beta(t-s) \right\rangle \equiv \mathrm{tr}_B \left\{ B_\alpha^\dagger(t) B_\beta(t-s) \rho_B \right\} , \tag{38}$$

which we assume to be time-translation invariant

$$\left\langle B_\alpha^\dagger(t) B_\beta(t-s) \right\rangle = \left\langle B_\alpha^\dagger(s) B_\beta(0) \right\rangle . \tag{39}$$

Under the third and final approximation, called *secular approximation*, we assume that transitions of different frequencies occur at sufficiently distinct frequencies so that $e^{i(\omega'-\omega)t}$ oscillates fast and can be neglected unless $\omega' = \omega$. This allows us to perform the sum over ω' and gives

$$\frac{d}{dt}\rho_S(t) = \sum_{\omega} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}(\omega) \left(A_\beta(\omega) \rho_S(t) A_\alpha^\dagger(\omega) - A_\alpha^\dagger(\omega) A_\beta(\omega) \rho_S(t) \right) + \text{ h.c. } , \tag{40}$$

We may further define the real and imaginary parts of $\Gamma_{\alpha\beta}(\omega)$ as follows

$$\Gamma_{\alpha\beta}(\omega) = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + i S_{\alpha\beta}(\omega), \tag{41}$$

or equivalently

$$S_{\alpha\beta}(\omega) = \frac{1}{2i} \left(\Gamma_{\alpha\beta}(\omega) - \Gamma_{\beta\alpha}^*(\omega) \right) , \tag{42}$$

and

$$\gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}^*(\omega) = \int_{-\infty}^{+\infty} ds e^{i\omega s} \left\langle B_\alpha^\dagger(s) B_\beta(0) \right\rangle . \tag{43}$$

Then the interaction picture Lindblad master equation is

$$\frac{d}{dt}\rho_S(t) = -i \left[H_{LS}, \rho_S(t) \right] + \mathcal{D} \left(\rho_S(t) \right) , \tag{44}$$

where the ‘Lamb-shift’ Hamiltonian is

$$H_{LS} = \sum_{\omega} \sum_{\alpha,\beta} S_{\alpha\beta}(\omega) A_\alpha^\dagger(\omega) A_\beta(\omega), \tag{45}$$

and has the property that it commutes with the system Hamiltonian

$$[H_S, H_{LS}] = 0. \tag{46}$$

In Eq. (44) the dissipator part is given by

$$\mathcal{D}(\rho_S) = \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho_S\right\}\right). \tag{47}$$

Provided that the matrix $\gamma_{\alpha\beta}(\omega)$ is positive and diagonalizable over the indices α, β, Eq. (44) can be brought to the so-called Lindblad form

$$\mathcal{D}(\rho_S) = \sum_{\omega}\sum_{\alpha}\bar{\gamma}_\alpha(\omega)\left(\bar{A}_\alpha(\omega)\rho_S \bar{A}_\alpha^\dagger(\omega) - \frac{1}{2}\left\{\bar{A}_\alpha^\dagger(\omega)\bar{A}_\alpha(\omega), \rho_S\right\}\right), \tag{48}$$

where the new operators $\bar{A}_\alpha(\omega)$ are related to the old ones via the unitary matrix that diagonalizes. Concretely, if the diagonal rate matrix is given by $\bar{\gamma}_\alpha(\omega)\delta_{\alpha\beta} \equiv \bar{\gamma}_{\alpha\beta}(\omega)$ where $\bar{\gamma}_{\alpha\beta}(\omega) = \sum_{\mu\nu} U_{\alpha\mu}\gamma_{\mu\nu}(\omega)(U^\dagger)_{\nu\beta} = \sum_{\mu\nu} U_{\alpha\mu}U^*_{\beta\nu}\gamma_{\mu\nu}(\omega)$, and equivalently $\gamma_{\alpha\beta}(\omega) = \sum_{\mu\nu}(U^\dagger)_{\alpha\mu}\bar{\gamma}_{\mu\nu}(\omega)U_{\nu\beta} = \sum_{\mu\nu} U^*_{\mu\alpha}U_{\nu\beta}\bar{\gamma}_{\mu\nu}(\omega)$, we plug in this latter form for $\gamma_{\alpha\beta}(\omega)$ into Eq. (47)

