# Flat systems, equivalence
# trajectory generation

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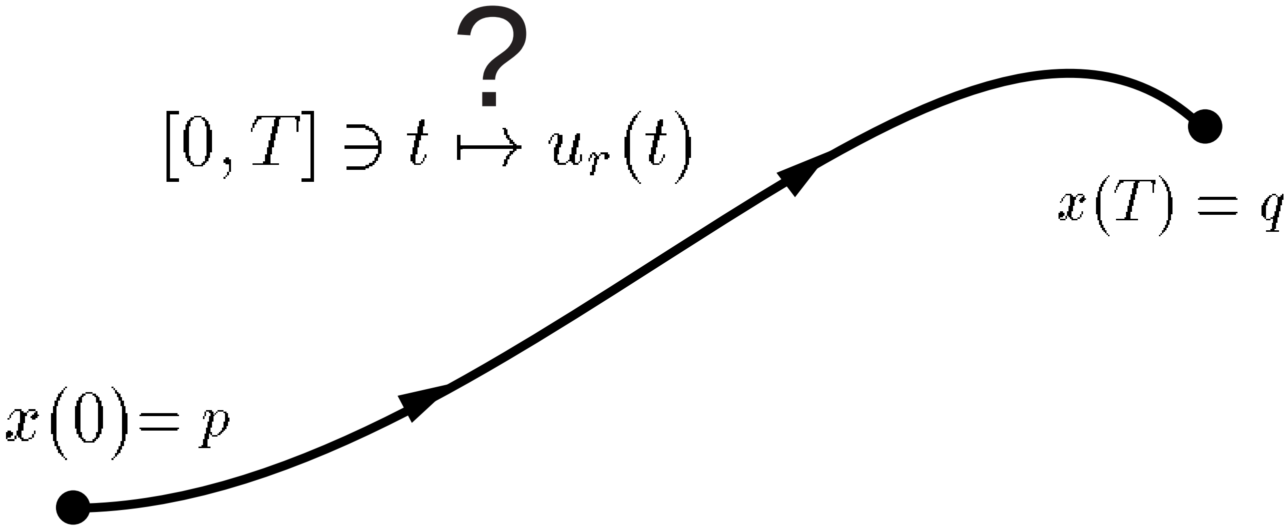
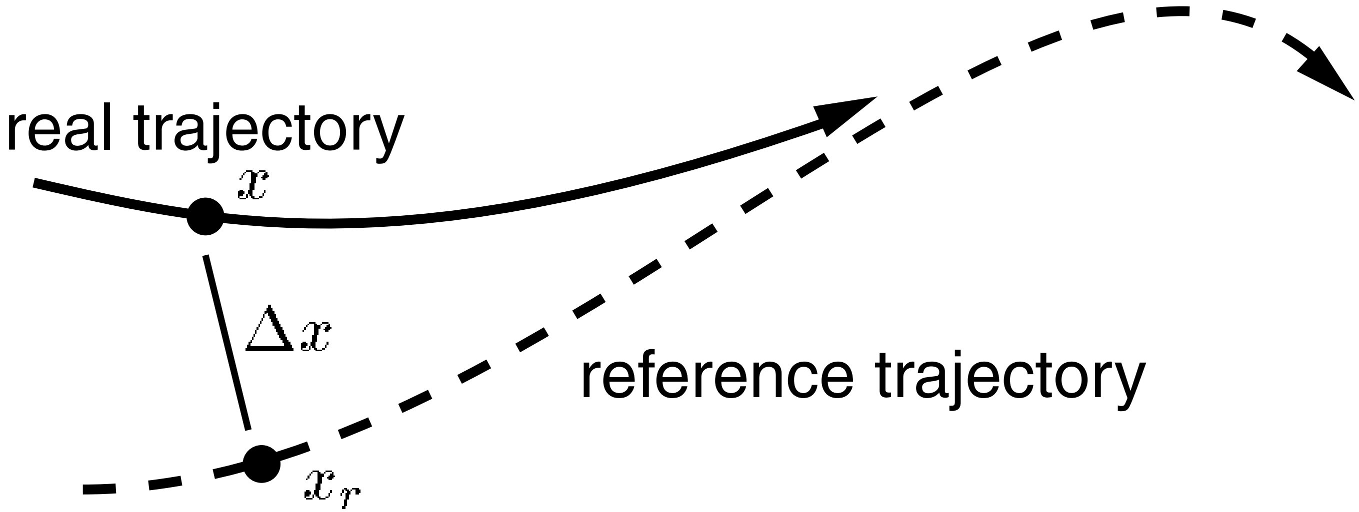
**Motion planning: controllability.**




Difficult problem because it requires, in general, the integration of

$$\frac{d}{dt}x = f(x, u(t)).$$

**Tracking for $\frac{d}{dt}x = f(x,u)$: stabilization.**




Compute $\Delta u$, $u = u_r + \Delta u$, such that $\Delta x = x - x_r$ tends to 0.

## Outline (flatness and ODE)

Linear systems: the Brunovsky canonical form

Static feedback linearization.

Inversion and the structure algorithm.

Differential flatness.

Examples of engineering interest.

## Linear time-invariant systems

$$\frac{dx}{dt} = Ax + Bu$$

Controllability: Kalman-criterion, Brunovsky canonical form.

