Mathematical methods for modeling and control of open quantum systems[1]

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[1] Lecture-notes, slides and Matlab simulation scripts available at: http://cas.ensmp.fr/~rouchon/LIASFMA/index.html

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Outline

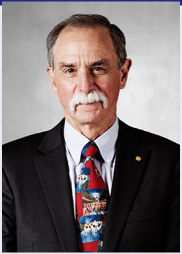
Second quantum revolution: Controlling quantum degrees of freedom

Some applications

- Nuclear Magnetic Resonance (NMR) applications;
- Quantum chemical synthesis;
- High resolution measurement devices (e.g. atomic/optic clocks);
- Quantum communication (BB84, . . .);
- Quantum computation and simulation.

Physics Nobel prize 2012

Serge Haroche



David J. Wineland

Nobel prize: ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems.

Outline of the lectures

Nov. 30 Quantum mechanics from scratch: two-level systems (qubits,spins), harmonic oscillators (modes, springs), the Haroche photon box.

Dec. 2 Dynamical models: Markov chains and Kraus maps (discrete time), Lindblad master equation and stochastic master equations (continuous time). Two key examples: quantum non demolition measurement of photons (discrete time), homodyne measurement of a qubit (continuous-time).

Dec. 7 Averaging (rotating wave approximation) and singular perturbations (adiabatic elimination): resonant control of qubits, dispersive and resonant coupling between qubits and harmonic oscillators, adiabatic elimination of a low-quality harmonic oscillator.

Dec. 9 Stabilization with a quantum controller: cat-qubit and how a low-quality harmonic oscillator can stabilize via coherent coupling the quantum information stored in a high-quality harmonic oscillator.

Reference books

1. Cohen-Tannoudji, C.; Diu, B. & Laloë, F.: Mécanique Quantique Hermann, Paris, 1977, I& II (*quantum physics: a well known and tutorial textbook*)
2. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006. (*quantum physics: spin/spring systems, decoherence, Schrödinger cats, entanglement.*)
3. C. Gardiner, P. Zoller: The Quantum World of Ultra-Cold Atoms and Light I& II. Imperial College Press, 2009. (*quantum physics, measurement and control*)
4. Barnett, S. M. & Radmore, P. M.: Methods in Theoretical Quantum Optics Oxford University Press, 2003. (*mathematical physics: many useful operator formulae for spin/spring systems*)
5. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976. (*mathematical physics: functional analysis aspects when the Hilbert space is of infinite dimension*)
6. Gardiner, C. W.: Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences [3rd ed], Springer, 2004. (*tutorial introduction to probability, Markov processes, stochastic differential equations and Ito calculus.*)
7. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000. (*tutorial introduction with a computer science and communication view point*)

Models of open quantum systems are based on three features[5]

1. Schrödinger: $\hbar = 1$, wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

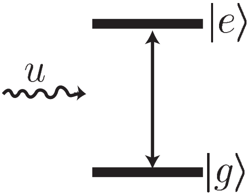
2. Entanglement and tensor product for composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

3. Randomness and irreversibility induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho \boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y \rho \boldsymbol{P}_y}{\text{Tr}\left(\rho \boldsymbol{P}_y\right)}$$

[5] S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006.

Outline



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and obey to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$, i.e. energy in frequency unit) :

$$\imath \frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle = \left(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|\right)|\psi\rangle$$

where $\boldsymbol{H}_0$ is the Hamiltonian, a Hermitian operator $\boldsymbol{H}_0^\dagger = \boldsymbol{H}_0$. Energy is defined up to a constant: $\boldsymbol{H}_0$ and $\boldsymbol{H}_0 + \varpi(t)\boldsymbol{I}$ ($\varpi(t) \in \mathbb{R}$ arbitrary) are attached to the same physical system. If $|\psi\rangle$ satisfies $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)}|\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\frac{d}{dt}|\chi\rangle = (\boldsymbol{H}_0 + \varpi\boldsymbol{I})|\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta}|\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time.

