Quantum Control

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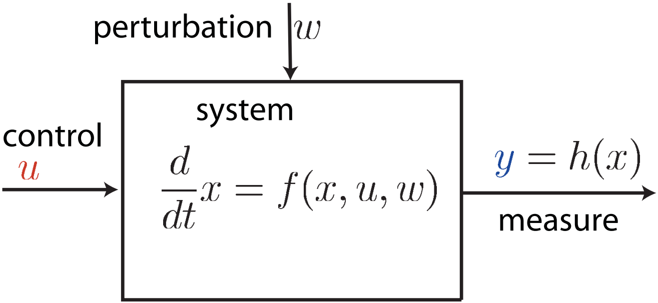
Slides and exercises on the web page
http://cas.ensmp.fr/~rouchon/index.html

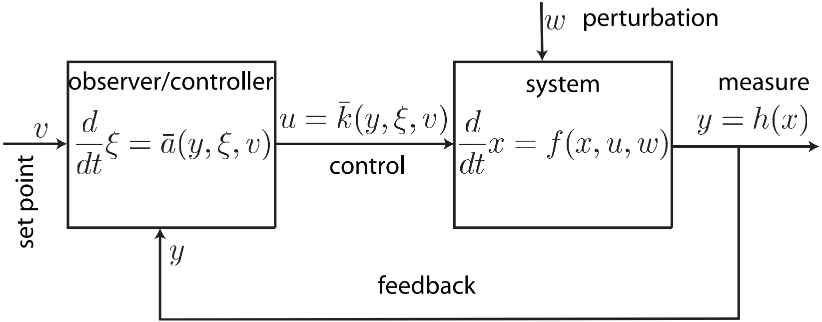
Model of classical systems



For the harmonic oscillator of pulsation ω with measured position y, controlled by the force u and subject to an additional unknown force w.

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad y = x_1$$

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = -\omega^2 x_1 + u + w$$



Proportional Integral Derivative (PID) for $\frac{d^2}{dt^2}y = -\omega^2 y + u + w$ with the set point $v = y^c$

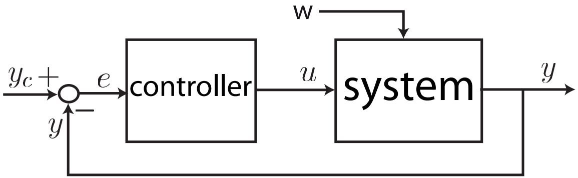
$$u = -K_p(y - y^c) - K_d \frac{d}{dt}(y - y^c) - K_{\text{int}} \int (y - y^c)$$

with the positive gains $(K_p, K_d, K_{\text{int}})$ tuned as follows ($0 < \Omega_0 \sim \omega$, $0 < \xi \sim 1$, $0 < \epsilon \ll 1$:

$$K_p = \Omega_0^2, \quad K_d = 2\xi\sqrt{\omega^2 + \Omega_0^2}, \quad , K_{\text{int}} = \epsilon(\omega^2 + \Omega_0^2)^{3/2}.$$

Quantum feedback: the back-action of the measurement.

A typical stabilizing feedback-loop for a classical system



Two kinds of stabilizing feedbacks for quantum systems

1. Measurement-based feedback: **controller is classical**; measurement back-action on the system $\mathcal{S}$ is stochastic (collapse of the wave-packet); the measured output y is a classical signal; the control input u is a classical variable appearing in some controlled Schrödinger equation; $u(t)$ depends on the past measurements $y(\tau)$, $\tau \leq t$.

2. Coherent/autonomous feedback and reservoir engineering: the system $\mathcal{S}$ is coupled to **the controller, another quantum system**; the composite system, $\mathcal{H}_S \otimes \mathcal{H}_{controller}$, is an open-quantum system relaxing to some target (separable) state.

Several reference books

1. Cohen-Tannoudji, C.; Diu, B. & Laloë, F.: Mécanique Quantique Hermann, Paris, 1977, I& II (*quantum physics: a well known and tutorial textbook*)
2. S. Haroche, J.M. Raimond: Exploring the Quantum: Atoms, Cavities and Photons. Oxford University Press, 2006. (*quantum physics: spin/spring systems, decoherence, Schrödinger cats, entanglement.*)
   See also lectures at Collège de France:
   `http://www.cqed.org/college/collegeparis.html`
3. H. Wiseman, G. Milburn: Quantum Measurement and Control. Cambridge University Press, 2009. (*quantum physics and control: estimation and feedback*)
4. C. Gardiner, P. Zoller: The Quantum World of Ultra-Cold Atoms and Light: Book I and Book II, Imperial College Press, London., 2014 and 2015 (*a full suite of theoretical techniques needed for quantum technologies*)
5. Barnett, S. M. & Radmore, P. M.: Methods in Theoretical Quantum Optics Oxford University Press, 2003. (*mathematical physics: many useful operator formulae for spin/spring systems*)
6. E. Davies: Quantum Theory of Open Systems. Academic Press, 1976. (*mathematical physics: functional analysis aspects when the Hilbert space is of infinite dimension*)
7. Gardiner, C. W.: Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences [3rd ed], Springer, 2004. (*tutorial introduction to probability, Markov processes, stochastic differential equations and Ito calculus.*)
8. M. Nielsen, I. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000. (*tutorial introduction with a computer science and communication view point*)

Outline of the lectures and exercises

Monday: feedback for classical and for quantum systems; the first experimental realization of a quantum-state feedback (LKB photon box); the quantum harmonic oscillator; three quantum features **Schrödinger deterministic evolution; stochastic collapse of the wave packet; tensor product for composite systems**; entanglement between the probe-qubit and the photons; qubit-measurement back-action on the photons; derivation of the discrete-time Markov model in the ideal case;
Matlab simulations with the wave function; how to cope with imperfections such as detection efficiency and detection error; passage to the density operator formulation; Matlab simulations with the density operator; discussion on the asymptotic behavior.

Tuesday: adding measurement imperfections; decoherence as unread fictitious measurements; creation annihilation operators; discrete-time Markov chain; quantum trajectories; QND measurement of photons; convergence analysis based on martingales and super-martingales.
Realistic Matlab simulation in open-loop including cavity decoherence and thermal photon; Lyapunov stabilization of photon-number state via a measurement-based feedback; closed-loop simulation in the ideal and realistic cases.

Outline of the lectures and exercises (end)

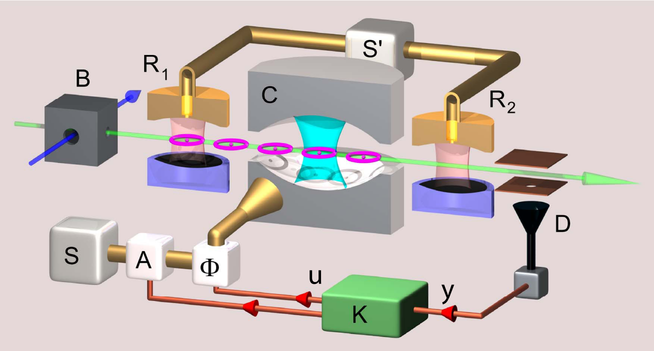
Wednesday: The structure of discrete-time models of open-quantum systems: hidden Markov chain; Kraus maps; quantum channels.
The structure of continuous-time models: stochastic master equation in the diffusive case; Ito calculus for dummies; infinitesimal Kraus maps and Lindblad master equations.

Thursday: half-spin system or qubit; Pauli operators; Bloch sphere representation of the density operator; QND measurement of a super-conducting qubit via homodyne or heterodyne measurements; the stochastic master equation; convergence analysis based on martingales. Decoherence attached to fluorescence and dephasing.
Simulation of the QND measurement of a super-conducting qubit; feedback stabilization via measurement-based feedback

Friday **(to be discussed with the participants)**: coherent (autonomous feedback) and reservoir engineering: the controller is another open quantum system highly dissipative; dispersive and resonant coupling for spin/pring systems; cooling;
Stabilization of a Schrödinger cat via an autonomous feedback scheme

The first experimental realization of a **quantum state feedback**

The photon box of the Laboratoire Kastler-Brossel (LKB):
group of S.Haroche (Nobel Prize 2012), J.M.Raimond and M. Brune.[1]



Stabilization of a quantum state with exactly $n = 0, 1, 2, 3, \ldots$ photon(s).
Experiment: C. Sayrin et. al., Nature 477, 73-77, September 2011.
Theory: I. Dotsenko et al., Physical Review A, 80: 013805-013813, 2009.
R. Somaraju et al., Rev. Math. Phys., 25, 1350001, 2013.
H. Amini et. al., Automatica, 49 (9): 2683-2692, 2013.

[1] Courtesy of Igor Dotsenko. **Sampling period 80 μs.**

Three quantum features emphasized by the LKB photon box[2]

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$ or density op. $\rho \sim |\psi\rangle\langle\psi|$

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H},\rho], \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1$$

2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome μ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle = \text{Tr}\left(\rho \boldsymbol{P}_\mu\right)$ depending on $|\psi\rangle$, ρ just before the measurement
   - measurement back-action if outcome $\mu = y$:

$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}, \quad \rho \mapsto \rho_+ = \frac{\boldsymbol{P}_y\rho\boldsymbol{P}_y}{\text{Tr}\left(\rho\boldsymbol{P}_y\right)}$$

3. Tensor product for the description of composite systems (S, M):
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system M only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[2] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

Composite system built with an harmonic oscillator and a qubit.

- System S corresponds to a quantized harmonic oscillator:

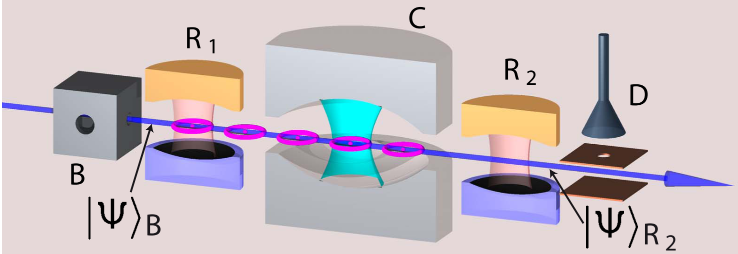
$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

  where $|n\rangle$ represents the Fock state associated to exactly n photons inside the cavity
- Meter M is a qu-bit, a 2-level system (idem 1/2 spin system) : $\mathcal{H}_M = \mathbb{C}^2$, each atom admits two energy levels and is described by a wave function $c_g|g\rangle + c_e|e\rangle$ with $|c_g|^2 + |c_e|^2 = 1$; atoms leaving B are all in state $|g\rangle$
- **State of the full system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:**

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \Psi_{ng}|n\rangle \otimes |g\rangle + \Psi_{ne}|n\rangle \otimes |e\rangle, \qquad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}.$$

  Ortho-normal basis: $(|n\rangle \otimes |g\rangle, |n\rangle \otimes |e\rangle)_{n\in\mathbb{N}}$.

The Markov ideal model (1)

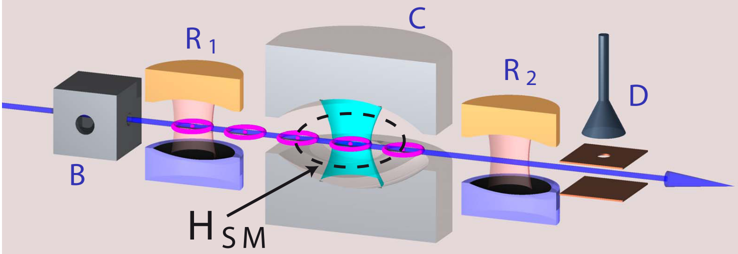


- When atom comes out B, $|\Psi\rangle_B$ of the full system is separable $|\Psi\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in D, the state is in general entangled (not separable):

$$|\Psi\rangle_{R_2} = \boldsymbol{U}_{SM}\big(|\psi\rangle \otimes |g\rangle\big) = \big(\boldsymbol{M}_g|\psi\rangle\big) \otimes |g\rangle + \big(\boldsymbol{M}_e|\psi\rangle\big) \otimes |e\rangle$$

  where $\boldsymbol{U}_{SM}$ is a unitary transformation (Schrödinger propagator) defining the linear measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

The Markov ideal model (2)



The unitary propagator $\boldsymbol{U}_{SM}$ is derived from Jaynes-Cummings Hamiltonian $\boldsymbol{H}_{SM}$ in the interaction frame.
Two kinds of qubit/cavity Hamiltonians:

resonant, $\boldsymbol{H}_{SM}/\hbar = i\big(\Omega(vt)/2\big)\ \big(\boldsymbol{a}^\dagger \otimes \boldsymbol{\sigma_-} - \boldsymbol{a} \otimes \boldsymbol{\sigma_+}\big)$,

dispersive, $\boldsymbol{H}_{SM}/\hbar = \big(\Omega^2(vt)/(2\delta)\big)\ \boldsymbol{N} \otimes \boldsymbol{\sigma_z}$,

where $\Omega(x) = \Omega_0 e^{-\frac{x^2}{w^2}}$, $x = vt$ with v atom velocity, Ω_0 vacuum Rabi pulsation, w radial mode-width and where $\delta = \omega_q - \omega_c$ is the detuning between qubit pulsation ω_q and cavity pulsation ω_c ($|\delta| \ll \Omega_0$).

Dispersive and resonant Jaynes-Cummings propagators

The solution of $i\frac{d}{dt}\boldsymbol{U} = -\frac{i}{\hbar}\boldsymbol{H}_{SM}(t)\boldsymbol{U}$, with $\boldsymbol{U}_0 = \boldsymbol{I}$ reads

- for $\boldsymbol{H}_{SM}(t)/\hbar = i\, f(t)\big(\boldsymbol{a}^\dagger \otimes |g\rangle\langle e| - \boldsymbol{a} \otimes |e\rangle\langle g|\big)$ (resonant)

$$\boldsymbol{U}_t = \cos\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right) \otimes |g\rangle\langle g| + \cos\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right) \otimes |e\rangle\langle e|$$
$$- \boldsymbol{a}\frac{\sin\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}} \otimes |e\rangle\langle g| + \frac{\sin\left(\frac{\theta_t}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger \otimes |g\rangle\langle e|.$$

- for $\boldsymbol{H}_{SM}(t)/\hbar = f(t)\ \boldsymbol{N} \otimes (|e\rangle\langle e| - |g\rangle\langle g|)$ (dispersive)

$$\boldsymbol{U}(t) = \exp\left(i\theta(t)\boldsymbol{N}\right) \otimes |g\rangle\langle g| + \exp\left(-i\theta(t)\boldsymbol{N}\right) \otimes |e\rangle\langle e|.$$

where $\theta(t) = \int_0^t f(\tau)\, d\tau$.

