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M2 Mathématiques & Applications
UE (ANEDP, COCV): Analyse et contrôle de systèmes quantiques
Contrôle des connaissances, durée 2 heures.
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Second-order averaging

We consider a particular type of interaction between two harmonic oscillators modeled by the following time-dependent Hamiltonian:

$$H(t) = \omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \omega_b \boldsymbol{b}^\dagger \boldsymbol{b} - \frac{\chi_{bb}}{2}\boldsymbol{b}^{\dagger 2}\boldsymbol{b}^2 + (g_1 e^{-i\omega_1 t}\boldsymbol{a}^{\dagger 2}\boldsymbol{b} + \text{h.c.}) + (g_2 e^{-i\omega_2 t}\boldsymbol{a}^{\dagger 2}\boldsymbol{b} + \text{h.c.}),$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the annihilation operators of the two harmonic oscillators, $\omega_1 = 2\omega_a - \omega_b - \Delta$ and $\omega_2 = 2\omega_a - \omega_b + \chi_{bb} + \Delta$. We also assume $|g_1|, |g_2| \ll |\Delta|, |\Delta + \chi_{bb}|$.

1. Write the Schrödinger equation in the rotating frame of $\boldsymbol{H}_0 = \omega_a \boldsymbol{a}^\dagger \boldsymbol{a} + \omega_b \boldsymbol{b}^\dagger \boldsymbol{b} - \frac{\chi_{bb}}{2}\boldsymbol{b}^{\dagger 2}\boldsymbol{b}^2$, i.e. with $|\phi\rangle = e^{it\boldsymbol{H}_0}|\psi\rangle$ (we will calculate $\boldsymbol{H}_{\text{int}} = e^{it\boldsymbol{H}_0}(\boldsymbol{H}(t) - \boldsymbol{H}_0)e^{-it\boldsymbol{H}_0}$).

2. For simplicity sakes, we will truncate the mode $\boldsymbol{b}$ up to the Fock state $|2\rangle$ ($\boldsymbol{b} = |0\rangle\langle 1| + \sqrt{2}|1\rangle\langle 2|$, $\boldsymbol{b}^\dagger = |1\rangle\langle 0| + \sqrt{2}|2\rangle\langle 1|$, $\boldsymbol{b}^\dagger\boldsymbol{b} = |1\rangle\langle 1| + 2|2\rangle\langle 2|$). Perform rotating-wave approximations of 1st and 2nd order. Interpret the result.

Stabilization of Fock states by reservoir engineering

We consider here the LKB photon box with resonant interaction atom/cavity. Each atom is associated to the Hilbert space $\mathcal{H}_M = \mathbb{C}^2$ with orthonormal basis $\{|g\rangle, |e\rangle\}$. The photons trapped inside the cavity are associated to the infinite dimensional Hilbert space $\mathcal{H}_S$ of orthonormal basis $\{|n\rangle\}_{n\geq 0}$ where each $|n\rangle$ represents the Fock state associated to exactly n photons. The wave function $|\Psi\rangle$ describing the composite system atom/cavity lives in $\mathcal{H}_S \otimes \mathcal{H}_M$. The passage of an atom through the cavity corresponds to the Schrödinger propagator

$$U_\theta = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - |e\rangle\langle g| \otimes \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger$$

where $\boldsymbol{a}$ and $\boldsymbol{N} = \boldsymbol{a}^\dagger\boldsymbol{a}$ are the usual annihilation and photon-number operators and where θ is a parameter that can be tuned. The objective is to stabilize the system to the Fock state $|\overline{n}\rangle$, for some number $\overline{n} > 0$. The atoms are sent one by one though the cavity and are labeled by the integer k. We denote by $|\Psi_k\rangle$ the wave function before the passage of atom k and its measurement.

1. Compute $U_\theta^\dagger$, the Hermitian conjugate of U_θ and verify that $U_\theta U_\theta^\dagger = \boldsymbol{I}$.

2. Assume that $|\Psi_k\rangle = (\cos u|g\rangle + \sin u|e\rangle) \otimes |\psi_k\rangle$ where u is another parameter that can be tuned. Compute the value of $|\Psi\rangle$ just after the passage of atom k.