The controlled 2-level system

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{\text{eg}} = \omega_e - \omega_g$

The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{\text{eg}}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{\text{eg}} t}{2}} |g\rangle + \psi_{e0} e^{\frac{-i\omega_{\text{eg}} t}{2}} |e\rangle.$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the coherent evolution the controlled Hamiltonian

$$\boldsymbol{H}(t) = \frac{\omega_{\text{eg}}}{2}\boldsymbol{\sigma_z} + \frac{u(t)}{2}\boldsymbol{\sigma_x} = \frac{\omega_{\text{eg}}}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u(t)}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

The controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{\text{eg}}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

The 3 Pauli Matrices[6]

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

[6] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

Pauli matrices and some formula

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\boldsymbol{\sigma_x}^2 = \boldsymbol{I}, \quad \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}, \quad [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ \text{circular permutation } \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\boldsymbol{\sigma_x}} = \cos\theta + i\sin\theta\boldsymbol{\sigma_x}$ (idem for $\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{I} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{\sigma_z}\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\boldsymbol{\sigma_\alpha} e^{i\theta\boldsymbol{\sigma_\beta}} = e^{-i\theta\boldsymbol{\sigma_\beta}}\boldsymbol{\sigma_\alpha}, \qquad \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{-1} = \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{\dagger} = e^{-i\theta\boldsymbol{\sigma_\alpha}}.$$

and also

$$e^{-\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} e^{\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}} = e^{-i\theta\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} = \boldsymbol{\sigma_\beta} e^{i\theta\boldsymbol{\sigma_\alpha}}$$

We start from $|\psi\rangle$ that obeys $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle$. We consider the orthogonal projector on $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$, called density operator. Then ρ is an Hermitian operator ≥ 0, that satisfies $\text{Tr}(\rho) = 1$, $\rho^2 = \rho$ and obeys to the Liouville equation:

$$\frac{d}{dt}\rho = -i[\boldsymbol{H}, \rho].$$

For a two level system $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and

$$\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$$

where $(x, y, z) = (2\Re(\psi_g\psi_e^*), 2\Im(\psi_g\psi_e^*), |\psi_e|^2 - |\psi_g|^2) \in \mathbb{R}^3$ represent a vector $\vec{M}$, the Bloch vector, that evolves on the unite sphere of $\mathbb{R}^3$, $\mathbb{S}^2$ called the the Bloch Sphere since $\text{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 = 1$.
The Liouville equation with $\boldsymbol{H} = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z} + \frac{u}{2}\boldsymbol{\sigma_x}$ reads

$$\frac{d}{dt}\vec{M} = (u\vec{i} + \omega_{eg}\vec{k}) \times \vec{M}.$$

Exercise

Consider $\boldsymbol{H} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/2$ with $(u, v, w) \in \mathbb{R}^3$.

1. For (u, v, w) constant and non zero, compute the solutions of

$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\boldsymbol{\sigma} = (u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y} + w\boldsymbol{\sigma_z})/\sqrt{u^2 + v^2 + w^2}$ and $\omega = \sqrt{u^2 + v^2 + w^2}$. Indication: use the fact that $\boldsymbol{\sigma}^2 = \boldsymbol{I}$.

2. Assume that, (u, v, w) depends on t according to $(u, v, w)(t) = \omega(t)(\bar{u}, \bar{v}, \bar{w})$ with $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3/\{0\}$ constant of length 1. Compute the solutions of

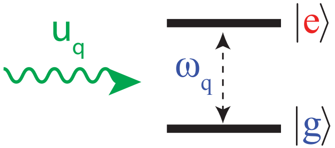
$$\frac{d}{dt}|\psi\rangle = -i\boldsymbol{H}(t)|\psi\rangle, \quad \frac{d}{dt}\boldsymbol{U} = -i\boldsymbol{H}(t)\boldsymbol{U} \text{ with } \boldsymbol{U}_0 = \boldsymbol{I}$$

   in term of $|\psi\rangle_0$, $\overline{\boldsymbol{\sigma}} = \bar{u}\boldsymbol{\sigma_x} + \bar{v}\boldsymbol{\sigma_y} + \bar{w}\boldsymbol{\sigma_z}$ and $\theta(t) = \int_0^t \omega$.

3. Explain why (u, v, w) colinear to the constant vector $(\bar{u}, \bar{v}, \bar{w})$ is crucial, for the computations in previous question.