$$\begin{aligned}
\mathcal{D}(\rho_S) =& \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho_S\right\}\right)\\
=& \sum_{\omega}\sum_{\alpha,\beta}\sum_{\mu,\nu}\bar{\gamma}_{\mu\nu}(\omega)U^*_{\mu\alpha}U_{\nu\beta}\left(A_\beta(\omega)\rho_S A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho_S\right\}\right)\\
=& \sum_{\omega}\sum_{\mu,\nu}\bar{\gamma}_{\mu\nu}(\omega)\left(\bar{A}_\mu(\omega)\rho_S \bar{A}_\nu^\dagger(\omega) - \frac{1}{2}\left\{\bar{A}_\nu^\dagger(\omega)\bar{A}_\mu(\omega), \rho_S\right\}\right)\\
=& \sum_{\omega}\sum_{\mu,\nu}\bar{\gamma}_{\mu}(\omega)\left(\bar{A}_\mu(\omega)\rho_S \bar{A}_\mu^\dagger(\omega) - \frac{1}{2}\left\{\bar{A}_\mu^\dagger(\omega)\bar{A}_\mu(\omega), \rho_S\right\}\right)\\
\equiv& \sum_{\omega}\sum_{\mu}\bar{\gamma}_\mu(\omega)\mathcal{D}[\bar{A}_\mu(\omega)]\rho_S(t),
\end{aligned} \tag{49}$$

where we have used the fact that the parenthesis is linear in both $A_\beta(\omega)$ and in $A_\alpha^\dagger(\omega)$ and the replacement $\bar{A}_\mu(\omega) \equiv \sum_\alpha U_{\mu\alpha}A_\alpha(\omega)$, and we have introduced the definition of the dissipator superoperator

$$\mathcal{D}[O]\rho = O\rho O^\dagger - \frac{1}{2}\{O^\dagger O, \rho\}. \tag{50}$$

That the quantities $\gamma_{\alpha\beta}(\omega) > 0$ is a reasonable requirement. One can put this on firmer footing by imposing conditions on the time dependence of the two-point correlation function

$\langle B_\alpha(t)^\dagger B_\beta(0)\rangle$ (see Breuer and Petruccione, page 136). The Lindblad form can be shown to be completely positive and trace-preserving (CPTP). The first property includes the fact that the action of the dissipator retains the positivity of the density matrix. The second property, which is immediate from the form of Eq. (44) (*exercise*), means that under the Lindblad master equation the trace of the reduced density matrix $\rho_S(t)$ remains unity, i.e. $\frac{d}{dt}\mathrm{tr}_S\{\rho_S(t)\} = 0$. Further details on this can be found in Chapters 2, 3 of Breuer and Petruccione.

From Eq. (44), the Schrödinger picture master equation is easy to obtain with the only modification that now the unitary dynamics is generated by $H_S + H_{LS}$, that is, Eq. (44) is replaced by

$$\frac{d}{dt}\rho_S(t) = -i\left[H_S + H_{LS}, \rho_S(t)\right] + \mathcal{D}\left(\rho_S(t)\right), \tag{51}$$

with now $\rho_S(t)$ the Schrödinger-picture reduced density matrix for the system.

III. BOSONIC BATH

In this section we give formulas for various bath correlation functions in the specific example of a bosonic bath. To this end, we consider

$$\begin{aligned} B =& \sum_l g_l(b_l + b_l^\dagger), \\ [b_l, b_m^\dagger] =& \delta_{lm}, \\ H_B =& \sum_l \hbar\omega_l b_l^\dagger b_l. \end{aligned} \tag{52}$$

In circuit quantum electrodynamics, for example, the bosonic modes annihilated by operators b_l can be thought of as the normal modes of a transmission line. The interaction-picture form of the above, with respect to $H_S + H_B$, whatever form H_S takes, is

$$B(t) = \sum_l g_l(b_l e^{-i\omega_l t} + b_l^\dagger e^{i\omega_l t}). \tag{53}$$