3. After its passage, atom k is measured with measurement outcome $y \in \{e, g\}$ (atom measurement operator $|e\rangle\langle e| - |g\rangle\langle g|$). Just after this measurement, $|\Psi\rangle$ becomes separable with a photon wave-function $|\psi_{k+1}\rangle$ given by

$$|\psi_{k+1}\rangle = \boldsymbol{M}_{y,\theta,u}|\psi_k\rangle \Big/ \sqrt{\langle\psi_k|\boldsymbol{M}_{y,\theta,u}^\dagger \boldsymbol{M}_{y,\theta,u}|\psi_k\rangle}.$$

   Give the explicit formulae for $\boldsymbol{M}_{y,\theta,u}$ and the probabilities to detect $y = g$ and $y = e$ knowing $|\psi_k\rangle$

4. Denote by $\rho_k \sim |\psi_k\rangle\langle\psi_k|$ the density operator for the photons before the passage of atom k. Denote by ρ_{k+1} the expectation value of $|\psi_{k+1}\rangle\langle\psi_{k+1}|$ knowing ρ_k. Compute the explicit formula of map $\boldsymbol{K}_{\theta,u}$ defined by $\rho_{k+1} = \boldsymbol{K}_{\theta,u}(\rho_k)$.

5. We assume in this question that $u = 0$ and $\theta = \pi/\sqrt{\overline{n}}$.

   (a) Set $p_k^n = \langle n|\rho_k|n\rangle$ and $p_k = (p_k^n)_{n\geq 0}$. Show that

   $$p_{k+1}^n = \cos^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right)p_k^n + \sin^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}}}\right)p_k^{n+1}.$$

   (b) Show that if $p_k^n \geq 0$ with $\sum_{n\geq 0} p_k^n = 1$, then $p_{k+1}^n \geq 0$ with $\sum_{n\geq 0} p_{k+1}^n = 1$.

   (c) Take $n^* > 0$. Show that if $\forall n \geq n^*$, $p_k^n = 0$, then $\forall n \geq n^*$, $p_{k+1}^n = 0$.

   (d) Show that $p_{k+1} = p_k$ if and only if $p_k^n = 0$ for $n \notin \{r^2\overline{n} \mid r \geq 0\}$.

   (e) Show that if $\forall n < \overline{n}$, $p_k^n = 0$, then $\forall n < \overline{n}$, $p_{k+1}^n = 0$.

   (f) For any integer $r > 0$, prove that if $\forall n < r^2\overline{n}$, $p_k^n = 0$, then $\forall n < r^2\overline{n}$, $p_{k+1}^n = 0$.

   (g) Prove that for any initial value of the density operator ρ_0 satisfying $\forall n \geq \overline{n}$, $\rho_0|n\rangle = 0$, we have $\lim_{k\mapsto+\infty} \rho_k = |0\rangle\langle 0|$ (indication: prove first that p_k^n tends to $\delta_{n,0}$ and conclude for ρ_k using its positivity[1]).

   (h) Prove that for any initial value of the density operator ρ_0 satisfying $\forall n \in [0, \overline{n} - 1] \cup [4\overline{n}, +\infty[$, $\rho_0|n\rangle = 0$, we have $\lim_{k\mapsto+\infty} \rho_k = |\overline{n}\rangle\langle\overline{n}|$ (indication: prove first that p_k^n tends to $\delta_{n,\overline{n}}$ and conclude for ρ_k using his positivity).

6. We assume in this question that $u = \pi/2$ and $\theta = \pi/\sqrt{\overline{n}+1}$.

   (a) Set $p_k^n = \langle n|\rho_k|n\rangle$ and $p_k = (p_k^n)_{n\geq 0}$. Show that

   $$p_{k+1}^n = \sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}+1}}\right)p_k^{n-1} + \cos^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}+1}}\right)p_k^n.$$

   (b) Show that if $p_k^n \geq 0$ with $\sum_{n\geq 0} p_k^n = 1$, then $p_{k+1}^n \geq 0$ with $\sum_{n\geq 0} p_{k+1}^n = 1$.