Summary: 2-level system, i.e. a qubit (spin-half system)

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ \psi_g |g\rangle + \psi_e |e\rangle,\ \psi_g, \psi_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho, \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|;$
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|;$
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|;$
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ldots$



- Hamiltonian: $\boldsymbol{H}_M = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \mid (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

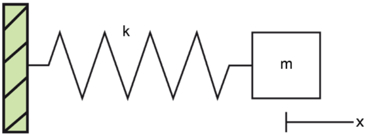
Outline

Harmonic oscillator

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

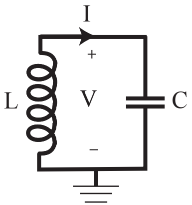
$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Mechanical oscillator



Frictionless spring: $\frac{d^2}{dt^2}x = -\frac{k}{m}x$.

Electrical oscillator:



LC oscillator:

$$\frac{d}{dt}I = \frac{V}{L}, \; \frac{d}{dt}V = -\frac{I}{C}, \quad \left(\frac{d^2}{dt^2}I = -\frac{1}{LC}I\right).$$

Quantum regime

$k_B T \ll \hbar\omega$: typically for the photon box experiment in these lectures, $\omega = 51 GHz$ and $T = 0.8K$.

Harmonic oscillator[7]: quantization and correspondence principle

$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \frac{\omega}{2}(\boldsymbol{P}^2 + \boldsymbol{X}^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where $\boldsymbol{H}$ results from $\mathbb{H}$ by replacing x by position operator $\boldsymbol{X}$ and p by momentum operator $\boldsymbol{P} = -i\frac{\partial}{\partial x}$. $\boldsymbol{H}$ is a Hermitian operator on $L^2(\mathbb{R},\mathbb{C})$, with its domain to be given.

PDE model: $i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$

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[7] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Harmonic oscillator: annihilation and creation operators

Average position $\langle \boldsymbol{X} \rangle_t = \langle \psi | \boldsymbol{X} | \psi \rangle$ and momentum $\langle \boldsymbol{P} \rangle_t = \langle \psi | \boldsymbol{P} | \psi \rangle$:

$$\langle \boldsymbol{X} \rangle_t = \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle \boldsymbol{P} \rangle_t = -i \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation $\boldsymbol{a}$ and creation operators $\boldsymbol{a}^\dagger$ (domains to be given):

$$\boldsymbol{a} = \tfrac{1}{\sqrt{2}}(\boldsymbol{X} + i\boldsymbol{P}) = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad \boldsymbol{a}^\dagger = \tfrac{1}{\sqrt{2}}(\boldsymbol{X} - i\boldsymbol{P}) = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[\boldsymbol{X}, \boldsymbol{P}] = i\boldsymbol{I}, \quad [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}, \quad \boldsymbol{H} = \frac{\omega}{2}(\boldsymbol{P}^2 + \boldsymbol{X}^2) = \omega \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2} \right).$$

Set $\boldsymbol{X}_\theta = \tfrac{1}{\sqrt{2}} \left(e^{-i\theta} \boldsymbol{a} + e^{i\theta} \boldsymbol{a}^\dagger \right)$ for any angle θ:

$$\left[\boldsymbol{X}_\theta, \boldsymbol{X}_{\theta + \frac{\pi}{2}} \right] = i\boldsymbol{I}.$$

Harmonic oscillator: spectral decomposition and Fock states

Spectrum of Hamiltonian $\boldsymbol{H} = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$:

$$E_n = \omega(n+\frac{1}{2}), \;\; \psi_n(x) = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \;\; H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Spectral decomposition of $\boldsymbol{a}^\dagger \boldsymbol{a}$ using $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$:

- If $|\psi\rangle$ is an eigenstate associated to eigenvalue λ, $\boldsymbol{a}|\psi\rangle$ and $\boldsymbol{a}^\dagger|\psi\rangle$ are also eigenstates associated to $\lambda - 1$ and $\lambda + 1$.
- $\boldsymbol{a}^\dagger \boldsymbol{a}$ is semi-definite positive.
- The ground state $|\psi_0\rangle$ is necessarily associated to eigenvalue 0 and is given by the Gaussian function $\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$.