The Markov ideal model (3)

Just before D, the field/atom state is **entangled**:

$$\boldsymbol{M}_g|\psi\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi\rangle \otimes |e\rangle$$

Denote by $\mu \in \{g, e\}$ the measurement outcome in detector D: with probability $\mathbb{P}_\mu = \langle\psi|\boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu|\psi\rangle$ we get μ. Just after the measurement outcome $\mu = y$, the state becomes separable:

$$|\Psi\rangle_D = \frac{1}{\sqrt{\mathbb{P}_y}} \left(\boldsymbol{M}_y|\psi\rangle\right) \otimes |y\rangle = \left(\frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle\right) \otimes |y\rangle.$$

Markov process (wave function formulation)

$$|\psi\rangle_+ = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\langle\psi|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi\rangle}}|\psi\rangle & \text{with probability } \mathbb{P}_g = \langle\psi|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\langle\psi|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi\rangle}}|\psi\rangle & \text{with probability } \mathbb{P}_e = \langle\psi|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi\rangle; \end{cases}$$

See the quantum Monte Carlo simulations of the Matlab script: `WaveModelPhotonBox.m`.

Monday exercise (1)

Passage to the density operator Show that the wave function formulation $|\psi\rangle_+ = \frac{\boldsymbol{M}_y}{\sqrt{\langle\psi|\boldsymbol{M}_y^\dagger \boldsymbol{M}_y|\psi\rangle}}|\psi\rangle$ becomes with the density operator $\rho = |\psi\rangle\langle\psi|$: $\rho_+ = \frac{\boldsymbol{M}_y \rho \boldsymbol{M}_y^\dagger}{\text{Tr}\left(\boldsymbol{M}_y \rho \boldsymbol{M}_y^\dagger\right)}$ where y is the measurement outcome.

Detection efficiency alone The probability to detect the atom is $\eta \in [0, 1]$. Thus we have 3 possible outcomes for y: $y = g$ if detection in g, $y = e$ if detection in e and $y = 0$ if no detection. By definition, ρ_+ is the expectation value of the density operator just after the measurement knowing the measurement outcome and the density operator just before the measurement. Show that

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\text{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)} & \text{if } y = g \equiv -1, \text{ probability } \eta \,\text{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right) \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\text{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)} & \text{if } y = e \equiv +1, \text{ probability } \eta \,\text{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right) \\ \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta \end{cases}$$

Matlab simulations with $\eta = 1/3$ Transform the wave function formulation of `WaveModelPhotonBox.m` into the density operator formulation with a detection efficiency $\eta = 1/3$; show that the photon populations correspond then to the diagonal of ρ; what is the main change versus `WaveModelPhotonBox.m`? Look at the evolution of the off-diagonal elements of ρ: what do you observe numerically ?

Monday exercise (2)

Detection errors alone We assume that the probability to detect $y = e$ knowing that the true collapse of the atom is g is denoted by $\mathbb{P}(y = e/\mu = g) = \eta_g \in [0, 1]$. Similarly $\mathbb{P}(y = g/\mu = e) = \eta_e \in [0, 1]$ the probability of erroneous assignation to g when the atom collapses in e. Show that ρ_+ is given by the following rule (use the Bayes law on conditional probabilities)

$$\rho_+ = \begin{cases} \frac{(1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = g, \text{ prob. } \mathrm{Tr}\left((1-\eta_g)\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + \eta_e\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right); \\ \frac{\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right)} \text{ if } y = e, \text{ prob. } \mathrm{Tr}\left(\eta_g\boldsymbol{M}_g\rho\boldsymbol{M}_g^\dagger + (1-\eta_e)\boldsymbol{M}_e\rho\boldsymbol{M}_e^\dagger\right). \end{cases}$$

Detection efficiency and errors What are the transition rules for ρ_+ with a detection efficiency η and errors rates η_g and η_e?

Matlab simulations with $\eta = 1/3$ and $\eta_g = \eta_e = 1/10$. Adapt the previous Matlab simulation with $\eta = 1/3$ to detection errors with rates $\eta_g = \eta_e = 1/10$. What do you observe on the convergence speed? Does it change the asymptotic values of the off diagonal elements of ρ?

Recall: quantum system under measurement (discrete-time)

Quantum state ρ summarizes our knowledge about the system (quantum equivalent of proba.distr. over possible configurations)

- Hamiltonian interaction of target system with measurement system: propagator in $\mathcal{H}_S \otimes \mathcal{H}_M$

$$U(|\psi_S\rangle \otimes |\psi_M\rangle) = \boldsymbol{M}_g|\psi_S\rangle \otimes |g\rangle + \boldsymbol{M}_e|\psi_S\rangle \otimes |e\rangle$$

with $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

- Collapse of measurement system (from quantum to classical) at detection implies stochastic evolution of target system:

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)} \text{if } y = g, \text{ prob. } \mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right); \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)} \text{if } y = e, \text{ prob. } \mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right) \ . \end{cases}$$

Here, QND measurement of photon number:

$$\begin{aligned} \boldsymbol{M}_g &= \sum_{n\in\mathbb{N}} \cos\phi_n \, |n\rangle\langle n| \\ \boldsymbol{M}_e &= \sum_{n\in\mathbb{N}} \sin\phi_n \, |n\rangle\langle n| \end{aligned}$$

QND martingales and super-martingales (1)

- For any real function f, $\text{Tr}\,(f(\boldsymbol{N})\rho)$ is a martingale:

$$\mathbb{E}\left(\,\text{Tr}\,(f(\boldsymbol{N})\rho_{k+1})\;|\;\rho_k\right) = \;\text{Tr}\,(f(\boldsymbol{N})\rho_k)\,.$$

Interpretation: in particular for $f(\boldsymbol{N}) = |n_{\text{target}}\rangle\langle n_{\text{target}}|$, we have

$$\mathbb{E}\left(\langle n_{\text{target}}|\rho_{k+1}|n_{\text{target}}\rangle\right) = \langle n_{\text{target}}|\rho_k|n_{\text{target}}\rangle$$

i.e. the probability to be at $|n_{\text{target}}\rangle$ stays constant.

QND martingales and super-martingales (2)

- $V(\rho) = 1 - \sum_{n\geq 0}(\langle n|\rho|n\rangle)^2$ is a super-martingale:

$$\mathbb{E}\left(V(\rho_{k+1}) \mid \rho_k\right) - V(\rho_k) = -W(\rho_k) \leq 0$$

since we have $W(\rho) = \sum_n W_n(\rho)$ with all $W_n(\rho)$ nonnegative:[3]

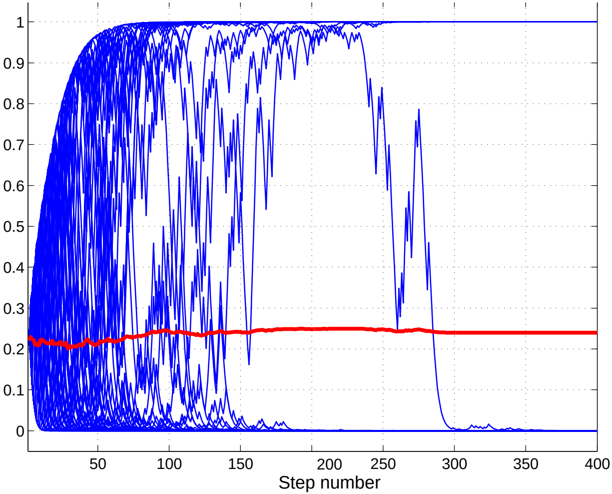
$$W_n(\rho) = \text{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right) \text{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right) \left(\frac{|\cos(\varphi_n)|^2 \langle n|\rho|n\rangle}{\text{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger\right)} - \frac{|\sin(\varphi_n)|^2 \langle n|\rho|n\rangle}{\text{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger\right)} \right)^2$$

Interpretation: ρ gets closer to satisfying $\sum_n \rho_{n,n}^2 = \sum_n \rho_{n,n} = 1$ i.e. to a form $\rho = |\bar{n}\rangle\langle\bar{n}|$ ("pure state" = maximal information state) for an a priori random n. Information extracted by measurement makes state "less uncertain" *a posteriori* but not more predictable *a priori*.

[3][Use the identity $px^2 + (1-p)y^2 - (px + (1-p)y)^2 = p(1-p)(x-y)^2$]

Asymptotic behavior: numerical simulations

100 Monte-Carlo simulations of $\mathrm{Tr}\left(\rho_k |3\rangle\langle 3|\right)$ versus k



This is an idealized situation: with pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\rho_+ = |\psi_+\rangle\langle\psi_+| = \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger \ / \ \operatorname{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$$

when the atom collapses in $\mu = g, e$ with proba. $\operatorname{Tr}\left(\boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)$.

We will now add perturbations from the environment.

Recall: LKB photon-box: Markov process with detection efficiency

Detection efficiency: the probability to detect the atom is $\eta \in [0,1]$.
Three possible outcomes for $y \in \{g, e, 0\}$.

The only possible update is based on ρ: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ and the outcome $y \in \{g, e, 0\}$.

$$\rho_+ = \begin{cases} \frac{\boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger}{\mathrm{Tr}(\boldsymbol{M}_g \rho \boldsymbol{M}_g)} & \text{if } y = g, \text{ probability } \eta \, \mathrm{Tr}\left(\boldsymbol{M}_g \rho \boldsymbol{M}_g\right) \\ \frac{\boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger}{\mathrm{Tr}(\boldsymbol{M}_e \rho \boldsymbol{M}_e)} & \text{if } y = e, \text{ probability } \eta \, \mathrm{Tr}\left(\boldsymbol{M}_e \rho \boldsymbol{M}_e\right) \\ \boldsymbol{M}_g \rho \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho \boldsymbol{M}_e^\dagger & \text{if } y = 0, \text{ probability } 1 - \eta \end{cases}$$

ρ_+ does not remain pure: the quantum state ρ_+ becomes a “mixed state” (rank > 1) reflecting a classical probability distribution.
$|\psi_+\rangle$ becomes physically inaccessible=irrelevant.

External perturbations seen as unread measurements

General viewpoint: add another measurement device with possible outcomes $\lambda \in \{...\}$, with operators $\tilde{\boldsymbol{M}}_\lambda$.
These measurement outcomes are inaccessible ($\eta = 0$): the associated information is lost into the environment.

The only possible update is based on ρ: expectation ρ_+ of $|\psi_+\rangle\langle\psi_+|$ knowing ρ, the (imperfect) detection y, and nothing about λ.

$$\rho_{+/2} = \sum_\lambda \tilde{\boldsymbol{M}}_\lambda \rho \tilde{\boldsymbol{M}}_\lambda^\dagger \quad \text{where } \sum_\lambda \tilde{\boldsymbol{M}}_\lambda^\dagger \tilde{\boldsymbol{M}}_\lambda = \boldsymbol{I}$$

$$\rho_+ = \begin{cases} \dfrac{\boldsymbol{M}_g \rho_{+/2} \boldsymbol{M}_g^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_g \rho_{+/2} \boldsymbol{M}_g\right)} & \text{if } y = g\text{, probability } \eta \,\mathrm{Tr}\left(\boldsymbol{M}_g \rho_{+/2} \boldsymbol{M}_g\right) \\ \dfrac{\boldsymbol{M}_e \rho_{+/2} \boldsymbol{M}_e^\dagger}{\mathrm{Tr}\left(\boldsymbol{M}_e \rho_{+/2} \boldsymbol{M}_e\right)} & \text{if } y = e\text{, probability } \eta \,\mathrm{Tr}\left(\boldsymbol{M}_e \rho_{+/2} \boldsymbol{M}_e\right) \\ \boldsymbol{M}_g \rho_{+/2} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \rho_{+/2} \boldsymbol{M}_e^\dagger & \text{if } y = 0\text{, probability } 1 - \eta \end{cases}$$

Under $\rho \mapsto \rho_{+/2}$ implied by the environment alone, $\rho = |\psi\rangle\langle\psi|$ does not remain pure. This has been called **decoherence.** Its effects, similar to damping in classical systems, are well-known historically.

LKB photon-box: Decoherence through Cavity decay

The field in the cavity interacts weakly with other fields in the universe. Overall Hilbert space (simplified model): $\mathcal{H}_S \otimes \mathcal{H}_E$.
Resonant interaction:

$$\boldsymbol{H}_{SE}/\hbar = i\,\sqrt{\gamma}\big(\boldsymbol{a}^\dagger \otimes \boldsymbol{b} - \boldsymbol{b}^\dagger \otimes \boldsymbol{a}\big)$$

Propagator over $dt = 1$ for $\gamma \ll 1$:

$$\boldsymbol{U} \simeq \boldsymbol{I} + i\sqrt{\gamma}(\boldsymbol{a}^\dagger \otimes \boldsymbol{b} - \boldsymbol{b}^\dagger \otimes \boldsymbol{a}) - \tfrac{\gamma}{2}(\boldsymbol{a}^\dagger \otimes \boldsymbol{b} - \boldsymbol{b}^\dagger \otimes \boldsymbol{a})^2$$

For environment at zero temperature, the initial environment state is $|\psi_E\rangle = |0\rangle$ such that
$\boldsymbol{b}|\psi_E\rangle = 0$ and $\boldsymbol{b}^\dagger|\psi_E\rangle = |1\rangle$. Thus:

$$\boldsymbol{U}(|\psi_S\rangle \otimes |\psi_E\rangle) = \tilde{\boldsymbol{M}}_{-1}|\psi_S\rangle \otimes |1\rangle_E + \tilde{\boldsymbol{M}}_0|\psi_S\rangle \otimes |0\rangle_E$$

with $\tilde{\boldsymbol{M}}_{-1} = \sqrt{\gamma}\boldsymbol{a}$ and $\tilde{\boldsymbol{M}}_0 = \boldsymbol{I} - \frac{\gamma}{2}\boldsymbol{a}^\dagger\boldsymbol{a}$ to first order (proba $O(\gamma)$).