   (c) Take $n_* > 0$. Show that if $\forall n < n_*$, $p_k^n = 0$, then $\forall n < n_*$, $p_{k+1}^n = 0$.

[1] Here δ_{n_1,n_2} stands for the Kronecker symbol: $\delta_{n_1,n_2} = 0$ for $n_1 \neq n_2$ and $\delta_{n,n} = 1$.

(d) Show that $p_{k+1} = p_k$ (with $p_k^n \geq 0$ and $\sum_{n\geq 0} p_k^n = 1$) if and only if $p_k^n = 0$ for $n \notin \{r^2(\overline{n}+1) - 1 \mid r \geq 1\}$.

(e) Show that if $\forall n > \overline{n}$, $p_k^n = 0$, then $\forall n > \overline{n}$, $p_{k+1}^n = 0$.

(f) For any integer $r > 0$, prove that if $\forall n > r^2(\overline{n}+1) - 1$, $p_k^n = 0$, then $\forall n > r^2(\overline{n}+1) - 1$, $p_{k+1}^n = 0$.

(g) Prove that for any initial value of the density operator ρ_0 satisfying $\forall n > \overline{n}$, $\rho_0|n\rangle = 0$, we have $\lim_{k\mapsto+\infty} \rho_k = |\overline{n}\rangle\langle\overline{n}|$.

(h) Prove that for any initial value of the density operator ρ_0 satisfying $\forall n \in [0, \overline{n}] \cup [4\overline{n}+4, +\infty[$, $\rho_0|n\rangle = 0$, we have $\lim_{k\mapsto+\infty} \rho_k = |4\overline{n}+3\rangle\langle 4\overline{n}+3|$.

7. Take $r \in]0, 1[$. For each atom k we consider a random number r_k in $[0, 1]$ (uniform law):
- if $r_k < r$ atom k is sent in excited state $|e\rangle$ $(u = \pi/2)$ with $\theta = \frac{\pi}{\sqrt{\overline{n}+1}}$;
- if $r_k > r$ atom k is sent in ground state $|g\rangle$ $(u = 0)$ with $\theta = \frac{\pi}{\sqrt{\overline{n}}}$.

(a) Compute the Kraux map $\boldsymbol{K}$ giving the expectation value of ρ_{k+1} knowing ρ_k: $\rho_{k+1} \triangleq \boldsymbol{K}(\rho_k)$.

(b) Set $p_k^n = \langle n|\rho_k|n\rangle$ and $p_k = (p_k^n)_{n\geq 0}$. Give the recurrence relation between p_{k+1} and p_k.

(c) Show that the unique solution to $p_{k+1} = p_k$ (with $p_k^n \geq 0$, $p_k^n = 0$ for $n \geq 4\overline{n}$ and $\sum_{n\geq 0} p_k^n = 1$) is $p_k^n = \delta_{n,\overline{n}}$. Deduce that $\boldsymbol{K}(\rho) = \rho$ admits a unique solution denoted by $\bar{\rho}$ among the set of density operators with support in the vector space spanned by $|0\rangle, \ldots, |4\overline{n}\rangle$.

(d) Super-bonus question: in the general case, investigate the fixed points of $\boldsymbol{K}$ and also the limit of ρ_k for k tending to $+\infty$ (as far as we know, this is an open mathematical issue).

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Second-order averaging

1. In this frame $\boldsymbol{a}$ become $e^{it\boldsymbol{H}_0}\boldsymbol{a}e^{-it\boldsymbol{H}_0} = e^{-i\omega_a t}\boldsymbol{a}$ and $\boldsymbol{b}$ becomes $e^{it\boldsymbol{H}_0}\boldsymbol{b}e^{-it\boldsymbol{H}_0} = (e^{i\chi_{bb}t\boldsymbol{b}^\dagger\boldsymbol{b}-i\omega_b t})\boldsymbol{b}$. Thus, we have

$$\boldsymbol{H}_{\text{int}} = (g_1 e^{i(\chi_{bb}\boldsymbol{b}^\dagger\boldsymbol{b}+\Delta)t}\boldsymbol{a}^{\dagger 2}\boldsymbol{b} + \text{h.c.}) + (g_2 e^{i(\chi_{bb}(\boldsymbol{b}^\dagger\boldsymbol{b}-1)-\Delta)t}\boldsymbol{a}^{\dagger 2}\boldsymbol{b} + \text{h.c.})$$