Harmonic oscillator: spectral decomposition and Fock states

$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$: spectrum of $\boldsymbol{a}^\dagger \boldsymbol{a}$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photons (phonons): the eigenstate of $\boldsymbol{a}^\dagger \boldsymbol{a}$ associated to the eigenvalue n ($|n\rangle \sim \psi_n(x)$):

$$\boldsymbol{a}^\dagger \boldsymbol{a}|n\rangle = n|n\rangle, \quad \boldsymbol{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ is called 0-photon state or vacuum state.

The operator $\boldsymbol{a}$ (resp. $\boldsymbol{a}^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Hilbert space of quantum system: $\mathcal{H} = \{\sum_n c_n|n\rangle \mid (c_n) \in l^2(\mathbb{C})\} \sim L^2(\mathbb{R}, \mathbb{C})$.
Domain of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^1(\mathbb{C})\}$.
Domain of $\boldsymbol{H}$ ot $\boldsymbol{a}^\dagger \boldsymbol{a}$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^2(\mathbb{C})\}$.

$$h^k(\mathbb{C}) = \{(c_n) \in l^2(\mathbb{C}) \mid \sum n^k |c_n|^2 < \infty\}, \qquad k = 1, 2.$$

Harmonic oscillator: displacement operator

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$, ($\mathbb{H} = \frac{\omega}{2}(p^2 + x^2) + \sqrt{2}ux$)

$$\boldsymbol{H} = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{\mathbf{I}}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger).$$

The associated controlled PDE

$$i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \left(\frac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator $\boldsymbol{D}_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a}} = e^{\sqrt{2}i\Im\alpha \boldsymbol{X} - \sqrt{2}i\Re\alpha \boldsymbol{P}}$$

From Baker-Campbell Hausdorf formula, for all operators $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$e^{\boldsymbol{A}}\boldsymbol{B}e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \frac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \frac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots$$

we get the Glauber formula[8] when $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$:

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\, e^{\boldsymbol{B}}\, e^{-\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]}.$$

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[8] Take s derivative of $e^{s(\boldsymbol{A}+\boldsymbol{B})}$ and of $e^{s\boldsymbol{A}}\, e^{s\boldsymbol{B}}\, e^{-\frac{s^2}{2}[\boldsymbol{A},\boldsymbol{B}]}$.

With $\boldsymbol{A} = \alpha \boldsymbol{a}^\dagger$ and $\boldsymbol{B} = -\alpha^* \boldsymbol{a}$, Glauber formula gives:

$$\boldsymbol{D}_\alpha = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} = e^{+\frac{|\alpha|^2}{2}} e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger}$$

$$\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}_{-\alpha} \boldsymbol{a}^\dagger \boldsymbol{D}_\alpha = \boldsymbol{a}^\dagger + \alpha^* \boldsymbol{I}.$$

With $\boldsymbol{A} = \sqrt{2} i \Im \alpha \boldsymbol{X} \sim i\sqrt{2} \Im \alpha x$ and $\boldsymbol{B} = -\sqrt{2} \imath \Re \alpha \boldsymbol{P} \sim -\sqrt{2} \Re \alpha \frac{\partial}{\partial x}$, Glauber formula gives[9]:

$$\boldsymbol{D}_\alpha = e^{-i \Re \alpha \Im \alpha} e^{i\sqrt{2} \Im \alpha x} e^{-\sqrt{2} \Re \alpha \frac{\partial}{\partial x}}$$

$$(\boldsymbol{D}_\alpha | \psi \rangle)_{x,t} = e^{-i \Re \alpha \Im \alpha} e^{i \sqrt{2} \Im \alpha x} \psi(x - \sqrt{2} \Re \alpha, t)$$