Then the unilateral spectral function is

$$\begin{aligned} \Gamma(\omega) &= \int_0^\infty ds e^{i\omega s}\left\langle B^\dagger(s)B(0)\right\rangle \\ &= \int_0^\infty ds e^{i\omega s}\left\langle \left[\sum_l g_l(b_l e^{-i\omega_l s} + b_l^\dagger e^{i\omega_l s})\right]\left[\sum_m g_m(b_m + b_m^\dagger)\right]\right\rangle. \end{aligned} \tag{54}$$

Since $\langle \ldots \rangle = \mathrm{Tr}_B\{\rho_B \ldots\} = \frac{\mathrm{Tr}_B\{e^{-\beta H_B}\ldots\}}{\mathrm{Tr}_B\{e^{-\beta H_B}\}}$, and since H_B is diagonal in the bosonic normal mode basis with annihilation operators $\{b_l | l = 1, 2, \ldots\}$, we have $\langle b_l^\dagger b_m \rangle \propto \delta_{lm}$, and $\langle b_l b_m \rangle = \langle b_l^\dagger b_m^\dagger \rangle = 0$.

$$\begin{aligned}\Gamma(\omega) &= \int_0^\infty ds e^{i\omega s} \left\langle \sum_l g_l^2 (b_l b_l^\dagger e^{-i\omega_l s} + b_l^\dagger b_l e^{i\omega_l s}) \right\rangle \\ &= \int_0^\infty ds e^{i\omega s} \sum_l g_l^2 (\langle b_l b_l^\dagger \rangle e^{-i\omega_l s} + \langle b_l^\dagger b_l \rangle e^{i\omega_l s}) \\ &= \sum_l g_l^2 \langle b_l b_l^\dagger \rangle \int_0^\infty ds e^{i(\omega-\omega_l)s} + \sum_l g_l^2 \langle b_l^\dagger b_l \rangle \int_0^\infty ds e^{i(\omega+\omega_l)s}\end{aligned} \tag{55}$$

Using

$$\langle b_l^\dagger b_l \rangle = n_B(\omega_l) = \frac{1}{e^{\beta\hbar\omega_l} - 1}, \langle b_l b_l^\dagger \rangle = 1 + n_B(\omega_l). \tag{56}$$

and Sokhotski-Plemelj

$$\int_0^\infty ds e^{-i(\omega \pm \nu)s} = \pi\delta(\omega \pm \nu) - i\mathcal{P}\left(\frac{1}{\omega \pm \nu}\right) \tag{57}$$

we have

$$\begin{aligned}\Gamma(\omega) =& \sum_l g_l^2 \left[1 + n_B(\omega_l)\right] \left[\pi\delta(\omega - \omega_l) + i\mathcal{P}\left(\frac{1}{\omega - \omega_l}\right)\right] \\ &+ \sum_l g_l^2 n_B(\omega_l) \left[\pi\delta(\omega + \omega_l) + i\mathcal{P}\left(\frac{1}{\omega + \omega_l}\right)\right]\end{aligned} \tag{58}$$

and therefore the real part of the above is

$$\frac{1}{2\pi}\gamma(\omega) = \sum_l g_l^2 \left[1 + n_B(\omega_l)\right] \delta(\omega - \omega_l) + \sum_l g_l^2 n_B(\omega_l) \delta(\omega + \omega_l), \tag{59}$$

and its imaginary part

$$S(\omega) = \sum_l g_l^2 \left[1 + n_B(\omega_l)\right] \mathcal{P}\left(\frac{1}{\omega - \omega_l}\right) + \sum_l g_l^2 n_B(\omega_l) \mathcal{P}\left(\frac{1}{\omega + \omega_l}\right). \tag{60}$$