2. The Hamiltonian in the rotating frame is given by

$$\begin{aligned}\boldsymbol{H}_{\text{int}} = (g_1 e^{i\Delta t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| + g_1\sqrt{2}e^{i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| + \text{h.c.})\\ + (g_2 e^{-i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| + g_2\sqrt{2}e^{-i\Delta t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| + \text{h.c.})\end{aligned}$$

The first order rotating wave approximation is given by averaging in time the above Hamiltonian giving rise to $\boldsymbol{H}_{\text{rwa}}^{\text{1st}} = 0$. The second order rotating wave approximation is therefore given by

$$\boldsymbol{H}_{\text{rwa}}^{\text{2nd}} = -i\overline{\boldsymbol{H}_{\text{int}}\int_t \boldsymbol{H}_{\text{int}}}$$

where

$$\begin{aligned}&-i\boldsymbol{H}_{\text{int}}\int_t \boldsymbol{H}_{\text{int}} =\\ &\Big(g_1 e^{i\Delta t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| + g_1\sqrt{2}e^{i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| + g_2 e^{-i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| + g_2\sqrt{2}e^{-i\Delta t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| + \text{h.c.}\Big)\\ &\left(-\frac{g_1}{\Delta}e^{i\Delta t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| - \frac{g_1\sqrt{2}}{\chi_{bb}+\Delta}e^{i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| + \frac{g_2}{\chi_{bb}+\Delta}e^{-i(\chi_{bb}+\Delta)t}\boldsymbol{a}^{\dagger 2}|0\rangle\langle 1| + \frac{g_2\sqrt{2}}{\Delta}e^{-i\Delta t}\boldsymbol{a}^{\dagger 2}|1\rangle\langle 2| - \text{h.c.}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\boldsymbol{H}_{\text{rwa}}^{\text{2nd}} &= \left(\frac{\sqrt{2}}{\Delta} - \frac{\sqrt{2}}{\Delta+\chi_{bb}}\right)\left(g_1 g_2 \boldsymbol{a}^{\dagger 4}|0\rangle\langle 2| + g_1^* g_2^* \boldsymbol{a}^4 |2\rangle\langle 0|\right)\\ &+ \left(\frac{|g_1|^2}{\Delta} - \frac{|g_2|^2}{\chi_{bb}+\Delta}\right)\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2|0\rangle\langle 0| + \left(\frac{2|g_1|^2 - |g_2|^2}{\chi_{bb}+\Delta} + \frac{|g_1|^2 - 2|g_2|^2}{\Delta}\right)\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2|1\rangle\langle 1| + \left(\frac{2|g_1|^2}{\chi_{bb}+\Delta} - \frac{2|g_2|^2}{\Delta}\right)\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2|2\rangle\langle 2|\\ &+ 4\left(\frac{|g_1|^2}{\Delta} - \frac{|g_2|^2}{\chi_{bb}+\Delta}\right)\boldsymbol{a}^\dagger\boldsymbol{a}|1\rangle\langle 1| + 8\left(\frac{|g_1|^2}{\chi_{bb}+\Delta} - \frac{|g_2|^2}{\Delta}\right)\boldsymbol{a}^\dagger\boldsymbol{a}|2\rangle\langle 2|\\ &+ 2\left(\frac{|g_1|^2}{\Delta} - \frac{|g_2|^2}{\chi_{bb}+\Delta}\right)|1\rangle\langle 1| + 4\left(\frac{|g_1|^2}{\chi_{bb}+\Delta} - \frac{|g_2|^2}{\Delta}\right)|2\rangle\langle 2|.\end{aligned}$$

The first term in the Hamiltonian models an exchange of 4 photons of the harmonic oscillator $\boldsymbol{a}$ with two excitations of the mode $\boldsymbol{b}$. The rest of the Hamiltonian is diagonal in the Fock states basis of the two harmonic oscillators and therefore do not lead to any exchange of energy.