Markov chain evolution operators:

- zero photon annihilation during ΔT: Kraus operator $\tilde{\boldsymbol{M}}_0 = \boldsymbol{I} - \frac{\Delta T}{2} \boldsymbol{L}_{-1}^\dagger \boldsymbol{L}_{-1}$, probability $\approx$ $\text{Tr}\left(\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_0^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_0^\dagger}{\text{Tr}\left(\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_O^\dagger\right)}$.
- one photon annihilation during ΔT: Kraus operator $\tilde{\boldsymbol{M}}_{-1} = \sqrt{\Delta T} \boldsymbol{L}_{-1}$, probability $\approx$ $\text{Tr}\left(\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger}{\text{Tr}\left(\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger\right)}$

where

$$\boldsymbol{L}_{-1} = \sqrt{\gamma} \boldsymbol{a}$$

is the Lindblad operator associated to cavity damping (see bellow the continuous time models) with $1/\gamma = T_{cav}$ the photon life time and $\Delta T \ll T_{cav}$ the sampling period ($T_{cav} = 100$ ms and $\Delta T \approx 100$ μs for the LKB photon Box).

At nonzero temperature, three possible outcomes:

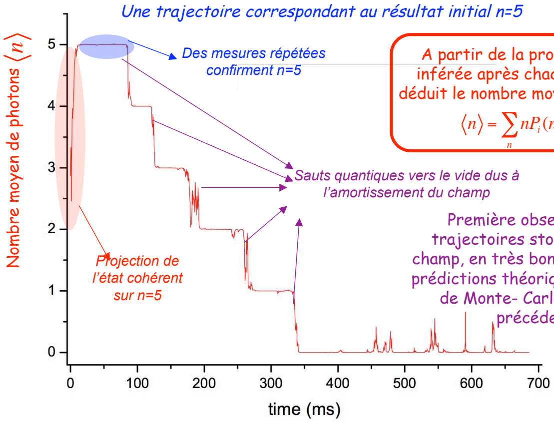
- zero photon annihilation during ΔT: Kraus operator $\tilde{\boldsymbol{M}}_0 = \boldsymbol{I} - \frac{\Delta T}{2}\boldsymbol{L}_{-1}^\dagger \boldsymbol{L}_{-1} - \frac{\Delta T}{2}\boldsymbol{L}_1^\dagger \boldsymbol{L}_1$, probability $\approx \text{Tr}\left(\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_0^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_0^\dagger}{\text{Tr}\left(\tilde{\boldsymbol{M}}_0 \rho_t \tilde{\boldsymbol{M}}_0^\dagger\right)}$.
- one photon annihilation during ΔT: Kraus operator $\tilde{\boldsymbol{M}}_{-1} = \sqrt{\Delta T}\boldsymbol{L}_{-1}$, probability $\approx \text{Tr}\left(\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger}{\text{Tr}\left(\tilde{\boldsymbol{M}}_{-1} \rho_t \tilde{\boldsymbol{M}}_{-1}^\dagger\right)}$
- one photon creation during ΔT: Kraus operator $\tilde{\boldsymbol{M}}_1 = \sqrt{\Delta T}\boldsymbol{L}_1$, probability $\approx \text{Tr}\left(\tilde{\boldsymbol{M}}_1 \rho_t \tilde{\boldsymbol{M}}_1^\dagger\right)$ with back action $\rho_{t+\Delta T} \approx \frac{\tilde{\boldsymbol{M}}_1 \rho_t \tilde{\boldsymbol{M}}_1^\dagger}{\text{Tr}\left(\tilde{\boldsymbol{M}}_1 \rho_t \tilde{\boldsymbol{M}}_1^\dagger\right)}$

where

$$\boldsymbol{L}_{-1} = \sqrt{\frac{1+n_{th}}{T_{cav}}}\boldsymbol{a}, \quad \boldsymbol{L}_1 = \sqrt{\frac{n_{th}}{T_{cav}}}\boldsymbol{a}^\dagger$$

are the Lindblad operators associated to cavity decoherence : n_{th} is the average presence of thermal photons ($n_{th} \approx 0.05$ for the LKB photon box).

Valeur moyenne du nombre de photons le long d'une longue séquence de mesure: observation d'une trajectoire stochastique



A partir de la probabilité $P_i(n)$ inférée après chaque atome, on déduit le nombre moyen de photons:

$$\langle n \rangle = \sum_n n P_i(n) \qquad (6-10)$$

Première observation des trajectoires stochastiques du champ, en très bon accord avec les prédictions théoriques (simulations de Monte- Carlo. Voir cours précédents).

[4] From Serge Haroche, Collège de France, notes de cours 2007/2008.

Summary: quantum measurement and the route to feedback

- The environment measuring our quantum system implies **decoherence**. The state moves stochastically; the best an external observer (we) can do is describe the **expected evolution** by

$$\rho_+ = \sum_\lambda \tilde{\boldsymbol{M}}_\lambda \rho \tilde{\boldsymbol{M}}_\lambda^\dagger \qquad \text{where } \sum_\lambda \tilde{\boldsymbol{M}}_\lambda^\dagger \tilde{\boldsymbol{M}}_\lambda = \boldsymbol{I}$$

- To correct this decoherence with measurement-based feedback, we couple the system to a measurement device. This is described, with left stochastic matrix $(\eta_{\mu',\mu})$ to model uncertainties, by

$$\rho_+ = \frac{\sum_\mu \eta_{y,\mu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_\mu \eta_{y,\mu} \boldsymbol{M}_\mu \rho \boldsymbol{M}_\mu^\dagger\right)} \qquad \text{when } y = \mu' \text{; proba=denominator.}$$

- Only measuring thus implies a stochastic evolution. On average:
  - The information extracted by measurement makes the state purer, “less uncertain”.
  - The probability to converge to a target $|n_{\text{target}}\rangle$ is not improved. (This is due to “QND type measurement”. It can in fact be improved, see reservoir engineering.)

Summary: quantum measurement and the route to feedback

- To actually get closer to target: apply feedback knowing system state ρ.

LKB actuator:

$u = 0$: dispersive interaction i.e. just measure, ideally
$\boldsymbol{M}_g(0) = \cos(\phi_{\boldsymbol{N}})$, $\boldsymbol{M}_e(0) = \sin(\phi_{\boldsymbol{N}})$

$u = 1$: resonant interaction with atom prepared in $|e\rangle$ (add energy)

$$\boldsymbol{M}_g(1) = \frac{\sin\left(\frac{\theta_{0+}}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}\boldsymbol{a}^\dagger \text{ and } \boldsymbol{M}_e(1) = \cos\left(\frac{\theta_{0+}}{2}\sqrt{\boldsymbol{N}+\boldsymbol{I}}\right)$$

$u = -1$: resonant interaction with atom prepared in $|g\rangle$ (subtract energy)

$$\boldsymbol{M}_g(-1) = \cos\left(\frac{\theta_{0-}}{2}\sqrt{\boldsymbol{N}}\right) \text{ and } \boldsymbol{M}_e(-1) = -\boldsymbol{a}\frac{\sin\left(\frac{\theta_{0-}}{2}\sqrt{\boldsymbol{N}}\right)}{\sqrt{\boldsymbol{N}}}$$

with θ_{0+}, θ_{0-} constant parameters.

Tuesday exercise (1)

Consider the model with $\eta = 1$ and $\eta_e = \eta_g = 0$ (template `FeedbackTemplate_0.m`)

Actuation effect Show that the control Lyapunov function

$$V(\rho) = \text{Tr}\left((\boldsymbol{N} - n_{\text{target}}\, \boldsymbol{I})^2 \rho \right)$$

evolves as follows with the LKB actuator:

$$\mathbb{E}(V(\rho_{k+1}|\rho_k, u = 1)) - V(\rho_k) = \text{Tr}\left(\rho_k \ \sin^2\left(\tfrac{\theta_{0+}}{2} \sqrt{\boldsymbol{N} + \boldsymbol{I}} \right) (1 + 2(\boldsymbol{N} - n_{\text{target}}\, \boldsymbol{I})) \right)$$

$$\mathbb{E}(V(\rho_{k+1}|\rho_k, u = -1)) - V(\rho_k) = \text{Tr}\left(\rho_k \ \sin^2\left(\tfrac{\theta_{0-}}{2} \sqrt{\boldsymbol{N}} \right) (1 - 2(\boldsymbol{N} - n_{\text{target}}\, \boldsymbol{I})) \right) .$$

How does it evolve when selecting $u = 0$?

Hints: Use the following commutation relation and its hermitian conjugate: $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$ for any $f(\boldsymbol{N}) = \sum_{n \geq 0} f(n) |n\rangle\langle n|$. If you want an easier intermediate step, check expected $\langle n|\rho_{k+1}|n\rangle$ as a function of $u \in \{-1, 0, +1\}$.

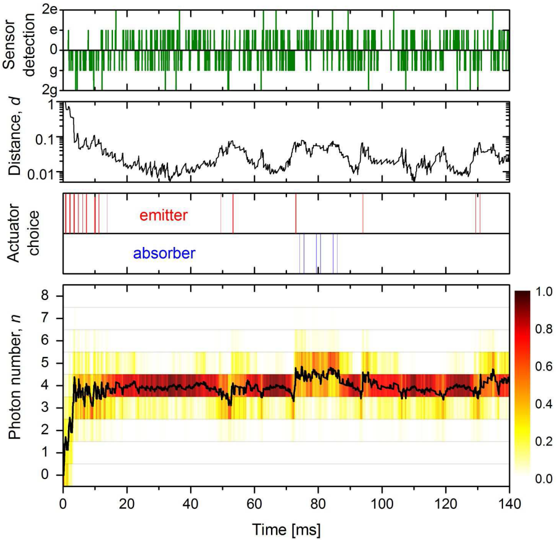
Tuesday exercise (2)

Feedback in idealized case Use the above formulas to define a feedback strategy: how select u knowing ρ_k, to drive the system towards $|n_{\text{target}}\rangle\langle n_{\text{target}}|$ with $n_{\text{target}} = 3$? Program this into the matlab template `FeedbackTemplate_0.m`, using $\phi_0 = \pi/7$, $\phi_R = 0$, $\theta_{0+} = 2\pi/\sqrt{n_{\text{target}} + 2}$, $\theta_{0-} = 2\pi/\sqrt{n_{\text{target}} - 1}$. Check how you converge to $|n_{\text{target}}\rangle$.
What can you guarantee analytically?

Parameter tuning Investigate the effect of ϕ_0, ϕ_R, θ_{0+} and θ_{0-}. One suggests to consider the special values $\theta_{0+} = 2\pi/\sqrt{n_{\text{target}} + 1}$ and $\theta_{0-} = 2\pi/\sqrt{n_{\text{target}}}$.
Can you understand why $\theta_{0+} = 2\pi/\sqrt{n_{\text{target}} + 2}$, $\theta_{0-} = 2\pi/\sqrt{n_{\text{target}} - 1}$ is a good choice for robustness issues?

Decoherence Add the effect of decoherence into the simulation. Observe its effect on the evolution both with and without feedback. Can you adapt the feedback law to get better results?

Closed-loop experimental results



Zhou et al. Field locked to Fock state by quantum feedback with single photon corrections. Physical Review Letter, 2012, 108, 243602.

See the closed-loop quantum Monte Carlo simulations of the Matlab script: `RealisticFeedbackPhotonBox.m`.

Stochastic Master Equation (SME) and quantum filtering

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\boldsymbol{K}_{y_k}(\rho_k)}{\mathrm{Tr}\big(\boldsymbol{K}_{y_k}(\rho_k)\big)}, \text{ with proba. } p_{y_k}(\rho_k) = \mathrm{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)$$

where each $\boldsymbol{K}_y$ is a linear completely positive map admitting the expression

$$\boldsymbol{K}_y(\rho) = \sum_{\mu} \boldsymbol{M}_{y,\mu} \rho \boldsymbol{M}_{y,\mu}^\dagger \quad \text{with} \quad \sum_{y,\mu} \boldsymbol{M}_{y,\mu}^\dagger \boldsymbol{M}_{y,\mu} = \boldsymbol{I}.$$

$\boldsymbol{K} = \sum_y \boldsymbol{K}_y$ corresponds to a Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1} | \rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_y \boldsymbol{K}_y(\rho_k).$$

Quantum filtering (Belavkin quantum filters)

data: initial quantum state ρ_0, past measurement outcomes $\boldsymbol{y_l}$ for $l \in \{0, \dots, k-1\}$;

goal: estimation of ρ_k via the recurrence (quantum filter)

$$\rho_{l+1} = \frac{\boldsymbol{K}_{\boldsymbol{y_l}}(\rho_l)}{\mathrm{Tr}\left(\boldsymbol{K}_{\boldsymbol{y_l}}(\rho_l)\right)}, \quad l = 0, \dots, k-1.$$

Continuous/discrete-time Stochastic Master Equation (SME)

Discrete-time models are Markov processes

$$\rho_{k+1} = \frac{\boldsymbol{K}_{y_k}(\rho_k)}{\text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)}, \text{ with proba. } p_{y_k}(\rho_k) = \text{Tr}\left(\boldsymbol{K}_{y_k}(\rho_k)\right)$$

associated to Kraus maps (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_y \boldsymbol{K}_y(\rho_k)$$

Continuous-time models are stochastic differential systems

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt + \sum_\nu \sqrt{\eta_\nu}\left(\boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \text{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t\right) dW_{\nu,t}$$

driven by Wiener process[5] $dW_{\nu,t} = dy_{\nu,t} - \sqrt{\eta_\nu}\, \text{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\, \rho_t\right) dt$ with measures $y_{\nu,t}$, detection efficiencies $\eta_\nu \in [0,1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

[5] and/or Poisson processes, see next slides.