If we distinguish between positive and negative frequencies, assuming bath modes all have positive frequency $\omega_l \geq 0$, then

$$\begin{aligned}\frac{1}{2\pi}\gamma(\omega \geq 0) &= \sum_l g_l^2 \left[1 + n_B(\omega_l)\right] \delta(\omega - \omega_l), \\ \frac{1}{2\pi}\gamma(\omega < 0) &= \sum_l g_l^2 n_B(\omega_l) \delta(\omega + \omega_l).\end{aligned} \tag{61}$$

Taking the zero-temperature limit $T \to 0$ equivalently $\beta \to \infty$, only relaxation can occur, i.e.

$$\begin{aligned} \frac{1}{2\pi}\gamma(\omega \geq 0)\Big|_{\beta\to\infty} =& \sum_l g_l^2 \delta(\omega - \omega_l) \equiv \frac{1}{2\pi} J(\omega), \\ \frac{1}{2\pi}\gamma(\omega < 0)\Big|_{\beta\to\infty} =& 0, \end{aligned} \tag{62}$$

where we have defined the zero-temperature bath spectral function in the spirit of Caldeira and Leggett [A. O. Caldeira and A. J. Leggett, *Ann. Phys.* **149**, 374 (1983)]. We can recast the nonzero-temperature result in terms of the zero-temperature spectral function as follows

$$\begin{aligned} \gamma(\omega) =& \Theta(\omega)\left[1 + n_B(\omega)\right] J(\omega) + \Theta(-\omega) n_B(|\omega|) J(|\omega|) \\ =& \left[\Theta(\omega) + n_B(|\omega|)\right] J(|\omega|). \end{aligned} \tag{63}$$

With this form, we can check detailed balance for all $\omega > 0$

$$\gamma(\omega) = \gamma(-\omega)\frac{1 + n_B(\omega)}{n_B(\omega)} = e^{\beta\omega}\gamma(-\omega). \tag{64}$$

We can define the bilateral power spectral density for $B(t)$

$$S_{BB}(\omega) = \int_{-\infty}^{\infty} ds e^{i\omega s}\langle B^\dagger(s)B(0)\rangle = \int_{-\infty}^{\infty} ds e^{i\omega s}\langle B(s)B(0)\rangle = \gamma(\omega), \tag{65}$$

as can be shown by changing variable $s \to -s$ in the integral $\int_{-\infty}^{0}$.

IV. FINITE-TEMPERATURE STEADY-STATE

If the bath is in a thermal state with inverse temperature $\beta = 1/k_B T$, with k_B the Boltzmann constant, i.e. $\rho_B = e^{-\beta H_B}/\mathrm{tr}_B(e^{-\beta H_B})$ then we expect under the Lindblad master equation Eq. (44) that the system will have a unique Gibbs steady state

$$\rho_{\mathrm{th}} = \frac{\exp\left(-\beta H_S\right)}{\mathrm{tr}_S \exp\left(-\beta H_S\right)}. \tag{66}$$

That is, under certain conditions detailed below, for any initial state $\rho_S(t=0)$, we have a unique steady state

$$\rho_S(t) \longrightarrow \rho_{\mathrm{th}}, \qquad \text{for } t \longrightarrow +\infty. \tag{67}$$

For that to be the case, one imposes the KMS (Kubo-Martin-Schwinger) condition on the correlation functions of the bath, which reads

$$\left\langle B_\alpha^\dagger(t)B_\beta(0)\right\rangle = \left\langle B_\beta(0)B_\alpha^\dagger(t+i\beta)\right\rangle . \tag{68}$$

The KMS condition holds for the canonical ensemble equilibrium density matrix for the bath

$$\rho_B = \frac{\exp\left(-\beta H_B\right)}{\mathrm{tr}_B \exp\left(-\beta H_B\right)}. \tag{69}$$