Stabilization of Fock states by reservoir engineering

1. We have

$$U_\theta^\dagger = |g\rangle\langle g| \otimes \cos(\theta\sqrt{\boldsymbol{N}}) + |e\rangle\langle e| \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}}) - |g\rangle\langle e| \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger + |e\rangle\langle g| \otimes \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}$$

   Using the identity $\boldsymbol{a}f(\boldsymbol{N}) = f(\boldsymbol{N}+\boldsymbol{I})\boldsymbol{a}$ and $\cos^2 + \sin^2 = 1$, we get the result.

2. We have

$$|\Psi\rangle = \cos u \left(|g\rangle \otimes \cos(\theta\sqrt{\boldsymbol{N}})|\psi_k\rangle - |e\rangle \otimes \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}|\psi_k\rangle \right) + \sin u \left(|e\rangle \otimes \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}})|\psi_k\rangle + |g\rangle \otimes \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger|\psi_k\rangle \right)$$

3. We have

$$\begin{aligned}
\boldsymbol{M}_{g,\theta,u} &= \cos u\, \cos(\theta\sqrt{\boldsymbol{N}}) + \sin u\, \frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}}\,\boldsymbol{a}^\dagger \\
\boldsymbol{M}_{e,\theta,u} &= -\cos u\, \boldsymbol{a}\,\frac{\sin(\theta\sqrt{\boldsymbol{N}})}{\sqrt{\boldsymbol{N}}} + \sin u\, \cos(\theta\sqrt{\boldsymbol{N}+\boldsymbol{I}})
\end{aligned}$$

   and the probabilities are $\langle\psi_k|\boldsymbol{M}_{y,\theta,u}^\dagger \boldsymbol{M}_{y,\theta,u}|\psi_k\rangle$ for $y = g, e$.

4. We have $\rho_{k+1} = \boldsymbol{M}_{g,\theta,u}\rho_k\boldsymbol{M}_{g,\theta,u}^\dagger + \boldsymbol{M}_{e,\theta,u}\rho_k\boldsymbol{M}_{e,\theta,u}^\dagger$.

5. (a) This comes from the fact that

$$\boldsymbol{K}_{0,\frac{\pi}{\sqrt{\overline{n}}}}(\rho) = \cos\left(\pi\sqrt{\tfrac{\boldsymbol{N}}{\overline{n}}}\right)\rho\cos\left(\pi\sqrt{\tfrac{\boldsymbol{N}}{\overline{n}}}\right) + \boldsymbol{a}\sin\left(\pi\sqrt{\tfrac{\boldsymbol{N}+1}{\overline{n}}}\right)\rho\sin\left(\pi\sqrt{\tfrac{\boldsymbol{N}+1}{\overline{n}}}\right)\boldsymbol{a}^\dagger.$$

   (b) Just use $\cos^2 + \sin^2 = 1$.

   (c) This comes from the fact that formally $p_{k+1} = Mp_k$ with M an infinite dimensional matrix where its nonzero entries are only on the diagonal and the upper diagonal.

   (d) The fact that such p are necessary stationary is obvious. To prove that they are the only ones, we have to solve the following infinite set of equations

$$p^n = \cos^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right)p^n + \sin^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}}}\right)p^{n+1}, \quad n = 0, 1, \ldots.$$

   Thus $\sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right)p^n = \sin^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}}}\right)p^{n+1}$. Since $\sin\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right) = 0$ when $n = r^2\overline{n}$, we have $p^{r^2\overline{n}+1} = 0$ since then $\sin\left(\pi\sqrt{\tfrac{r^2\overline{n}+1}{\overline{n}}}\right) \neq 0$. It is then clear that for m such that $\pi\sqrt{\tfrac{r^2\overline{n}+m}{\overline{n}}} \in]r\pi, (r+1)\pi[$ we have $p^{r^2\overline{n}+m} = 0$.