Exercise: Prove that, for any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have

$$\boldsymbol{D}_{\alpha+\beta} = e^{\frac{\alpha^* \beta - \alpha \beta^*}{2}} \boldsymbol{D}_\alpha \boldsymbol{D}_\beta$$

$$\boldsymbol{D}_{\alpha+\epsilon} \boldsymbol{D}_{-\alpha} = \left(1 + \frac{\alpha \epsilon^* - \alpha^* \epsilon}{2}\right) \boldsymbol{I} + \epsilon \boldsymbol{a}^\dagger - \epsilon^* \boldsymbol{a} + \boldsymbol{O}(|\epsilon|^2)$$

$$\left(\frac{d}{dt} \boldsymbol{D}_\alpha\right) \boldsymbol{D}_{-\alpha} = \left(\frac{\alpha \frac{d}{dt} \alpha^* - \alpha^* \frac{d}{dt} \alpha}{2}\right) \boldsymbol{I} + \left(\frac{d}{dt} \alpha\right) \boldsymbol{a}^\dagger - \left(\frac{d}{dt} \alpha^*\right) \boldsymbol{a}.$$

[9] Remember that $e^{r \partial / \partial x}(f(x)) \equiv f(x + r)$.

Harmonic oscillator: lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right) + u(\boldsymbol{a}+\boldsymbol{a}^\dagger)\right)|\psi\rangle$. Set $\langle\boldsymbol{a}\rangle = \langle\psi|\boldsymbol{a}|\psi\rangle$. Then

$$\frac{d}{dt}\langle\boldsymbol{a}\rangle = -i\omega\langle\boldsymbol{a}\rangle - iu.$$

From $\boldsymbol{a} = \frac{\boldsymbol{X}+i\boldsymbol{P}}{\sqrt{2}}$, we have $\langle\boldsymbol{a}\rangle = \frac{\langle\boldsymbol{X}\rangle + i\langle\boldsymbol{P}\rangle}{\sqrt{2}}$ where $\langle\boldsymbol{X}\rangle = \langle\psi|\boldsymbol{X}|\psi\rangle \in \mathbb{R}$ and $\langle\boldsymbol{P}\rangle = \langle\psi|\boldsymbol{P}|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle\boldsymbol{X}\rangle = \omega\langle\boldsymbol{P}\rangle, \quad \frac{d}{dt}\langle\boldsymbol{P}\rangle = -\omega\langle\boldsymbol{X}\rangle - \sqrt{2}u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t}\boldsymbol{D}_{\langle\boldsymbol{a}\rangle_t}|\chi\rangle$ with

$$\theta_t = \int_0^t \left(\omega|\langle\boldsymbol{a}\rangle|^2 + u\Re(\langle\boldsymbol{a}\rangle)\right), \quad D_{\langle\boldsymbol{a}\rangle_t} = e^{\langle\boldsymbol{a}\rangle_t\boldsymbol{a}^\dagger - \langle\boldsymbol{a}\rangle_t^*\boldsymbol{a}},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle\boldsymbol{a}\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

Coherent states

$$|\alpha\rangle = \boldsymbol{D}_\alpha |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigenstate of $\boldsymbol{a}$:

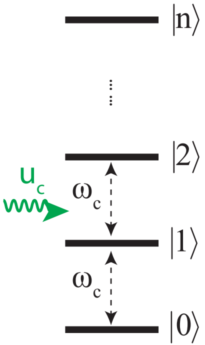
$$\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

A widely known result in quantum optics[10]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum.

[10] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.

Summary for the quantum harmonic oscillator

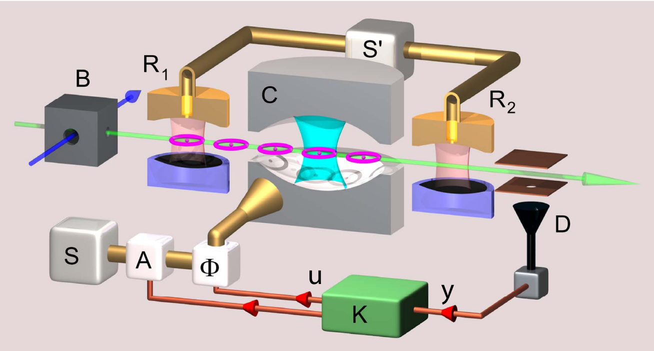
- Hilbert space:
  $\mathcal{H} = \left\{ \sum_{n\geq 0} \psi_n |n\rangle, \ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathbb{D} = \{\rho \in \mathcal{L}(\mathcal{H}), \rho^\dagger = \rho, \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\,|\boldsymbol{n}\text{-}\boldsymbol{1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \frac{\boldsymbol{X} + i\boldsymbol{P}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}$.
- Hamiltonian: $\boldsymbol{H}/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).