The proof is immediate using Eq. (69), Eq. (34), and the cyclic property of the trace (with respect to bath degrees of freedom)

$$\begin{aligned}\left\langle B_\alpha^\dagger(t)B_\beta(0)\right\rangle &= \mathrm{tr}\{e^{-\beta H_B}e^{iH_Bt}B_\alpha^\dagger e^{-iH_Bt}B_\beta\} = \mathrm{tr}\{e^{-\beta H_B}e^{iH_Bt}B_\alpha^\dagger e^{-iH_Bt}e^{\beta H_B}e^{-\beta H_B}B_\beta\}\\ &= \mathrm{tr}\{e^{iH_B(t+i\beta)}B_\alpha^\dagger e^{-iH_B(t+i\beta)}e^{-\beta H_B}B_\beta\} = \mathrm{tr}\{B_\alpha^\dagger(t+i\beta)e^{-\beta H_B}B_\beta\}\\ &= \mathrm{tr}\{e^{-\beta H_B}B_\beta(0)B_\alpha^\dagger(t+i\beta)\} = \left\langle B_\beta(0)B_\alpha^\dagger(t+i\beta)\right\rangle .\end{aligned} \tag{70}$$

If the KMS condition Eq. (68) holds, then via Eq. (43) we have

$$\gamma_{\alpha\beta}(-\omega) = \exp(-\beta\omega)\gamma_{\beta\alpha}(\omega). \tag{71}$$

Moreover, using Eq. (27) we can prove, by analogy with Eq. (28), that

$$\begin{aligned}\rho_{\mathrm{th}}A_\alpha(\omega) &= e^{\beta\omega}A_\alpha(\omega)\rho_{\mathrm{th}},\\ \rho_{\mathrm{th}}A_\alpha^\dagger(\omega) &= e^{-\beta\omega}A_\alpha^\dagger(\omega)\rho_{\mathrm{th}}.\end{aligned} \tag{72}$$

To prove that Eq. (66) is a steady state of Eq. (44), simply plug in $\rho_S(t) \to \rho_{\mathrm{th}}$ on the right-hand side of Eq. (44). Since the system Hamiltonian and the Lamb shift commute according to Eq. (66), it follows that also $[H_{LS}, \rho_{\mathrm{th}}] = 0$, and hence the first term of the right-hand side of Eq. (44) vanishes. We just need to ensure that second term on the right-hand side, the dissipator term, also vanishes

$$\begin{aligned}
\mathcal{D}\left(\rho_{\rm th}\right) &= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(A_{\beta}(\omega)\rho_{\rm th}A_{\alpha}^{\dagger}(\omega)-\frac{1}{2}\left\{A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega),\rho_{\rm th}\right\}\right)\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)\rho_{\rm th}-\frac{1}{2}A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)\rho_{\rm th}-\frac{1}{2}\rho_{\rm th}A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)\right)\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)-\frac{1}{2}A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)-\frac{1}{2}A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)\right)\rho_{\rm th}\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)-A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)\right)\rho_{\rm th}\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)-A_{\alpha}^{\dagger}(\omega)A_{\beta}(\omega)\right)\rho_{\rm th}\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)\rho_{\rm th}-\sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(-\omega)A_{\alpha}(\omega)A_{\beta}^{\dagger}(\omega)\rho_{\rm th}\\
&= \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)e^{-\beta\omega}A_{\beta}(\omega)A_{\alpha}^{\dagger}(\omega)\rho_{\rm th}-\sum_{\omega}\sum_{\alpha,\beta}\gamma_{\beta\alpha}(\omega)e^{-\beta\omega}A_{\alpha}(\omega)A_{\beta}^{\dagger}(\omega)\rho_{\rm th}=0.
\end{aligned} \tag{73}$$

In the first to last row, we changed $\omega \to -\omega$ everywhere in the sum and used the property Eq. (30) of collapse operators, together with Eq. (71). This completes the proof that ρ_{th} is a steady state of Eq. (44). It is also trivially a steady state for the Lindblad master equation in the Schrödinger picture Eq. (51), since $[H_S, \rho_{th}] = 0$.