   (e) This results from $p_{k+1}^{\overline{n}-1} = \cos^2\left(\pi\sqrt{\tfrac{\overline{n}-1}{\overline{n}}}\right)p_k^{\overline{n}-1}$ where $p_k^{\overline{n}}$ has disappeared.

(f) Similarly this results from $p_{k+1}^{r^2\overline{n}-1} = \cos^2\left(\pi\sqrt{\frac{r^2\overline{n}-1}{\overline{n}}}\right) p_k^{r^2\overline{n}-1}$ where $p_k^{r^2\overline{n}}$ has disappeared.

(g) Set $X = (p^0, \ldots, p^{\overline{n}-1})^T$. We have $X_{k+1} = MX_k$ where M is an $\overline{n}\times\overline{n}$ upper diagonal matrix with diagonal $\left[1, \cos^2\left(\pi\sqrt{\frac{1}{\overline{n}}}\right), \ldots, \cos^2\left(\pi\sqrt{\frac{\overline{n}-1}{\overline{n}}}\right)\right]$ corresponding to its eigenvalues. Since the eigenvector associated to 1 is $X = (1, 0, \ldots, 0)$, we know, form the spectral decomposition of $M = P\Delta P^1$ where Δ is the diagonal matrix made of its eigenvalues, that $M^k = P\Delta^k P^{-1}$ converges to $P\,\mathrm{diag}(1, 0, \ldots, 0)\,P^{-1}$. Thus X_k converges to $(\bar{x}, 0, \ldots, 0)$. Since $1 = \sum_{n=0}^{\overline{n}-1} p_k^n = \sum_{n=0}^{\overline{n}-1} p_{k+1}^n$ we have $\bar{x} = 1$ and thus p_k^n converges to $\delta_{n,0}$.

ρ_k coincides with a $\overline{n}\times\overline{n}$ non-negative matrice of trace one and those diagonal elements tend to zero except the first one converging towards 1. This implies that ρ_k converges towards $|0\rangle\langle 0|$.

(h) It is enough to take $X = (p^{\overline{n}}, \ldots, p^{4\overline{n}-1})^T$ and $X_{k+1} = MX_k$ where M is an $(3\overline{n})\times(3\overline{n})$ upper diagonal matrix with diagonal $\left[1, \cos^2\left(\pi\sqrt{\frac{\overline{n}+1}{\overline{n}}}\right), \ldots, \cos^2\left(\pi\sqrt{\frac{4\overline{n}-1}{\overline{n}}}\right)\right]$ and to reproduce the argument of the previous question.

6. (a) This comes from the fact that

$$\boldsymbol{K}_{\pi/2, \frac{\pi}{\sqrt{\overline{n}+1}}}(\rho) = \frac{\sin\left(\pi\sqrt{\frac{\boldsymbol{N}}{\overline{n}+1}}\right)}{\sqrt{\boldsymbol{N}}}\, \boldsymbol{a}^\dagger \rho \boldsymbol{a}\, \frac{\sin\left(\pi\sqrt{\frac{\boldsymbol{N}}{\overline{n}+1}}\right)}{\sqrt{\boldsymbol{N}}} + \cos\left(\pi\sqrt{\frac{\boldsymbol{N}+1}{\overline{n}+1}}\right) \rho \cos\left(\pi\sqrt{\frac{\boldsymbol{N}+1}{\overline{n}+1}}\right).$$

(b) Obvious

(c) Contrarily to question 5c, M is an under diagonal matrix.

(d) The fact that such p are necessary stationary is obvious. To prove that they are the only ones, we have to solve $\sin^2\left(\pi\sqrt{\frac{n+1}{\overline{n}+1}}\right) p^n = \sin^2\left(\pi\sqrt{\frac{n}{\overline{n}+1}}\right) p^{n-1}$. Since $\sin\left(\pi\sqrt{\frac{n+1}{\overline{n}+1}}\right) = 0$ when $n = r^2(\overline{n}+1) - 1$, we have $p^{r^2(\overline{n}+1)-2} = 0$ since then $\sin\left(\pi\sqrt{\frac{r^2(\overline{n}+1)-1}{\overline{n}+1}}\right) \neq 0$.