Given a SDE

$$dX_t = F(X_t, t)dt + \sum_\nu G_\nu(X_t, t)dW_{\nu,t},$$

we have the following chain rule summarized by the heuristic formulae:

$$dW_{\nu,t} = O(\sqrt{dt}), \quad dW_{\nu,t}dW_{\nu',t} = \delta_{\nu,\nu'}dt.$$

Itō's rule Defining $f_t = f(X_t)$ a C^2 function of X, we have

$$df_t = \left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right) dt + \sum_\nu \frac{\partial f}{\partial X}\Big|_{X_t} G_\nu(X_t, t)dW_{\nu,t}.$$

Furthermore

$$\mathbb{E}\left(\frac{d}{dt} f_t \;\middle|\; X_t \right) = \mathbb{E}\left(\frac{\partial f}{\partial X}\Big|_{X_t} F(X_t, t) + \frac{1}{2}\sum_\nu \frac{\partial^2 f}{\partial X^2}\Big|_{X_t} (G_\nu(X_t, t), G_\nu(X_t, t)) \right).$$

Continuous/discrete-time diffusive SME

With a single imperfect measure $dy_t = \sqrt{\eta}\,\text{Tr}\left(\left(L + L^\dagger\right)\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0,1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-\frac{i}{\hbar}[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}\left(L^\dagger L\rho_t + \rho_t L^\dagger L\right)\right)dt \\ + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \text{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right)dW_t$$

driven by the Wiener process dW_t (Gaussian law of mean 0 and variance dt).

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt}{\text{Tr}\left(M_{dy_t}\rho_t M_{dy_t}^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}$$

with $M_{dy_t} = I + \left(-\frac{i}{\hbar}H - \frac{1}{2}\left(L^\dagger L\right)\right)dt + \sqrt{\eta}\,dy_t L$. The probability to detect dy_t is given by the following density

$$\mathbb{P}\left(dy_t \in [s, s+ds]\right) = \frac{\text{Tr}\left(M_s\rho_t M_s^\dagger + (1-\eta)L\rho_t L^\dagger dt\right)}{\sqrt{2\pi}}\,e^{-\frac{s^2}{2dt}}\,ds$$

close to a Gaussian law of variance dt and mean $\sqrt{\eta}\,\text{Tr}\left(\left(L+L^\dagger\right)\rho_t\right)\,dt$.

Continuous/discrete-time jump SME

With Poisson process $\boldsymbol{N(t)}$, $\langle \boldsymbol{dN(t)} \rangle = \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt$, and detection imperfections modeled by $\overline{\theta} \geq 0$ and $\overline{\eta} \in [0, 1]$, the quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H, \rho_t] + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\right) dt$$
$$+ \left(\frac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\,(V\rho_t V^\dagger)} - \rho_t\right)\left(\boldsymbol{dN(t)} - \left(\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right)\right) dt\right)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta} V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\,\mathrm{Tr}\,(V\rho_t V^\dagger)}$.
For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_0\rho_t M_0^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt}{\mathrm{Tr}\left(M_0\rho_t M_0^\dagger + (1-\overline{\eta})V\rho_t V^\dagger dt\right)}$$

with $M_0 = I + \left(-iH + \frac{1}{2}\left(\overline{\eta}\,\mathrm{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V\right)\right) dt$.

Continuous/discrete-time diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \Big(-i[H,\rho_t] + L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L) + V\rho_t V^\dagger - \frac{1}{2}(V^\dagger V\rho_t + \rho_t V^\dagger V)\Big)\ dt$$
$$+\sqrt{\eta}\Big(L\rho_t + \rho_t L^\dagger - \operatorname{Tr}\big((L+L^\dagger)\rho_t\big)\,\rho_t\Big)\,\boldsymbol{dW_t}$$
$$+\left(\frac{\overline{\theta}\rho_t + \overline{\eta}V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)} - \rho_t\right)\Big(\boldsymbol{dN(t)} - \big(\overline{\theta} + \overline{\eta}\operatorname{Tr}\big(V\rho_t V^\dagger\big)\big)\,dt\Big)$$

For $\boldsymbol{N(t+dt) - N(t) = 1}$ we have $\rho_{t+dt} = \dfrac{\overline{\theta}\rho_t + \overline{\eta}V\rho_t V^\dagger}{\overline{\theta} + \overline{\eta}\operatorname{Tr}(V\rho_t V^\dagger)}$.

For $\boldsymbol{dN(t) = 0}$ we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt}{\operatorname{Tr}\Big(M_{\boldsymbol{dy_t}}\rho_t M_{\boldsymbol{dy_t}}^\dagger + (1-\eta)L\rho_t L^\dagger dt + (1-\overline{\eta})V\rho_t V^\dagger dt\Big)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2}L^\dagger L + \frac{1}{2}\left(\overline{\eta}\operatorname{Tr}\left(V\rho_t V^\dagger\right) I - V^\dagger V\right)\right)dt + \sqrt{\eta}\,\boldsymbol{dy_t}L$.

Continuous/discrete-time general diffusive-jump SME

The quantum state ρ_t is usually mixed and obeys to

$$d\rho_t = \left(-i[H,\rho_t] + \sum_\nu L_\nu \rho_t L_\nu^\dagger - \tfrac{1}{2}(L_\nu^\dagger L_\nu \rho_t + \rho_t L_\nu^\dagger L_\nu) + V_\mu \rho_t V_\mu^\dagger - \tfrac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu) \right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left(L_\nu \rho_t + \rho_t L_\nu^\dagger - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t \right) \rho_t \right) \boldsymbol{dW_{\nu,t}}$$
$$+ \sum_\mu \left(\frac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_\mu \rho_t V_\mu^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)} - \rho_t \right) \left(\boldsymbol{dN_\mu(t)} - \left(\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right) \right) dt \right)$$

where $\eta_\nu \in [0,1]$, $\overline{\theta}_\mu, \overline{\eta}_{\mu,\mu'} \geq 0$ with $\overline{\eta}_{\mu'} = \sum_\mu \overline{\eta}_{\mu,\mu'} \leq 1$ are parameters modelling measurements imperfections.

If, for some μ, $\boldsymbol{N_\mu(t+dt) - N_\mu(t) = 1}$, we have $\rho_{t+dt} = \dfrac{\overline{\theta}_\mu \rho_t + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} V_{\mu'} \rho_t V_{\mu'}^\dagger}{\overline{\theta}_\mu + \sum_{\mu'} \overline{\eta}_{\mu,\mu'} \operatorname{Tr}\left(V_{\mu'} \rho_t V_{\mu'}^\dagger \right)}$.

When $\forall \mu,\ \boldsymbol{dN_\mu(t) = 0}$, we have

$$\rho_{t+dt} = \frac{M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt}{\operatorname{Tr}\left(M_{\boldsymbol{dy_t}} \rho_t M_{\boldsymbol{dy_t}}^\dagger + \sum_\nu (1-\eta_\nu) L_\nu \rho_t L_\nu^\dagger dt + \sum_\mu (1-\overline{\eta}_\mu) V_\mu \rho_t V_\mu^\dagger dt \right)}$$

with $M_{\boldsymbol{dy_t}} = I + \left(-iH - \frac{1}{2}\sum_\nu L_\nu^\dagger L_\nu + \frac{1}{2}\sum_\mu \left(\overline{\eta}_\mu \operatorname{Tr}\left(V_\mu \rho_t V_\mu^\dagger \right) I - V_\mu^\dagger V_\mu \right) \right) dt + \sum_\nu \sqrt{\eta_\nu}\, \boldsymbol{dy_{\nu,t}} L_\nu$ and where $\boldsymbol{dy_{\nu,t}} = \sqrt{\eta_\nu} \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\, \rho_t \right) dt + \boldsymbol{dW_{\nu,t}}$.

The Lindblad master differential equation (finite dimensional case)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\boldsymbol{H},\rho] + \sum_{\nu} \boldsymbol{L}_{\nu}\rho\boldsymbol{L}_{\nu}^{\dagger} - \frac{1}{2}(\boldsymbol{L}_{\nu}^{\dagger}\boldsymbol{L}_{\nu}\rho + \rho\boldsymbol{L}_{\nu}^{\dagger}\boldsymbol{L}_{\nu}) \triangleq \mathcal{L}(\rho)$$

where

- $\boldsymbol{H}$ is the Hamiltonian that could depend on t (Hermitian operator on the underlying Hilbert space $\mathcal{H}$)
- the $\boldsymbol{L}_{\nu}$'s are operators on $\mathcal{H}$ that are not necessarily Hermitian.

Qualitative properties:

1. Positivity and trace conservation: if ρ_0 is a density operator, then $\rho(t)$ remains a density operator for all $t > 0$.
2. For any $t \geq 0$, the propagator $e^{t\mathcal{L}}$ is a Kraus map: exists a collection of operators (M_μ) such that $\sum_\mu M_\mu^\dagger M_\mu = I$ with $e^{t\mathcal{L}}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$ (Kraus theorem characterizing completely positive linear maps).
3. Contraction for many distances such as the nuclear distance: take two trajectories ρ and ρ'; for any $0 \leq t_1 \leq t_2$,

$$\text{Tr}\left(|\rho(t_2) - \rho'(t_2)|\right) \leq \text{Tr}\left(|\rho(t_1) - \rho'(t_1)|\right)$$

where for any Hermitian operator A, $|A| = \sqrt{A^2}$ and $\text{Tr}(|A|)$ corresponds to the sum of the absolute values of its eigenvalues.

Properties of the trace distance $D(\rho, \rho') = \text{Tr}\left(|\rho - \rho'|\right)/2$.

1. Unitary invariance: for any unitary operator U ($U^\dagger U = I$), $D\big(U\rho U^\dagger, U\rho' U^\dagger\big) = D(\rho, \rho')$.
2. For any density operators ρ and ρ',

$$D(\rho, \rho') = \max_{\substack{P \text{such that} \\ 0 \leq P = P^\dagger \leq I}} \text{Tr}\left(P(\rho - \rho')\right).$$

3. Triangular inequality: for any density operators ρ, ρ' and ρ''

$$D(\rho, \rho'') \leq D(\rho, \rho') + D(\rho', \rho'').$$

Kraus maps are contractions for several "distances"[6]

For any Kraus map $\rho \mapsto \boldsymbol{K}(\rho) = \sum_\mu M_\mu \rho M_\mu^\dagger$ ($\sum_\mu M_\mu^\dagger M_\mu = I$) $d(\boldsymbol{K}(\rho), \boldsymbol{K}(\sigma)) \leq d(\rho, \sigma)$ with

- trace distance: $d_{tr}(\rho, \sigma) = \frac{1}{2}\operatorname{Tr}(|\rho - \sigma|)$.
- Bures distance: $d_B(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ with fidelity $F(\rho, \sigma) = \operatorname{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$.
- Chernoff distance: $d_C(\rho, \sigma) = \sqrt{1 - Q(\rho, \sigma)}$ where $Q(\rho, \sigma) = \min_{0 \leq s \leq 1} \operatorname{Tr}\left(\rho^s \sigma^{1-s}\right)$.
- Relative entropy: $d_S(\rho, \sigma) = \sqrt{\operatorname{Tr}\left(\rho(\log\rho - \log\sigma)\right)}$.
- χ^2-divergence: $d_{\chi^2}(\rho, \sigma) = \sqrt{\operatorname{Tr}\left((\rho - \sigma)\sigma^{-\frac{1}{2}}(\rho - \sigma)\sigma^{-\frac{1}{2}}\right)}$.
- Hilbert's projective metric: if $\operatorname{supp}(\rho) = \operatorname{supp}(\sigma)$ $d_h(\rho, \sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$ otherwise $d_h(\rho, \sigma) = +\infty$.

[6] A good summary in M.J. Kastoryano PhD thesis: Quantum Markov Chain Mixing and Dissipative Engineering. University of Copenhagen, December 2011.

Non-commutative consensus and Hilbert's metric[7] [8]

The Schrödinger approach $d_h(\rho,\sigma) = \log\left(\left\|\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}\right\|_\infty \left\|\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right\|_\infty\right)$

$$\boldsymbol{K}(\rho) = \sum M_\mu \rho M_\mu^\dagger, \quad \sum M_\mu^\dagger M_\mu = I$$
$$\frac{d}{dt}\rho = -i[H,\rho] + \sum L_\mu \rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu$$

Contraction ratio: $\tanh\left(\frac{\Delta(\boldsymbol{K})}{4}\right)$ with $\Delta(\boldsymbol{K}) = \max_{\rho,\sigma>0} d_h(\boldsymbol{K}(\rho), \boldsymbol{K}(\sigma))$

The Heisenberg approach (dual of Schrödinger approach):

$$\boldsymbol{K}^*(A) = \sum M_\mu^\dagger A M_\mu, \quad \boldsymbol{K}^*(I) = I$$
$$\frac{d}{dt}A = i[H,A] + \sum L_\mu^\dagger A L_\mu - \frac{1}{2}L_\mu^\dagger L_\mu A - \frac{1}{2}A L_\mu^\dagger L_\mu, \quad A = I \text{ steady-state.}$$

"Contraction of the spectrum":

$$\lambda_{min}(A) \le \lambda_{min}(\boldsymbol{K}^*(A)) \le \lambda_{max}(\boldsymbol{K}^*(A)) \le \lambda_{max}(A).$$

[7] R. Sepulchre et al.: Consensus in non-commutative spaces. CDC 2010.
[8] D. Reeb et al.: Hilbert's projective metric in quantum information theory. J. Math. Phys. 52, 082201 (2011).