If the spectrum of the system Hamiltonian H_S is nondegenerate with energies ϵ_m corresponding to eigenkets $|n\rangle$, then we can define the population of state $|n\rangle$ as a function of time as follows

$$P(n,t) = \langle n\,|\rho_S(t)|\,n\rangle \tag{74}$$

Then immediately the Lindblad master equation implies the following rate equations

$$\frac{d}{dt}P(n,t) = \sum_m [W(n\mid m)P(m,t) - W(m\mid n)P(n,t)], \tag{75}$$

with

$$W(n\mid m) = \sum_{\alpha,\beta}\gamma_{\alpha\beta}\left(\varepsilon_m - \varepsilon_n\right)\langle m\,|A_\alpha|\,n\rangle\,\langle n\,|A_\beta|\,m\rangle\,. \tag{76}$$

This latter equation is consistent with Fermi's Golden Rule introduced above in the context of time-dependent perturbation theory. The form of the master equation presented in this subsection involves only the diagonal elements of the density matrix. The rates of transition satisfy detailed balance,

$$W(m \mid n) \exp\left(-\beta\varepsilon_n\right) = W(n \mid m) \exp\left(-\beta\varepsilon_m\right), \tag{77}$$

from which the steady state populations can be easily obtained to correspond to those of the Gibbs state introduced above in Eq. (67)

$$P_s(n) = \text{ const } \times \exp\left(-\beta\varepsilon_n\right). \tag{78}$$

V. EXAMPLES

In this section we will cover a couple of concrete examples of the more formal derivation above. Carmichael (see abstract for reference) opts to derive these equations first. We have taken above the approach of Breuer and Petruccione, and use the general form of the Lindblad master equation to then derive particular cases.

A. Simple harmonic oscillator

The hamiltonian of a simple harmonic oscillator is given by

$$\begin{aligned} H_S =& \hbar\omega_a a^\dagger a, \\ A =& (a + a^\dagger), \end{aligned} \tag{79}$$

with the second line containing the system operator that couples to a bath (bosonic or otherwise). The eigenspectrum of the Hamiltonian is given by

$$E_n = n\hbar\omega_a, |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle. \tag{80}$$

For convenience, set $\hbar = 1$ hereafter.

Recall the definition of the collapse operator at frequency ω

$$A(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi(\epsilon)(a + a^\dagger)\Pi(\epsilon'). \tag{81}$$

For the case of the harmonic oscillator, the summand is nonzero whenever the states at ϵ and ϵ' differ by one excitation only, that is $\epsilon' - \epsilon = \pm\omega_a$.

$$\begin{aligned} A(\omega_a) &= \sum_{n\geq 0} |n\rangle\langle n| (a + a^\dagger) |n+1\rangle\langle n+1| = \sum_{n\geq 0} \sqrt{n+1}\, |n\rangle\langle n+1| = a, \\ A(-\omega_a) &= \sum_{n\geq 0} |n+1\rangle\langle n+1| (a + a^\dagger) |n\rangle\langle n| = \sum_{n\geq 0} \sqrt{n+1}\, |n+1\rangle\langle n| = a^\dagger. \end{aligned} \tag{82}$$

Then the dissipator part of the Lindblad master equation reads

$$\begin{aligned} \mathcal{D}(\rho_S) =& \gamma(\omega_a) \left(A(\omega_a)\rho_S A(\omega_a)^\dagger - \frac{1}{2}\left\{ A(\omega_a)^\dagger A(\omega_a), \rho_S \right\} \right) \\ &+ \gamma(-\omega_a) \left(A(\omega_a)^\dagger \rho_S A(\omega_a) - \frac{1}{2}\left\{ A(\omega_a) A(\omega_a)^\dagger, \rho_S \right\} \right) \\ =& \gamma(\omega_a) \left(a\rho_S a^\dagger - \frac{1}{2}\left\{ a^\dagger a, \rho_S \right\} \right) + \gamma(-\omega_a) \left(a^\dagger \rho_S a - \frac{1}{2}\left\{ a a^\dagger, \rho_S \right\} \right) \end{aligned} \tag{83}$$

The Lamb shift contribution is a frequency shift of the oscillator

$$\begin{aligned} H_{LS} =& \sum_{\omega} \sum_{\alpha,\beta} S_{\alpha\beta}(\omega) A_\alpha^\dagger(\omega) A_\beta(\omega) \\ =& S(\omega_a) a^\dagger a + S(-\omega_a) a a^\dagger \\ =& [S(\omega_a) + S(-\omega_a)] a^\dagger a + \text{const.} \end{aligned} \tag{84}$$