- Quasi-classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}: \ |\alpha\rangle = \sum_{n\geq 0} \left(e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) |n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath \sqrt{2} x \Im \alpha} e^{-\frac{(x - \sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

Outline

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche (Nobel Prize 2012), J.M.Raimond and M. Brune.[11]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).

Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

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[11] Courtesy of Igor Dotsenko. **Sampling period $\Delta t \approx 80 \mu s$.**

- System S corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \mathcal{H}_c = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \;\middle|\; (c_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity

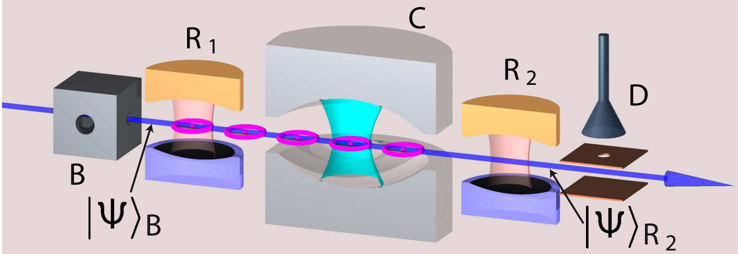
- Meter M is a qu-bit, a 2-level system (idem 1/2 spin system) : $\mathcal{H}_M = \mathcal{H}_a = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$

- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M = \mathcal{H}_c \otimes \mathcal{H}_a$**:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} c_{ng}|n\rangle \otimes |g\rangle + c_{ne}|n\rangle \otimes |e\rangle, \quad c_{ne}, c_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n \in \mathbb{N}}$.

The Markov model (1)



- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

The Markov model (2)

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \left\langle \psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi \right\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle.$$

Markov process: $|\psi_k\rangle \equiv |\psi\rangle_{t=k\Delta t}$, $k \in \mathbb{N}$, Δt sampling period,

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g|\psi_k\rangle}{\sqrt{\left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k \right\rangle}} & \text{with } y_k = g \text{, probability } \mathbb{P}_g = \left\langle \psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e|\psi_k\rangle}{\sqrt{\left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k \right\rangle}} & \text{with } y_k = e \text{, probability } \mathbb{P}_e = \left\langle \psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k \right\rangle. \end{cases}$$

Markov process with detection inefficiency

- With pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we have

$$\boldsymbol{\rho}_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$.

- Detection efficiency: the probability to detect the atom is $\eta \in [0, 1]$. Three possible outcomes for y: $y = g$ if detection in g, $y = e$ if detection in e and $y = 0$ if no detection.

The only possible update is based on ρ: expectation $\boldsymbol{\rho}_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the outcome $y \in \{g, e, 0\}$.

$$\boldsymbol{\rho}_+ = \begin{cases} \frac{\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger}{\text{Tr}(\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g)} & \text{if } y = g, \text{ probability } \eta \, \text{Tr}\left(\boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g\right) \\ \frac{\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger}{\text{Tr}(\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e)} & \text{if } y = e, \text{ probability } \eta \, \text{Tr}\left(\boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e\right) \\ \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta \end{cases}$$

For $\eta = 0$: $\boldsymbol{\rho}_+ = \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger = \mathbb{K}(\boldsymbol{\rho}) = \mathbb{E}\left(\boldsymbol{\rho}_+ \mid \boldsymbol{\rho}\right)$ defines a Kraus map.

- With pure state $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, we have

$$\boldsymbol{\rho}_+ = |\psi_+\rangle\langle\psi_+| = \frac{1}{\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)} \boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger$$

  when the atom collapses in $\mu = g, e$ with proba. $\text{Tr}\left(\boldsymbol{M}_\mu \boldsymbol{\rho} \boldsymbol{M}_\mu^\dagger\right)$.

- Detection error rates: $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$ the probability of erroneous assignation to e when the atom collapses in g; $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ (given by the contrast of the Ramsey fringes).