Recall: Continuous-time quantum SME

$$d\rho_t = \left(-\frac{i}{\hbar}[\boldsymbol{H}(u), \rho_t] + \sum_\mu \boldsymbol{L}_\mu \rho_t \boldsymbol{L}_\mu^\dagger - \frac{1}{2}\left(\boldsymbol{L}_\mu^\dagger \boldsymbol{L}\mu\rho_t + \rho_t \boldsymbol{L}_\mu^\dagger \boldsymbol{L}_\mu \right) \right) dt$$
$$+ \sqrt{\eta_\mu} \left(\boldsymbol{L}_\mu \rho_t + \rho_t \boldsymbol{L}_\mu^\dagger - \operatorname{Tr}\left((\boldsymbol{L}_\mu + \boldsymbol{L}_\mu^\dagger)\rho_t \right) \rho_t \right) \boldsymbol{dW_t}$$

$$\boldsymbol{dy_t^\mu} = \sqrt{\eta_\mu} \operatorname{Tr}\left((\boldsymbol{L}_\mu + \boldsymbol{L}_\mu^\dagger)\, \rho_t \right) dt + \boldsymbol{dW_t^\mu}$$

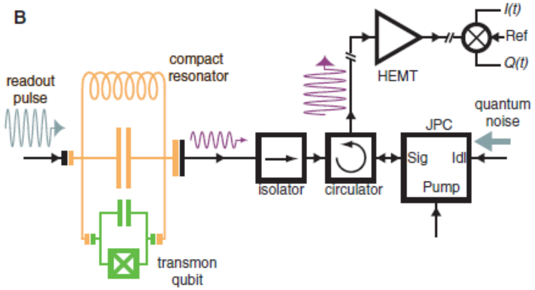
with

independent Wiener processes $\boldsymbol{dW_t^\mu}$ (Gaussian law of mean 0 and variance dt)
detection efficiencies $\eta_\mu \in [0, 1]$.

This SME must be understood in the Itō sense, compute with Itō rules.

Possibly $\eta_\mu = 0$ for some μ. This describes **decoherence** implied by external perturbations from the environment.

A key physical example in circuit QED: QND measure of σ_z[9]



Superconducting qubit dispersively coupled to a cavity traversed by a microwave signal (input/output theory). The back-action on the qubit state of a single measurement of both output field quadratures I_t and Q_t is described by a simple SME for the qubit density operator.

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right) dt$$
$$+ \sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right)\boldsymbol{dW_t^I} + i\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}$$

with I_t and Q_t given by $\boldsymbol{dI_t} = \sqrt{\eta\gamma/2}\,\mathrm{Tr}\left(2\boldsymbol{\sigma_z}\rho_t\right) dt + \boldsymbol{dW_t^I}$ and $\boldsymbol{dQ_t} = \boldsymbol{dW_t^Q}$, where $\gamma \geq 0$ is related to the measurement strength and $\eta \in [0,1]$ is the detection efficiency. u and v are the two control inputs.

[9] M. Hatridge et al. Quantum Back-Action of an Individual Variable-Strength Measurement. Science, 2013, 339, 178-181.

Qubit with QND measure of σ_z: asymptotic behavior in open-loop

Consider the following SME with $u = v = 0$ and $\eta > 0$:

$$
\begin{aligned}
d\rho_t &= \Big(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\Big) dt \\
&+ \sqrt{\eta\gamma/2}\big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\operatorname{Tr}(\boldsymbol{\sigma_z}\rho_t)\,\rho_t\big)\boldsymbol{dW_t^I} + i\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}
\end{aligned}
$$

Almost sure convergence:

- For any initial state ρ_0, the solution ρ_t converges almost surely as $t \to \infty$ to one of the states $|g\rangle\langle g|$ or $|e\rangle\langle e|$.
- The probability of convergence to $|g\rangle\langle g|$ (respectively $|e\rangle\langle e|$) is given by $p_g = \operatorname{Tr}(|g\rangle\langle g|\rho_0)$ (respectively $\operatorname{Tr}(|e\rangle\langle e|\rho_0)$).

Proof:

- martingale $V_e(\rho) = \operatorname{Tr}(|e\rangle\langle e|\rho) = (1+z)/2 \Rightarrow \mathbb{E}(dV_e|\rho_t) = 0$
- sub-martingale $V(\rho) = \operatorname{Tr}^2(\boldsymbol{\sigma_z}\rho) = z^2$
  $\Rightarrow \mathbb{E}(dV|\rho_t) = 2\eta\gamma\left(1 - z^2\right)^2 dt \geq 0.$

Confirmed by the quantum Monte Carlo simulations:
`TemplateQubit_0.m`

Adding decoherence due to spontaneous emission

$$
\begin{aligned}
d\rho_t = \Big(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\Big) dt \\
+ \sqrt{\eta\gamma/2}\big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t\big)\,\boldsymbol{dW_t^I} + i\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q} \\
+ \left(\boldsymbol{L_e}\rho_t\boldsymbol{L_e^\dagger} - \tfrac{1}{2}\left(\boldsymbol{L_e^\dagger}\boldsymbol{L_e}\rho_t + \rho_t\boldsymbol{L_e^\dagger}\boldsymbol{L_e}\right)\right) dt
\end{aligned}
$$

where $\boldsymbol{L_e} = \sqrt{1/T_1}\boldsymbol{\sigma_-}$ and T_1 is the average lifetime of the excited state $|e\rangle$.

For $u = v = 0$: all trajectories converge towards $|g\rangle$, the ground state.
Proof:

- super-martingale $V_e(\rho) = \mathrm{Tr}\,(|e\rangle\langle e|\rho) = (1+z)/2$

$$\Rightarrow \mathbb{E}(dV_e|\rho_t) = -\tfrac{1}{T_1} V_e\, dt$$

Confirmed by quantum Monte Carlo simulations and by experiments.

Thursday exercise

Feedback stabilization of the excited state

Actuation effect Consider the ideal model

$$d\rho_t = \left(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\right) dt$$
$$+ \sqrt{\eta\gamma/2}\left(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\right) \boldsymbol{dW_t^I} + i\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW_t^Q}$$

with u and v arbitrary. Show that the control Lyapunov function $V(\rho) = 1 - V_e(\rho) = (1 - z)/2$ evolves in expectation as

$$\mathbb{E}(dV_t|\rho_t) = v\,\mathrm{Tr}\left(\boldsymbol{\sigma_x}\rho_t\right)/2 - u\,\mathrm{Tr}\left(\boldsymbol{\sigma_y}\rho_t\right)/2 = vx/2 - uy/2.$$

Feedback design Using this observation, design a feedback law to stabilize the target state $\rho = |e\rangle\langle e|$ (i.e. $z = 1$ in the Bloch sphere representation).

Implement this feedback into the simulation `TemplateQubit_0.m`

Decoherence effect Add the decoherence due to spontaneous emission into the simulation. (See Wednesday's lecture about discretizing the SDE.)

Questioning observer-based feedback

So far we have made “observer-based feedback”:

- On the basis of detection results y_t, we update ρ_t which describes everything an external observer can now about the quantum system’s state. This is the “quantum filter”.
- We take control decisions u_t on the basis of the value of ρ_t

Quantum control is useful for building “quantum IT devices”.

These devices are supposed to do things that classical systems cannot. In particular, the quantum state is supposed to evolve in a way that cannot be efficiently simulated in a classical system.

This is not compatible with running an observer of ρ on a classical computer for control purposes.
$\Rightarrow$ need controllers of lower complexity

The quantum P[ID] controller a.k.a. “Markovian feedback”[10]

$$d\rho_t = \Big(-\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\Big)\, dt$$
$$+ \sqrt{\eta\gamma/2}\big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\,(\boldsymbol{\sigma_z}\rho_t)\,\rho_t\big)\,\boldsymbol{dW^I_t} + i\sqrt{\eta\gamma/2}[\boldsymbol{\sigma_z}, \rho_t]\boldsymbol{dW^Q_t}$$
$$+ \left(\boldsymbol{L}_e\rho_t\boldsymbol{L}_e^\dagger - \tfrac{1}{2}\left(\boldsymbol{L}_e^\dagger\boldsymbol{L}_e\rho_t + \rho_t\boldsymbol{L}_e^\dagger\boldsymbol{L}_e\right)\right) dt$$

with outputs:

$$\boldsymbol{dI_t} = \sqrt{\eta\gamma/2}\,\mathrm{Tr}\,(2\boldsymbol{\sigma_z}\rho_t)\, dt + \boldsymbol{dW^I_t} \quad \text{and} \quad \boldsymbol{dQ_t} = \boldsymbol{dW^Q_t}\ .$$

Proportional Control:

$$u_t\, dt = u_0\, dt + g_{u,I}\boldsymbol{dI_t} + g_{u,Q}\boldsymbol{dQ_t}\ , \quad v_t\, dt = v_0\, dt + g_{v,I}\boldsymbol{dI_t} + g_{v,Q}\boldsymbol{dQ_t}\ .$$

[10] H.Wiseman & G.Milburn, Phys.Rev.A, 1990s

Closed-loop equation under Markovian feedback

Remarkably, the closed-loop system follows a canonical quantum SME with modified noise operators. Proof on simplified case (SISO):

$$d\rho_t = \big(-\tfrac{i}{2}[H_0 + H_1(t), \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\big)dt \\ + \sqrt{\eta\gamma}\big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\big)\boldsymbol{dW_t^I}$$

with $H_0 = u_0\boldsymbol{\sigma_x}$ and
with $H_1(t)\,dt = g_{u,I}\boldsymbol{dI_t}\,\boldsymbol{\sigma_x} = g_{u,I}\left(\sqrt{\eta\gamma}\,\mathrm{Tr}\left(2\boldsymbol{\sigma_z}\rho_t\right)dt + \boldsymbol{dW_t^I}\right)\boldsymbol{\sigma_x}$.

Itō formulation takes causality into account: first we measure, then we apply feedback associated to that measurement. Thus:

$$\rho_{t+dt} = e^{-\frac{i}{2}H_1(t)dt}\Big\{\rho_t - \tfrac{i}{2}dt[H_0, \rho_t] + \gamma(\boldsymbol{\sigma_z}\rho\boldsymbol{\sigma_z} - \rho_t)\big)dt \\ + \sqrt{\eta\gamma}\big(\boldsymbol{\sigma_z}\rho_t + \rho_t\boldsymbol{\sigma_z} - 2\,\mathrm{Tr}\left(\boldsymbol{\sigma_z}\rho_t\right)\rho_t\big)\boldsymbol{dW_t^I}\Big\}\, e^{+\frac{i}{2}H_1(t)dt}$$

Closed-loop equation under Markovian feedback

Use the Baker-Campbell-Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + [A, [A, B]]/2 + O(||A||^3)$$

with Itō calculus and neglect terms of order $O(dt^{3/2})$. We get:

$$\begin{aligned}\rho_{t+dt} - \rho_t = \Big(-\tfrac{i}{2}[H_0 + H_b, \rho_t] + (\boldsymbol{L}_1 \rho \boldsymbol{L}_1^\dagger - \boldsymbol{L}_1^\dagger \boldsymbol{L}_1 \rho_t/2 - \rho_t \boldsymbol{L}_1^\dagger \boldsymbol{L}_1/2) + \\ (\boldsymbol{L}_2 \rho \boldsymbol{L}_2^\dagger - \boldsymbol{L}_2^\dagger \boldsymbol{L}_2 \rho_t/2 - \rho_t \boldsymbol{L}_2^\dagger \boldsymbol{L}_2/2)\Big)\, dt \\ + \Big(\sqrt{\eta}(\boldsymbol{L}_1 \rho_t + \rho_t \boldsymbol{L}_1^\dagger - \operatorname{Tr}\left(\boldsymbol{L}_1 \rho_t + \rho_t \boldsymbol{L}_1^\dagger\right) \rho_t) \\ + \sqrt{1-\eta}(\boldsymbol{L}_2 \rho_t + \rho_t \boldsymbol{L}_2^\dagger - \operatorname{Tr}\left(\boldsymbol{L}_2 \rho_t + \rho_t \boldsymbol{L}_2^\dagger\right) \rho_t)\Big)\, dW_t\end{aligned}$$

with

- $H_b = \frac{g\sqrt{\gamma}}{2}(\boldsymbol{\sigma_x}\boldsymbol{\sigma_z} + \boldsymbol{\sigma_z}\boldsymbol{\sigma_x}) = 0$
- $\boldsymbol{L}_1 = \sqrt{\gamma}\boldsymbol{\sigma_z} - i\sqrt{\eta} g_{u,l}\, \boldsymbol{\sigma_x}/2$
- $\boldsymbol{L}_2 = -i\sqrt{1-\eta} g_{u,l}\, \boldsymbol{\sigma_x}/2$.

Closed-loop equation: perfect case

For $\eta = 1$ we get the expected evolution:

$$\mathbb{E}(d\rho|\rho_t) = \Big(-\tfrac{i}{2}[H_0,\rho_t] + (\boldsymbol{L}_1\rho\boldsymbol{L}_1^\dagger - \boldsymbol{L}_1^\dagger\boldsymbol{L}_1\rho_t/2 - \rho_t\boldsymbol{L}_1^\dagger\boldsymbol{L}_1/2)\Big)\,dt$$

with $\boldsymbol{L}_1 = \sqrt{\gamma}\boldsymbol{\sigma_z} - i\sqrt{\eta}g_{u,l}\,\boldsymbol{\sigma_x}/2$.

This is a canonical Lindblad master equation with decoherence operator $\boldsymbol{L}_1$ tunable through $g_{u,l}$.

For instance taking $g_{u,l} = 2\sqrt{\gamma/\eta}$ we get

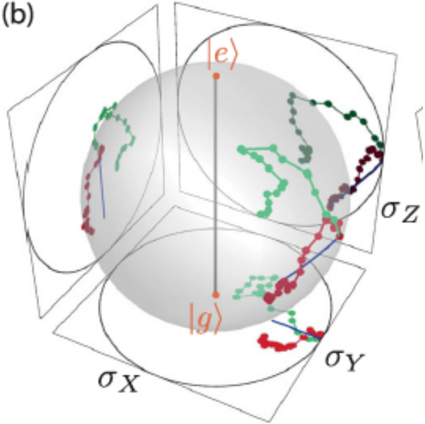
$$\boldsymbol{L}_1 = 2\sqrt{\gamma}\,U\,(|g\rangle\langle e|)\,U^\dagger = 2\sqrt{\gamma}\,U\,\boldsymbol{\sigma_-}\,U^\dagger$$

with $U|g\rangle = (|e\rangle - i|g\rangle)/\sqrt{2}$ and $U|e\rangle = (|e\rangle + i|g\rangle)/\sqrt{2}$.

This closed-loop system stabilizes $|\psi\rangle = (|e\rangle - i|g\rangle)/\sqrt{2}$ much like $\boldsymbol{\sigma_-}$ stabilizes $|g\rangle$. Other $g_{u,l}$ allow to stabilize other states.

group of B.Huard, ENS Paris.