Putting these two together, the full master equation in the lab frame is

$$\frac{d}{dt}\rho_S(t) = -i\left[\omega_a' a^\dagger a, \rho_S(t)\right] + \gamma(\omega_a)\mathcal{D}[a]\rho_S(t) + \gamma(-\omega_a)\mathcal{D}[a^\dagger]\rho_S(t), \tag{85}$$

where we have included the Lamb shift in $\omega_a' = \omega_a + S(\omega_a) + S(-\omega_a)$, and introduced the dissipator superoperator

$$\mathcal{D}[O]\rho = O\rho O^\dagger - \frac{1}{2}\{O^\dagger O, \rho\}. \tag{86}$$

B. Spin-1/2

The treatment of the spin-1/2 follows along the same lines, with system Hamiltonian and operators coupling to baths

$$\begin{aligned} H_S &= \frac{1}{2}\hbar\omega_{01}\sigma_z, \\ A_1 &= \sigma_x = \sigma_+ + \sigma_-, \\ A_\varphi &= \sigma_z. \end{aligned} \tag{87}$$

We have set ω_{01} is the positive transition frequency of the qubit. We also consider that the system couples to two different baths, one for spin relaxation via the operator A_1 and one primarily responsible for dephasing via the operator A_φ. To be concrete, we may imagine the following system-bath Hamiltonian in the laboratory frame

$$H_I = \sum_l g_{1,l} A_1 \otimes (b_{1,l}^\dagger + b_{1,l}) + \sum_l g_{\varphi,l} A_\varphi \otimes (b_{\varphi,l}^\dagger + b_{\varphi,l}), \tag{88}$$

corresponding to two bosonic baths each described by $H_{B,\alpha} = \sum_l \omega_{\alpha,l} b_{\alpha,l}^\dagger b_{\alpha,l}$ with $\alpha = 1, \varphi$. However the content of the bath need not be specified, and more generally we could have in the laboratory frame

$$H_I = A_\alpha \otimes B_\alpha, \tag{89}$$

as specified before, with $\alpha = 1, \varphi$.

We are interested in the possible transitions in the system spectrum, and these are given by the nonzero collapse operators

$$\begin{aligned} A_1(\omega_{01}) =& \sum_{\epsilon' - \epsilon = \omega_{01}} \Pi(\epsilon)(\sigma_+ + \sigma_-)\Pi(\epsilon') = \left|\downarrow\right\rangle \left\langle\uparrow\right| = \sigma_-, \\ A_1(-\omega_{01}) =& \sigma_+, \\ A_\varphi(0) =& \sum_{\epsilon' - \epsilon = 0} \Pi(\epsilon)\sigma_z\Pi(\epsilon') = \sigma_z, \end{aligned} \tag{90}$$

With these, the Lindblad master equation (neglecting the Lamb shift) is

$$\frac{d}{dt}\rho_S(t) = -i\left[\frac{1}{2}\omega_{01}\sigma_z, \rho_S(t)\right] + \gamma_1(\omega_{01})\mathcal{D}[\sigma_-]\rho_S(t) + \gamma_1(-\omega_{01})\mathcal{D}[\sigma_+]\rho_S(t) + \gamma_\varphi(0)\mathcal{D}[\sigma_z]\rho_S(t), \tag{91}$$

assuming no cross-correlation between the two baths, where $\gamma_\alpha(\omega) = S_{B_\alpha B_\alpha}(\omega)$, the bilateral power spectral density of the bath corresponding to $\alpha = 1, \varphi$, as defined above in Eq. (65). Notice how, by virtue of the operators chosen to each bath, there are two separate effects. The fluctuations of the operators in the bath φ can only act upon the system via a dissipator $\mathcal{D}[\sigma_z]$, and vice-versa, the fluctuations of operators in bath 1 can only induce relaxation or excitation of the two-level system. We will fix these notions in the problem set as we show the correspondence between Eq. (91) and the Bloch equations for a spin-1/2.