Bayesian law: expectation $\boldsymbol{\rho}_+$ of $|\psi_+\rangle\langle\psi_+|$ knowing $\boldsymbol{\rho}$ and the imperfect detection y.

$$\boldsymbol{\rho}_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger}{\text{Tr}\left((1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)} \text{ if } y = g, \text{ prob. Tr}\left((1-\eta_g)\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \eta_e \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger}{\text{Tr}\left(\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right)} \text{ if } y = e, \text{ prob. Tr}\left(\eta_g\boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + (1-\eta_e) \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger\right). \end{cases}$$

$\boldsymbol{\rho}_+$ does not remain pure: the quantum state $\boldsymbol{\rho}_+$ becomes a mixed state; $|\psi_+\rangle$ becomes physically irrelevant.

We get

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)}, & \text{with prob. } \mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} & \text{with prob. } \mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

Key point:

$$\mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+\eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right) \text{ and } \mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger+(1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)$$

are the probabilities to detect $y = g$ and e, knowing ρ.
Generalization by merging a Kraus map $\boldsymbol{K}(\rho) = \sum_\mu \boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger$ where $\sum_\mu \boldsymbol{M}_\mu^\dagger\boldsymbol{M}_\mu = \boldsymbol{I}$ with a left stochastic matrix $(\eta_{\mu',\mu})$:

$$\rho_+ = \frac{\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)} \quad \text{when we detect } y = \mu'.$$

The probability to detect $y = \mu'$ knowing ρ is $\mathrm{Tr}\left(\sum_\mu \eta_{y,\mu}\boldsymbol{M}_\mu\rho\boldsymbol{M}_\mu^\dagger\right)$.

Photon-box full model: 6×21 left stochastic matrix $(\eta_{\mu',\mu})$

$$\rho_{k+1} = \frac{1}{\mathrm{Tr}\left(\sum_\mu \eta_{y_k,\mu} M_\mu \rho_k M_\mu^\dagger\right)} \left(\sum_\mu \eta_{y_k,\mu} M_\mu \rho_k M_\mu^\dagger\right) \text{ where}$$

- we have a total of $m = 3 \times 7 = 21$ Kraus operators M_μ. The "jumps" are labeled by $\mu = (\mu^a, \mu^c)$ with $\mu^a \in \{no, g, e, gg, ge, eg, ee\}$ labeling atom related jumps and $\mu^c \in \{o, +, -\}$ cavity decoherence jumps.
- we have only $m' = 6$ real detection possibilities $y = \mu' \in \{no, g, e, gg, ge, ee\}$ corresponding respectively to no detection, a single detection in g, a single detection in e, a double detection both in g, a double detection one in g and the other in e, and a double detection both in e.

$\mu' \setminus \mu$	(no, μ^c)	(g, μ^c)	(e, μ^c)	(gg, μ^c)	(ee, μ^c)	(ge, μ^c) (eg, μ^c)
no	1	$1-\epsilon_d$	$1-\epsilon_d$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$	$(1-\epsilon_d)^2$
g	0	$\epsilon_d(1-\eta_g)$	$\epsilon_d\eta_e$	$2\epsilon_d(1-\epsilon_d)(1-\eta_g)$	$2\epsilon_d(1-\epsilon_d)\eta_e$	$\epsilon_d(1-\epsilon_d)(1-\eta_g+\eta_e)$
e	0	$\epsilon_d\eta_g$	$\epsilon_d(1-\eta_e)$	$2\epsilon_d(1-\epsilon_d)\eta_g$	$2\epsilon_d(1-\epsilon_d)(1-\eta_e)$	$\epsilon_d(1-\epsilon_d)(1-\eta_e+\eta_g)$
gg	0	0	0	$\epsilon_d^2(1-\eta_g)^2$	$\epsilon_d^2\eta_e^2$	$\epsilon_d^2\eta_e(1-\eta_g)$
ge	0	0	0	$2\epsilon_d^2\eta_g(1-\eta_g)$	$2\epsilon_d^2\eta_e(1-\eta_e)$	$\epsilon_d^2((1-\eta_g)(1-\eta_e)+\eta_g\eta_e)$
ee	0	0	0	$\epsilon_d^2\eta_g^2$	$\epsilon_d^2(1-\eta_e)^2$	$\epsilon_d^2\eta_g(1-\eta_e)$