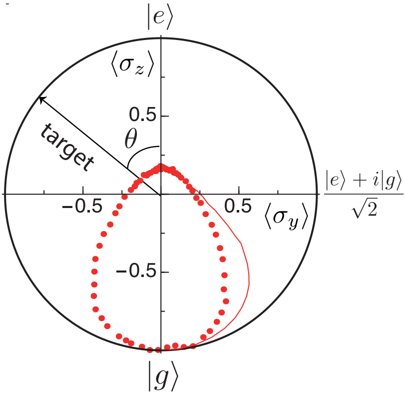
Measurement $\boldsymbol{L}$ operator: $\boldsymbol{\sigma_-}$ and $i\boldsymbol{\sigma_-}$ (fluorescence field) instead of $\boldsymbol{\sigma_z}$ and $i\boldsymbol{\sigma_z}$ (field sent to interact with the setup).



Open-loop: system always eventually converges to $|g\rangle$

group of B.Huard, ENS Paris.

Measurement $\boldsymbol{L}$ operator: $\boldsymbol{\sigma_-}$ and $i\boldsymbol{\sigma_-}$ (fluorescence field) instead of $\boldsymbol{\sigma_z}$ and $i\boldsymbol{\sigma_z}$ (field sent to interact with the setup).



Closed-loop: various states stabilized by Markovian feedback, $\eta = 0.35$.

The driven and damped classical oscillator

Dynamics in the (x', p') phase plane with $\omega \gg \kappa, \sqrt{u_1^2 + u_2^2}$:

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

Define the frame rotating at ω by $(x', p') \mapsto (x, p)$ with

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Removing highly oscillating terms (rotating wave approximation), from

$$\frac{d}{dt}x = -\kappa \sin^2(\omega t)x + 2u_1 \sin^2(\omega t) + (\kappa p - 2u_2) \sin(\omega t) \cos(\omega t)$$
$$\frac{d}{dt}p = -\kappa \cos^2(\omega t)p + 2u_2 \cos^2(\omega t) + (\kappa x - 2u_1) \sin(\omega t) \cos(\omega t)$$

we get, with $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}\alpha + u.$$

From $x' + ip' = \alpha' = e^{-i\omega t}\alpha$, we have $\frac{d}{dt}\alpha' = -(\frac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t}$

Driven and damped quantum oscillator ($n_{th} = 0$)

- The Lindblad master equation:

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

- Change of frame $\rho = \boldsymbol{D}_{\overline{\alpha}}\xi\boldsymbol{D}_{-\overline{\alpha}}$ with $\boldsymbol{D}_{\overline{\alpha}} = e^{\overline{\alpha}\boldsymbol{a}^\dagger - \overline{\alpha}^*\boldsymbol{a}}$. We get

$$\frac{d}{dt}\xi = \kappa\left(\boldsymbol{a}\xi\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\xi - \tfrac{1}{2}\xi\boldsymbol{a}^\dagger\boldsymbol{a}\right)$$

  since $\boldsymbol{D}_{-\overline{\alpha}}\boldsymbol{a}\boldsymbol{D}_{\overline{\alpha}} = \boldsymbol{a} + \overline{\alpha}$.

- Informal convergence proof with the strict Lyapunov function $V(\xi) = \mathrm{Tr}\left(\xi\boldsymbol{N}\right)$:

$$\frac{d}{dt}V(\xi) = -\kappa V(\xi) \Rightarrow V(\xi(t)) = V(\xi_0)e^{-\kappa t}.$$

  Since $\xi(t)$ is Hermitian and non-negative, $\xi(t)$ tends to $|0\rangle\langle 0|$ when $t \mapsto +\infty$.

Theorem

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

Assume that the initial state ρ_0 is a density operator with finite energy $Tr(\rho_0\boldsymbol{N}) < +\infty$. Then exists a unique solution to the Cauchy problem in the the Banach space $\mathcal{K}^1(\mathcal{H})$. It is defined for all $t > 0$ with $\rho(t)$ a density operator (Hermitian, non-negative and trace-class) that remains in the domain of the Lindblad super-operator

$$\rho \mapsto [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right).$$

This means that $t \mapsto \rho(t)$ is differentiable in the Banach space $\mathcal{K}^1(\mathcal{H})$. Moreover $\rho(t)$ converges for the trace-norm towards $|\overline{\alpha}\rangle\langle\overline{\alpha}|$ when t tends to $+\infty$, where $|\overline{\alpha}\rangle$ is the coherent state of complex amplitude $\overline{\alpha} = \frac{2u}{\kappa}$.

Lemma

Consider with $u \in \mathbb{C}$, $\kappa > 0$, the following Cauchy problem

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + \kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right), \quad \rho(0) = \rho_0.$$

1. *for any initial density operator ρ_0 with $Tr(\rho_0 \boldsymbol{N}) < +\infty$, we have $\frac{d}{dt}\alpha = -\frac{\kappa}{2}(\alpha - \overline{\alpha})$ where $\alpha = Tr(\rho\boldsymbol{a})$.*
2. *Assume that $\rho_0 = |\beta_0\rangle\langle\beta_0|$ where β_0 is some complex amplitude. Then for all $t \geq 0$, $\rho(t) = |\beta(t)\rangle\langle\beta(t)|$ remains a coherent state of amplitude $\beta(t)$ solution of the following equation: $\frac{d}{dt}\beta = -\frac{\kappa}{2}(\beta - \overline{\alpha})$ with $\beta(0) = \beta_0$.*

Statement 2 relies on:

$$\boldsymbol{a}|\beta\rangle = \beta|\beta\rangle, \quad |\beta\rangle = e^{-\frac{\beta\beta^*}{2}} e^{\beta\boldsymbol{a}^\dagger}|0\rangle \quad \frac{d}{dt}|\beta\rangle = \left(-\tfrac{1}{2}(\beta^*\dot{\beta} + \beta\dot{\beta}^*) + \dot{\beta}\boldsymbol{a}^\dagger\right)|\beta\rangle.$$

Driven and damped quantum oscillator with thermal photon

Parameters $\omega \gg \kappa, |u|$ and $n_{\text{th}} \geq 0$:

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\rho\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\rho - \tfrac{1}{2}\rho\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

Key issue: $\lim_{t\mapsto+\infty}\rho(t) = ?$.
The passage to another representation via the Wigner function:

- Since $\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}$ bounded and Hermitian operator (the dual of $\mathcal{K}^1(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$),

  $$W^{\{\rho\}}(x,p) = \tfrac{2}{\pi}\,\text{Tr}\left(\rho\boldsymbol{D}_\alpha e^{i\pi\boldsymbol{N}}\boldsymbol{D}_{-\alpha}\right) \quad \text{with} \quad \alpha = x+ip \in \mathbb{C},$$

  defines a real and bounded function $|W^{\{\rho\}}(x,p)| \leq \frac{2}{\pi}$.
- For a coherent state $\rho = |\beta\rangle\langle\beta|$ with $\beta \in \mathbb{C}$:

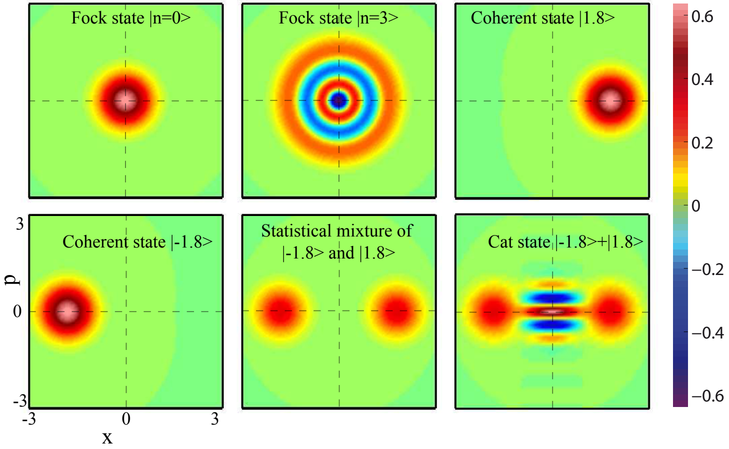
  $$W^{\{|\beta\rangle\langle\beta|\}}(x,p) = \tfrac{2}{\pi}e^{-2|\beta-(x+ip)|^2}.$$

Coherent state of amplitude $\beta \in \mathbb{C}$: $|\beta\rangle = \sum_{n\geq 0}\left(e^{-|\beta|^2/2}\frac{\beta^n}{\sqrt{n!}}\right)|n\rangle$;

Phase-cat states: $\mathcal{N}\left(|\beta\rangle + |-\beta\rangle\right)$.

Wigner function W^ρ associated ρ:

$W^\rho : \mathbb{C} \ni x + ip \to \frac{2}{\pi}\,\mathrm{Tr}\left(\rho \boldsymbol{D}_{x+ip} e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-(x+ip)}\right)$



The partial differential equation satisfied by the Wigner function (1)

With $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{-\alpha\alpha^*/2} = e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger} e^{\alpha\alpha^*/2}$ we have:

$$\tfrac{\pi}{2} W^{\{\rho\}}(\alpha, \alpha^*) = \text{ Tr}\left(\rho e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} e^{i\pi \boldsymbol{N}} e^{\alpha^* \boldsymbol{a}} e^{-\alpha \boldsymbol{a}^\dagger}\right)$$

where α and α^* are seen as independent variables:

$$\frac{\partial}{\partial \alpha} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial p}\right), \quad \frac{\partial}{\partial \alpha^*} = \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial p}\right)$$

We have $\frac{\pi}{2} \frac{\partial}{\partial \alpha} W^{\{\rho\}}(\alpha, \alpha^*) = \text{ Tr}\left((\rho \boldsymbol{a}^\dagger - \boldsymbol{a}^\dagger \rho) \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha}\right)$ Since $\boldsymbol{a}^\dagger \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha} = \boldsymbol{D}_\alpha e^{i\pi \boldsymbol{N}} \boldsymbol{D}_{-\alpha} (2\alpha^* - \boldsymbol{a}^\dagger)$, we get

$$\frac{\partial}{\partial \alpha} W^{\{\rho\}}(\alpha, \alpha^*) = 2\alpha^* W^{\{\rho\}}(\alpha, \alpha^*) - 2 W^{\{\boldsymbol{a}^\dagger \rho\}}(\alpha, \alpha^*).$$

Thus $W^{\{\boldsymbol{a}^\dagger \rho\}}(\alpha, \alpha^*) = \alpha^* W^{\{\rho\}}(\alpha, \alpha^*) - \frac{1}{2} \frac{\partial}{\partial \alpha} W^{\{\rho\}}(\alpha, \alpha^*)$, i.e.

$$W^{\{\boldsymbol{a}^\dagger \rho\}} = \left(\alpha^* - \tfrac{1}{2} \frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}.$$

Similar computations yield to the following correspondence rules:

$$W^{\{\rho \boldsymbol{a}\}} = \left(\alpha - \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}, \quad W^{\{\boldsymbol{a}\rho\}} = \left(\alpha + \tfrac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W^{\{\rho\}}$$

$$W^{\{\rho \boldsymbol{a}^\dagger\}} = \left(\alpha^* + \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}, \quad W^{\{\boldsymbol{a}^\dagger \rho\}} = \left(\alpha^* - \tfrac{1}{2}\frac{\partial}{\partial \alpha}\right) W^{\{\rho\}}.$$

Thus

$$\frac{d}{dt}\rho = [u\boldsymbol{a}^\dagger - u^*\boldsymbol{a}, \rho] + (1+n_{\text{th}})\kappa\left(\boldsymbol{a}\rho\boldsymbol{a}^\dagger - \tfrac{1}{2}\boldsymbol{a}^\dagger\boldsymbol{a}\rho - \tfrac{1}{2}\rho\boldsymbol{a}^\dagger\boldsymbol{a}\right) + n_{\text{th}}\kappa\left(\boldsymbol{a}^\dagger\rho\boldsymbol{a} - \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a}^\dagger\rho - \tfrac{1}{2}\rho\boldsymbol{a}\boldsymbol{a}^\dagger\right).$$

becomes

$$\frac{\partial}{\partial t}W^{\{\rho\}} = \frac{\kappa}{2}\left(\frac{\partial}{\partial \alpha}(\alpha - \overline{\alpha}) + \frac{\partial}{\partial \alpha^*}(\alpha^* - \overline{\alpha}^*) + (1+2n_{\text{th}})\frac{\partial^2}{\partial \alpha \partial \alpha^*}\right) W^{\{\rho\}}$$

Solutions of the quantum Fokker-Planck equation

Since the Green function of

$$\frac{\partial}{\partial t} W^{\{\rho\}} = \frac{\kappa}{2}\Big(\frac{\partial}{\partial x}\Big((x-\overline{x}) W^{\{\rho\}}\Big) + \frac{\partial}{\partial p}\Big((p-\overline{p}) W^{\{\rho\}}\Big) + \tfrac{1+2n_{\text{th}}}{4}\left(\frac{\partial^2 W^{\{\rho\}}}{\partial x^2} + \frac{\partial^2 W^{\{\rho\}}}{\partial p^2}\right)\Big)$$

is the following time-varying Gaussian function

$$G(x,p,t,x_0,p_0) = \frac{\exp\left(- \frac{\left(x-\overline{x}-(x_0-\overline{x})e^{-\frac{\kappa t}{2}}\right)^2 + \left(p-\overline{p}-(p_0-\overline{p})e^{-\frac{\kappa t}{2}}\right)^2}{(n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}\right)}{\pi (n_{\text{th}}+\frac{1}{2})(1-e^{-\kappa t})}$$

we can compute $W_t^{\{\rho\}}$ from $W_0^{\{\rho\}}$ for all $t > 0$:

$$W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p') G(x,p,t,x',p')\, dx' dp'.$$

Combining

- $W_t^{\{\rho\}}(x,p) = \int_{\mathbb{R}^2} W_0^{\{\rho\}}(x',p') G(x,p,t,x',p')\, dx'dp'$.
- G uniformly bounded and $\lim_{t\mapsto+\infty} G(x,p,t,x',p') = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right)$
- $W_0^{\{\rho\}}$ in L^1 with $\iint_{\mathbb{R}^2} W_0^{\{\rho\}} = 1$
- dominate convergence theorem

shows that all the solutions converge to a unique steady-state Gaussian density function, centered in $(\overline{x},\overline{p})$ with variance $\frac{1}{2}+n_{\text{th}}$:

$$\forall (x,p)\in\mathbb{R}^2, \quad \lim_{t\mapsto+\infty} W_t^{\{\rho\}}(x,p) = \frac{1}{\pi(n_{\text{th}}+\frac{1}{2})} \exp\left(-\frac{(x-\overline{x})^2+(p-\overline{p})^2}{(n_{\text{th}}+\frac{1}{2})}\right).$$

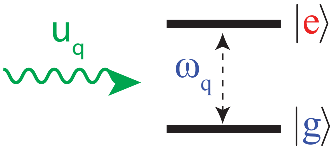
Friday exercise

Two-photon losses for the quantum harmonic oscillator correspond to $\rho(t)$ governed by $\frac{d}{dt}\rho = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho + \rho\boldsymbol{L}^\dagger\boldsymbol{L}) \triangleq \mathcal{L}(\rho), \quad \rho(0) = \rho_0$ with $\boldsymbol{L} = \boldsymbol{a}^2$. We recall that for any scalar function f, $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+1)\boldsymbol{a}$, and that for any integer $n \geq 1$, $\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\boldsymbol{a}|0\rangle = 0$ ($\left(|n\rangle\right)_{n\in\mathbb{N}}$ is the Hilbert basis corresponding to photon-number states).