(e) This results from

$$p_{k+1}^{\overline{n}+1} = \cos^2\left(\pi\sqrt{\frac{\overline{n}+2}{\overline{n}+1}}\right) p_k^{\overline{n}+1}.$$

(f) This results from

$$p_{k+1}^{r^2(\overline{n}+1)} = \cos^2\left(\pi\sqrt{\frac{r^2(\overline{n}+1)+1}{\overline{n}+1}}\right) p_k^{r^2(\overline{n}+1)}.$$

(g) Use the method of question 5g, set $X = (p^0, \ldots, p^{\overline{n}})^T$ with $X_{k+1} = MX_k$, M being an $(\overline{n}+1)\times(\overline{n}+1)$ under diagonal matrix with diagonal $\left[\cos^2\left(\pi\sqrt{\frac{1}{\overline{n}+1}}\right), \ldots, \cos^2\left(\pi\sqrt{\frac{\overline{n}}{\overline{n}+1}}\right), 1\right]$

(h) Prove that for any initial value of the density operator ρ_0 satisfying $\forall n \in [0, \overline{n}] \cup [4\overline{n}+4, +\infty[$, $\rho_0|n\rangle = 0$, we have $\lim_{k\mapsto+\infty} \rho_k = |4\overline{n}+3\rangle\langle 4\overline{n}+3|$.

Use $X = (p^{\overline{n}+1}, \ldots, p^{4\overline{n}+3})^T$ with $X_{k+1} = MX_k$, M being an $(3\overline{n}+3)\times(3\overline{n}+3)$ under diagonal matrix with diagonal $\left[\cos^2\left(\pi\sqrt{\frac{\overline{n}+2}{\overline{n}+1}}\right), \ldots, \cos^2\left(\pi\sqrt{\frac{4\overline{n}+3}{\overline{n}+1}}\right), 1\right]$.

7. (a) We have
$$\rho_{k+1} \triangleq \boldsymbol{K}(\rho_k) = r\boldsymbol{K}_{\frac{2\pi}{\sqrt{\overline{n}+1}},\frac{\pi}{2}}(\rho_k) + (1-r)\boldsymbol{K}_{\frac{2\pi}{\sqrt{\overline{n}}},0}(\rho_k).$$

(b) We have
$$p^n_{k+1} = r\sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}+1}}\right)p_k^{n-1} + \left(r\cos^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}+1}}\right) + (1-r)\cos^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right)\right)p_k^n + (1-r)\sin^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}}}\right)p_k^{n+1}.$$

(c) We have to solve
$$p^n = r\sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}+1}}\right)p^{n-1} + \left(r\cos^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}+1}}\right) + (1-r)\cos^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right)\right)p^n + (1-r)\sin^2\left(\pi\sqrt{\tfrac{n+1}{\overline{n}}}\right)p^{n+1}. \quad (1)$$

For $n = \overline{n}$ we get
$$r\sin^2\left(\pi\sqrt{\tfrac{\overline{n}}{\overline{n}+1}}\right)p^{\overline{n}-1} + (1-r)\sin^2\left(\pi\sqrt{\tfrac{\overline{n}+1}{\overline{n}}}\right)p^{\overline{n}+1} = 0.$$

Since each $p^n \geq 0$ and $r \in]0,1[$, we conclude that $p^{\overline{n}-1} = p^{\overline{n}+1} = 0$. With $n = \overline{n}-1$, equation (1) yields $0 = r\sin^2\left(\pi\sqrt{\tfrac{\overline{n}-1}{\overline{n}+1}}\right)p^{\overline{n}-2}$. Thus $p^{\overline{n}-2} = 0$ and by recurrence we have $p^n = 0$ for $n < \overline{n}$ since $r\sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}+1}}\right) > 0$ for $n \leq \overline{n}-1$.

Similarly we get $p^n = 0$ for $\overline{n} < n < 4\overline{n}$ since $(1-r)\sin^2\left(\pi\sqrt{\tfrac{n}{\overline{n}}}\right) > 0$.

(d) ...