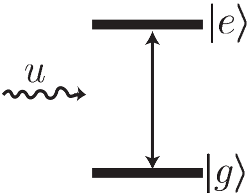
1. Show that $\boldsymbol{L}^\dagger\boldsymbol{L} = \boldsymbol{N}(\boldsymbol{N}-1)$. Set $p_n = \langle n|\rho|n\rangle$ for $n \geq 0$. Show that $\frac{d}{dt}p_n = (n+1)(n+2)p_{n+2} - n(n-1)p_n$. Deduce that the density operators $\bar{\rho}$ such that $\mathcal{L}(\bar{\rho}) = 0$ have their supports in span($|0\rangle, |1\rangle$):
$\exists \bar{p}_0 \in [0,1], \exists\, c \in \mathbb{C},\ \bar{\rho} = \bar{p}_0|0\rangle\langle 0| + (1-\bar{p}_0)|1\rangle\langle 1| + \bar{c}|1\rangle\langle 0| + \bar{c}^*|0\rangle\langle 1|$.
2. For any operator J (not necessarily Hermitian) prove that $\frac{d}{dt}\left(\text{Tr}\,(\rho J)\right) = \text{Tr}\left(\rho\mathcal{L}^*(J)\right)$ where $\mathcal{L}^*(J) = \boldsymbol{L}^\dagger J\boldsymbol{L} - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}J + J\boldsymbol{L}^\dagger\boldsymbol{L})$.
3. For any increasing scalar function f, prove that $\mathcal{L}^*(f(\boldsymbol{N})) \leq 0$. Deduce that $V(\rho) = \text{Tr}\,(N\rho)$ is a Lyapunov function and prove that, formally, for any initial density operator ρ_0, $\lim_{t\mapsto+\infty}\rho(t)$ exists and corresponds to a steady state $\bar{\rho}$ characterized in question 1. Show that $\bar{\rho}$ depends linearly on the initial condition ρ_0. Such dependence is denoted by $\bar{\rho} = \boldsymbol{K}(\rho_0)$.The remaining part of the exercise consists in providing an explicit formulation of this map.
4. An operator J is said to be invariant iff $\mathcal{L}^*(J) = 0$. Show that, for any invariant operator J, $\text{Tr}\,(\rho J)$ is a first integral.
5. Prove that $f(\boldsymbol{N})$ is an invariant operator if f is 2-periodic. Show that $J_0 = \sum_{n\geq 0}|2n\rangle\langle 2n|$ is invariant and deduce that $\langle 0|\boldsymbol{K}(\rho_0)|0\rangle = \text{Tr}\,(J_0\rho_0)$ and $\langle 1|\boldsymbol{K}(\rho_0)|1\rangle = 1 - \text{Tr}\,(J_0\rho_0)$.
6. Prove that $f(\boldsymbol{N})\boldsymbol{a}$ is an invariant operator if $f(1) = 0$ and for all integer $n \geq 2$ we have $nf(n) = (n-1)f(n-2)$.
7. Consider a real function f such that $f(0) = 1$ and, for all $n \geq 1$, $f(2n-1) = 0$ with $f(2n) = \prod_{k=1}^{n}\frac{2k-1}{2k}$. Check that $J_1 = f(\boldsymbol{N})\boldsymbol{a}$ is a bounded and invariant operator. Deduce that
$$\text{Tr}\,(\rho_0 J_1) = \sum_{n\geq 0}\sqrt{2n+1}f(2n)\langle 2n+1|\rho_0|2n\rangle = \langle 1|\boldsymbol{K}(\rho_0)|0\rangle.$$
8. Conclude that
$$\boldsymbol{K}(\rho_0) = \text{Tr}\,(J_0\rho_0)\,|0\rangle\langle 0| + \left(1 - \text{Tr}\,(J_0\rho_0)\right)|1\rangle\langle 1| + \text{Tr}\,(\rho_0 J_1)\,|1\rangle\langle 0| + \text{Tr}\left(\rho_0 J_1^\dagger\right)|0\rangle\langle 1|.$$

2-level system, i.e. a qubit (half-spin system)

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g|g\rangle + c_e|e\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}\ .$
- Operators and commutations:
  $\boldsymbol{\sigma_-} = |g\rangle\langle e|$, $\boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = |e\rangle\langle g|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = |g\rangle\langle e| + |e\rangle\langle g|$;
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i|g\rangle\langle e| - i|e\rangle\langle g|$;
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = |e\rangle\langle e| - |g\rangle\langle g|$;
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I}$, $\boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}$, $[\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}$, ...



- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}$.
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \,\middle|\, (x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$



The simplest quantum system: a ground state $|g\rangle$ of energy ω_g; an excited state $|e\rangle$ of energy ω_e. The quantum state $|\psi\rangle \in \mathbb{C}^2$ is a linear superposition $|\psi\rangle = \psi_g|g\rangle + \psi_e|e\rangle$ and obey to the Schrödinger equation (ψ_g and ψ_e depend on t).

Schrödinger equation for the uncontrolled 2-level system ($\hbar = 1$) :

$$\imath \frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle = \big(\omega_e|e\rangle\langle e| + \omega_g|g\rangle\langle g|\big)\,|\psi\rangle$$

where $\boldsymbol{H}_0$ is the Hamiltonian, a Hermitian operator $\boldsymbol{H}_0^\dagger = \boldsymbol{H}_0$. Energy is defined up to a constant: $\boldsymbol{H}_0$ and $\boldsymbol{H}_0 + \varpi(t)\boldsymbol{I}$ ($\varpi(t) \in \mathbb{R}$ arbitrary) are attached to the same physical system. If $|\psi\rangle$ satisfies $i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}_0|\psi\rangle$ then $|\chi\rangle = e^{-i\vartheta(t)}|\psi\rangle$ with $\frac{d}{dt}\vartheta = \varpi$ obeys to $i\frac{d}{dt}|\chi\rangle = (\boldsymbol{H}_0 + \varpi\boldsymbol{I})|\chi\rangle$. Thus for any ϑ, $|\psi\rangle$ and $e^{-i\vartheta}|\psi\rangle$ represent the same physical system: The global phase of a quantum system $|\psi\rangle$ can be chosen arbitrarily at any time.

Take origin of energy such that ω_g (resp. ω_e) becomes $-\frac{\omega_e-\omega_g}{2}$ (resp. $\frac{\omega_e-\omega_g}{2}$) and set $\omega_{eg} = \omega_e - \omega_g$
The solution of $i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle = \frac{\omega_{eg}}{2}(|e\rangle\langle e| - |g\rangle\langle g|)|\psi\rangle$ is

$$|\psi\rangle_t = \psi_{g0} e^{\frac{i\omega_{eg}t}{2}}|g\rangle + \psi_{e0} e^{\frac{-i\omega_{eg}t}{2}}|e\rangle.$$

With a classical electromagnetic field described by $u(t) \in \mathbb{R}$, the coherent evolution the controlled Hamiltonian

$$\boldsymbol{H}(t) = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}+\frac{u(t)}{2}\boldsymbol{\sigma_x} = \frac{\omega_{eg}}{2}(|e\rangle\langle e|-|g\rangle\langle g|)+\frac{u(t)}{2}(|e\rangle\langle g|+|g\rangle\langle e|)$$

The controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = (\boldsymbol{H}_0 + u(t)\boldsymbol{H}_1)|\psi\rangle$ reads:

$$i\frac{d}{dt}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} = \frac{\omega_{eg}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix} + \frac{u(t)}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_g\end{pmatrix}.$$

The 3 Pauli Matrices[11]

$$\boldsymbol{\sigma_x} = |e\rangle\langle g|+|g\rangle\langle e|,\ \boldsymbol{\sigma_y} = -i|e\rangle\langle g|+i|g\rangle\langle e|,\ \boldsymbol{\sigma_z} = |e\rangle\langle e|-|g\rangle\langle g|$$

[11] They correspond, up to multiplication by i, to the 3 imaginary quaternions.

$$\boldsymbol{\sigma_x} = |e\rangle\langle g| + |g\rangle\langle e|, \ \boldsymbol{\sigma_y} = -i|e\rangle\langle g| + i|g\rangle\langle e|, \ \boldsymbol{\sigma_z} = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\boldsymbol{\sigma_x}^2 = \boldsymbol{I}, \quad \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z}, \quad [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z}, \ \text{circular permutation } \ldots$$

- Since for any $\theta \in \mathbb{R}$, $e^{i\theta\boldsymbol{\sigma_x}} = \cos\theta + i\sin\theta\boldsymbol{\sigma_x}$ (idem for $\boldsymbol{\sigma_y}$ and $\boldsymbol{\sigma_z}$), the solution of $i\frac{d}{dt}|\psi\rangle = \frac{\omega_{eg}}{2}\boldsymbol{\sigma_z}|\psi\rangle$ is

$$|\psi\rangle_t = e^{\frac{-i\omega_{eg}t}{2}\boldsymbol{\sigma_z}}|\psi\rangle_0 = \left(\cos\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{I} - i\sin\left(\frac{\omega_{eg}t}{2}\right)\boldsymbol{\sigma_z}\right)|\psi\rangle_0$$

- For $\alpha, \beta = x, y, z$, $\alpha \neq \beta$ we have

$$\boldsymbol{\sigma_\alpha} e^{i\theta\boldsymbol{\sigma_\beta}} = e^{-i\theta\boldsymbol{\sigma_\beta}}\boldsymbol{\sigma_\alpha}, \qquad \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^{-1} = \left(e^{i\theta\boldsymbol{\sigma_\alpha}}\right)^\dagger = e^{-i\theta\boldsymbol{\sigma_\alpha}}.$$

  and also

$$e^{-\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta}e^{\frac{i\theta}{2}\boldsymbol{\sigma_\alpha}} = e^{-i\theta\boldsymbol{\sigma_\alpha}}\boldsymbol{\sigma_\beta} = \boldsymbol{\sigma_\beta}e^{i\theta\boldsymbol{\sigma_\alpha}}$$

Qubit model: Bloch sphere representation

ρ is a nonnegative Hermitian operator on span($|g\rangle, |e\rangle$) $\simeq \mathbb{C}^2$ such that $\text{Tr}(\rho) = 1$

We can write any such ρ as

$$\rho = \frac{\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}}{2}$$

and ρ positive is equivalent to $\text{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 \leq 1$. We have

$$x = \text{Tr}\left(\boldsymbol{\sigma_x}\rho\right), \; y = \text{Tr}\left(\boldsymbol{\sigma_y}\rho\right) \text{ and } z = \text{Tr}\left(\boldsymbol{\sigma_z}\rho\right).$$

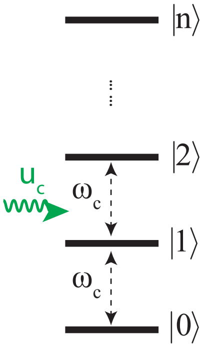
Thus ρ can be represented by $(x, y, z) \in \mathbb{R}^3$, cartesian coordinates of vector $\vec{M}$ inside the Bloch sphere ($\text{Tr}\left(\rho^2\right) = x^2 + y^2 + z^2 \leq 1$):

$$\frac{d}{dt}\rho_t = -\tfrac{i}{2}[u\boldsymbol{\sigma_x} + v\boldsymbol{\sigma_y}, \rho_t] \quad \Leftrightarrow \quad \frac{d}{dt}\vec{M} = (u\vec{e}_x + v\vec{e}_y) \times \vec{M}.$$

Here u and v stand for the rotation speed around x-axis and y-axis.

Quantum harmonic oscillator (spring system)

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n |n\rangle,\ (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho,\ \mathrm{Tr}(\rho) = 1, \rho \geq 0\}$.
- Operators and commutations:
  $\boldsymbol{a}|n\rangle = \sqrt{n}\ |\boldsymbol{n\text{-}1}\rangle$, $\boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N}|n\rangle = n|n\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = i\boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).
- Classical pure state $\equiv$ coherent state $|\alpha\rangle$
  $\alpha \in \mathbb{C}:\ |\alpha\rangle = \sum_{n\geq 0}\left(e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)|n\rangle$; $|\alpha\rangle \equiv \frac{1}{\pi^{1/4}} e^{i\sqrt{2}x\Im\alpha} e^{-\frac{(x-\sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\boldsymbol{D}_\alpha|0\rangle = |\alpha\rangle$.

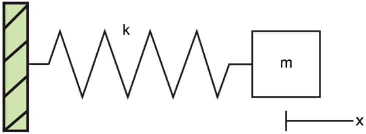


Harmonic oscillator

Classical Hamiltonian formulation of $\frac{d^2}{dt^2}x = -\omega^2 x$

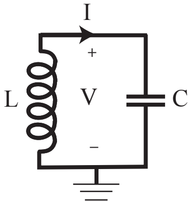
$$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$$

Mechanical oscillator



Frictionless spring: $\frac{d^2}{dt^2}x = -\frac{k}{m}x$.

Electrical oscillator:



LC oscillator:

$$\frac{d}{dt}I = \frac{V}{L}, \frac{d}{dt}V = -\frac{I}{C}, \quad \left(\frac{d^2}{dt^2}I = -\frac{1}{LC}I\right).$$

Quantum regime

$k_B T \ll \hbar\omega$: typically for the photon box experiment in these lectures, $\omega = 51 GHz$ and $T = 0.8K$.

$\frac{d}{dt}x = \omega p = \frac{\partial \mathbb{H}}{\partial p}, \quad \frac{d}{dt}p = -\omega x = -\frac{\partial \mathbb{H}}{\partial x}, \quad \mathbb{H} = \frac{\omega}{2}(p^2 + x^2).$

Quantization: probability wave function $|\psi\rangle_t \sim (\psi(x,t))_{x\in\mathbb{R}}$ with $|\psi\rangle_t \sim \psi(.,t) \in L^2(\mathbb{R},\mathbb{C})$ obeys to the Schrödinger equation ($\hbar = 1$ in all the lectures)

$$i\frac{d}{dt}|\psi\rangle = \boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$$

where $\boldsymbol{H}$ results from $\mathbb{H}$ by replacing x by position operator $\sqrt{2}\boldsymbol{X}$ and p by momentum operator $\sqrt{2}\boldsymbol{P} = -i\frac{\partial}{\partial x}$. $\boldsymbol{H}$ is a Hermitian operator on $L^2(\mathbb{R},\mathbb{C})$, with its domain to be given.

PDE model: $i\frac{\partial\psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2\psi}{\partial x^2}(x,t) + \frac{\omega}{2}x^2\psi(x,t), \quad x \in \mathbb{R}.$

[12] Two references: C. Cohen-Tannoudji, B. Diu, and F. Laloë. *Mécanique Quantique*, volume I& II. Hermann, Paris, 1977.
M. Barnett and P. M. Radmore. *Methods in Theoretical Quantum Optics*. Oxford University Press, 2003.

Harmonic oscillator: annihilation and creation operators

Average position $\langle \boldsymbol{X} \rangle_t = \langle \psi | \boldsymbol{X} | \psi \rangle$ and momentum $\langle \boldsymbol{P} \rangle_t = \langle \psi | \boldsymbol{P} | \psi \rangle$:

$$\langle \boldsymbol{X} \rangle_t = \tfrac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} x |\psi|^2 dx, , \quad \langle \boldsymbol{P} \rangle_t = -\tfrac{i}{\sqrt{2}} \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx.$$

Annihilation $\boldsymbol{a}$ and creation operators $\boldsymbol{a}^\dagger$ (domains to be given):

$$\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right), \quad \boldsymbol{a}^\dagger = \boldsymbol{X} - i\boldsymbol{P} = \tfrac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right)$$

Commutation relationships:

$$[\boldsymbol{X}, \boldsymbol{P}] = \tfrac{i}{2} \boldsymbol{I}, \quad [\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}, \quad \boldsymbol{H} = \omega(\boldsymbol{P}^2 + \boldsymbol{X}^2) = \omega \left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{1}{2} \right).$$

Set $\boldsymbol{X}_\lambda = \tfrac{1}{2} \left(e^{-i\lambda} \boldsymbol{a} + e^{i\lambda} \boldsymbol{a}^\dagger \right)$ for any angle λ:

$$\left[\boldsymbol{X}_\lambda, \boldsymbol{X}_{\lambda + \frac{\pi}{2}} \right] = \tfrac{i}{2} \boldsymbol{I}.$$

Spectrum of Hamiltonian $\boldsymbol{H} = -\frac{\omega}{2}\frac{\partial^2}{\partial x^2} + \frac{\omega}{2}x^2$:

$$E_n = \omega(n+\frac{1}{2}),\ \ \psi_n(x) = \left(\frac{1}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x),\ \ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Spectral decomposition of $\boldsymbol{a}^\dagger \boldsymbol{a}$ using $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$:

- If $|\psi\rangle$ is an eigenstate associated to eigenvalue λ, $\boldsymbol{a}|\psi\rangle$ and $\boldsymbol{a}^\dagger|\psi\rangle$ are also eigenstates associated to $\lambda - 1$ and $\lambda + 1$.
- $\boldsymbol{a}^\dagger \boldsymbol{a}$ is semi-definite positive.
- The ground state $|\psi_0\rangle$ is necessarily associated to eigenvalue 0 and is given by the Gaussian function $\psi_0(x) = \frac{1}{\pi^{1/4}} \exp(-x^2/2)$.

Harmonic oscillator: spectral decomposition and Fock states

$[\boldsymbol{a}, \boldsymbol{a}^\dagger] = 1$: spectrum of $\boldsymbol{a}^\dagger \boldsymbol{a}$ is non-degenerate and is $\mathbb{N}$.

Fock state with n photons (phonons): the eigenstate of $\boldsymbol{a}^\dagger \boldsymbol{a}$ associated to the eigenvalue n ($|n\rangle \sim \psi_n(x)$):

$$\boldsymbol{a}^\dagger \boldsymbol{a}|n\rangle = n|n\rangle, \quad \boldsymbol{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \boldsymbol{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

The ground state $|0\rangle$ is called 0-photon state or vacuum state.

The operator $\boldsymbol{a}$ (resp. $\boldsymbol{a}^\dagger$) is the annihilation (resp. creation) operator since it transfers $|n\rangle$ to $|n-1\rangle$ (resp. $|n+1\rangle$) and thus decreases (resp. increases) the quantum number n by one unit.

Hilbert space of quantum system: $\mathcal{H} = \{\sum_n c_n|n\rangle \mid (c_n) \in l^2(\mathbb{C})\} \sim L^2(\mathbb{R}, \mathbb{C})$.
Domain of $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^1(\mathbb{C})\}$.
Domain of $\boldsymbol{H}$ ot $\boldsymbol{a}^\dagger \boldsymbol{a}$: $\{\sum_n c_n|n\rangle \mid (c_n) \in h^2(\mathbb{C})\}$.

$$h^k(\mathbb{C}) = \{(c_n) \in l^2(\mathbb{C}) \mid \sum n^k |c_n|^2 < \infty\}, \qquad k = 1, 2.$$

Quantization of $\frac{d^2}{dt^2}x = -\omega^2 x - \omega\sqrt{2}u$, ($\mathbb{H} = \frac{\omega}{2}(p^2 + x^2) + \sqrt{2}ux$)

$$\boldsymbol{H} = \omega\left(\boldsymbol{a}^\dagger \boldsymbol{a} + \frac{1}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger).$$

The associated controlled PDE

$$i\frac{\partial \psi}{\partial t}(x,t) = -\frac{\omega}{2}\frac{\partial^2 \psi}{\partial x^2}(x,t) + \left(\tfrac{\omega}{2}x^2 + \sqrt{2}ux\right)\psi(x,t).$$

Glauber displacement operator $\boldsymbol{D}_\alpha$ (unitary) with $\alpha \in \mathbb{C}$:

$$\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^* \boldsymbol{a}} = e^{2i\Im\alpha \boldsymbol{X} - 2i\Re\alpha \boldsymbol{P}}$$

From Baker-Campbell Hausdorf formula, for all operators $\boldsymbol{A}$ and $\boldsymbol{B}$,

$$e^{\boldsymbol{A}}\boldsymbol{B}e^{-\boldsymbol{A}} = \boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \tfrac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \tfrac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots$$

we get the Glauber formula[13] when $[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] = [\boldsymbol{B},[\boldsymbol{A},\boldsymbol{B}]] = 0$:

$$e^{\boldsymbol{A}+\boldsymbol{B}} = e^{\boldsymbol{A}}\, e^{\boldsymbol{B}}\, e^{-\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]}.$$

[13] Take s derivative of $e^{s(\boldsymbol{A}+\boldsymbol{B})}$ and of $e^{s\boldsymbol{A}}\, e^{s\boldsymbol{B}}\, e^{-\frac{s^2}{2}[\boldsymbol{A},\boldsymbol{B}]}$.

With $\boldsymbol{A} = \alpha \boldsymbol{a}^\dagger$ and $\boldsymbol{B} = -\alpha^* \boldsymbol{a}$, Glauber formula gives:

$$\boldsymbol{D}_\alpha = e^{-\frac{|\alpha|^2}{2}} \; e^{\alpha \boldsymbol{a}^\dagger} e^{-\alpha^* \boldsymbol{a}} = e^{+\frac{|\alpha|^2}{2}} \; e^{-\alpha^* \boldsymbol{a}} e^{\alpha \boldsymbol{a}^\dagger}$$

$$\boldsymbol{D}_{-\alpha} \boldsymbol{a} \boldsymbol{D}_\alpha = \boldsymbol{a} + \alpha \boldsymbol{I} \quad \text{and} \quad \boldsymbol{D}_{-\alpha} \boldsymbol{a}^\dagger \boldsymbol{D}_\alpha = \boldsymbol{a}^\dagger + \alpha^* \boldsymbol{I}.$$

With $\boldsymbol{A} = 2i\Im\alpha \boldsymbol{X} \sim i\sqrt{2}\Im\alpha x$ and $\boldsymbol{B} = -2\imath\Re\alpha \boldsymbol{P} \sim -\sqrt{2}\Re\alpha \frac{\partial}{\partial x}$, Glauber formula gives[14]:

$$\boldsymbol{D}_\alpha = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} e^{-\sqrt{2}\Re\alpha \frac{\partial}{\partial x}}$$

$$(\boldsymbol{D}_\alpha |\psi\rangle)_{x,t} = e^{-i\Re\alpha\Im\alpha} \; e^{i\sqrt{2}\Im\alpha x} \psi(x - \sqrt{2}\Re\alpha, t)$$

Exercise: Prove that, for any $\alpha, \beta, \epsilon \in \mathbb{C}$, we have

$$\boldsymbol{D}_{\alpha+\beta} = e^{\frac{\alpha^*\beta - \alpha\beta^*}{2}} \boldsymbol{D}_\alpha \boldsymbol{D}_\beta$$

$$\boldsymbol{D}_{\alpha+\epsilon} \boldsymbol{D}_{-\alpha} = \left(1 + \frac{\alpha\epsilon^* - \alpha^*\epsilon}{2}\right) \boldsymbol{I} + \epsilon \boldsymbol{a}^\dagger - \epsilon^* \boldsymbol{a} + \boldsymbol{O}(|\epsilon|^2)$$

$$\left(\frac{d}{dt} \boldsymbol{D}_\alpha\right) \boldsymbol{D}_{-\alpha} = \left(\frac{\alpha \frac{d}{dt}\alpha^* - \alpha^* \frac{d}{dt}\alpha}{2}\right) \boldsymbol{I} + \left(\frac{d}{dt}\alpha\right) \boldsymbol{a}^\dagger - \left(\frac{d}{dt}\alpha^*\right) \boldsymbol{a}.$$

[14] Note that the operator $e^{-r\partial/\partial x}$ corresponds to a translation of x by r.

Harmonic oscillator: lack of controllability

Take $|\psi\rangle$ solution of the controlled Schrödinger equation $i\frac{d}{dt}|\psi\rangle = \left(\omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{1}{2}\right) + u(\boldsymbol{a} + \boldsymbol{a}^\dagger)\right)|\psi\rangle$. Set $\langle\boldsymbol{a}\rangle = \langle\psi|\boldsymbol{a}|\psi\rangle$. Then

$$\frac{d}{dt}\langle\boldsymbol{a}\rangle = -i\omega\langle\boldsymbol{a}\rangle - iu.$$

From $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P}$, we have $\langle\boldsymbol{a}\rangle = \langle\boldsymbol{X}\rangle + i\langle\boldsymbol{P}\rangle$ where $\langle\boldsymbol{X}\rangle = \langle\psi|\boldsymbol{X}|\psi\rangle \in \mathbb{R}$ and $\langle\boldsymbol{P}\rangle = \langle\psi|\boldsymbol{P}|\psi\rangle \in \mathbb{R}$. Consequently:

$$\frac{d}{dt}\langle\boldsymbol{X}\rangle = \omega\langle\boldsymbol{P}\rangle, \quad \frac{d}{dt}\langle\boldsymbol{P}\rangle = -\omega\langle\boldsymbol{X}\rangle - u.$$

Consider the change of frame $|\psi\rangle = e^{-i\theta_t}\boldsymbol{D}_{\langle\boldsymbol{a}\rangle_t}|\chi\rangle$ with

$$\theta_t = \int_0^t \left(\omega|\langle\boldsymbol{a}\rangle|^2 + u\Re(\langle\boldsymbol{a}\rangle)\right), \quad D_{\langle\boldsymbol{a}\rangle_t} = e^{\langle\boldsymbol{a}\rangle_t\boldsymbol{a}^\dagger - \langle\boldsymbol{a}\rangle_t^*\boldsymbol{a}},$$

Then $|\chi\rangle$ obeys to autonomous Schrödinger equation

$$i\frac{d}{dt}|\chi\rangle = \omega\left(\boldsymbol{a}^\dagger\boldsymbol{a} + \frac{\boldsymbol{I}}{2}\right)|\chi\rangle.$$

The dynamics of $|\psi\rangle$ can be decomposed into two parts:

- a controllable part of dimension two for $\langle\boldsymbol{a}\rangle$
- an uncontrollable part of infinite dimension for $|\chi\rangle$.

Coherent states

$$|\alpha\rangle = \boldsymbol{D}_\alpha|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha \in \mathbb{C}$$

are the states reachable from vacuum set. They are also the eigenstate of $\boldsymbol{a}$: $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$.
A widely known result in quantum optics[15]: classical currents and sources (generalizing the role played by u) only generate classical light (quasi-classical states of the quantized field generalizing the coherent state introduced here)
We just propose here a control theoretic interpretation in terms of reachable set from vacuum.

[15] See complement B_{III}, page 217 of C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1989.