Interpretation in terms of free modules (torsion-free implies free for such modules).

Non trivial example: two oscillators in parallel.

Similar results for time-varying systems

$$\frac{d}{dt}x = A(t)y + B(t)u$$

with $A(t)$ and $B(t)$ meromorphic functions of $t$.

**Two linear simple examples**

A single oscillator :

$$\frac{d^2}{dt^2}x = -\omega^2(x-u)$$

(trajectory, motion planing, tracking, feedback, ....).

A less simple example: two oscillators in parallel.

**Two oscillators (see the video)**

Dynamics

$$\frac{d^2}{dt^2}x_1 = \omega_1^2(u - x_1), \quad \frac{d^2}{dt^2}x_2 = \omega_2^2(u - x_2).$$

Main trick: Brunovsky output via $u$ elimination:

$$\omega_2^2\frac{d^2}{dt^2}x_1 - \omega_1^2\frac{d^2}{dt^2}x_2 = \omega_1^2\omega_2^2(x_2 - x_1).$$

Controllable when $\omega_1 \neq \omega_2$ with Brunovsky (flat) output $y = \omega_2^2 x_1 - \omega_1^2 x_2$.

For linear SISO system $z(s) = \frac{P(s)}{Q(s)}u(s)$ ($z$ is the output, $s = \frac{d}{dt}$):

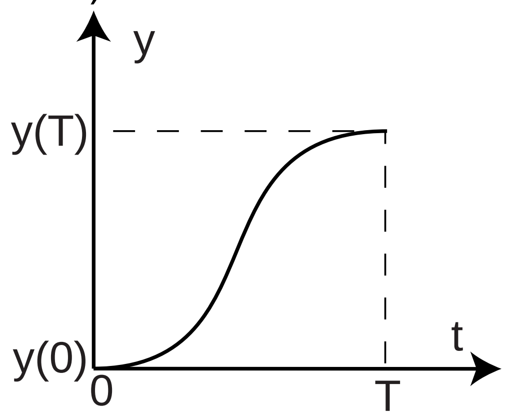
$$z(s) = P(s)y(s), \quad u(s) = Q(s)y(s).$$

Since $P$ and $Q$ have no common divisor, exist $R$ and $S$ such that $PR + QS = 1$, i.e., $y = Ry + Su$ is the “flat” output.

We have then

$$(x_1, x_2, u) = \text{linear combination of } (y, \ddot{y}, y^{(4)}).$$

Steady-state to steady-state steering via $y$ with the following shape (take, e.g., a polynomial of degree 9):




$$\dot{y} = \ddot{y} = y^{(3)} = y^{(4)} = 0 \text{ for } t = 0$$
$$\dot{y} = \ddot{y} = y^{(3)} = y^{(4)} = 0 \text{ for } t = T$$

## Nonlinear control systems

The simplest non linear robot:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta + u$$

The nonlinear inverted pendulum.

$$\frac{d}{dt}(\dot{D}\cos\theta + l\dot{\theta}) = g\sin\theta - \dot{D}\dot{\theta}\sin\theta.$$

**Static feedback linearization**

The nonlinear version of Brunovsky Canonical form.

The Lie Brackets characterization.

Typical examples: fully actuated robots; exothermic chemical reactors.

**Lie Bracket characterization of static feedback linearization**

Definition: $\frac{d}{dt}x = f(x, u)$ is static-feedback linearizable, iff, exists a change of variables $z = \phi(x)$ and a static invertible feedback $v = k(x, u)$ that made the equations linear: $\frac{d}{dt}z = Az + Bv$ with $(A, B)$ controllable (local definition when $f(0, 0) = 0$).

Proposition: $\frac{d}{dt}x = f(x, u)$ static feedback linearizable, iff, the extended system $\frac{d}{dt}x = f(x, u)$, $\frac{d}{dt}u = \bar{u}$ is static feedback linearizable.

**Proof of the proposition when** $\dim(u) = 1$**.**

If $x = \phi(z_1, \ldots, z_n)$ and $u = k(z, v)$ transform $\frac{d}{dt}x = f(x, u)$ into the controllable system $\frac{d}{dt}z = Az + Bv$ then $(x, u) = (\phi(z), k(z, v))$ and $\bar{u} = \frac{\partial k}{\partial z}(Az + Bv) + \frac{\partial k}{\partial v}\bar{v}$ transform the extended system into $\frac{d}{dt}z = Az + Bv$, $\frac{d}{dt}v = \bar{v}$.

If $(x, u) = (\phi(z, v), \psi(z, v))$ and $\bar{u} = k(z, v, \bar{v})$ transform the extended system into $z_1^{(n)} = v$, $\frac{d}{dt}v = \bar{v}$ (Brunovsky form) then for all $(z, v, \bar{v})$, we have ($z = (z_1, \ldots, z_{n-1}, z_n)$, $\dot{z} = (z_2, \ldots, z_n, v)$)

$$\frac{\partial \phi}{\partial z}(z, v) \cdot (z_2, \ldots, z_n, v) + \frac{\partial \phi}{\partial v}(z, v)\bar{v} \equiv f(\phi(z, v), \psi(z, v)).$$

thus $\phi$ is independent of $v$: $x = \phi(z)$ and $u = \psi(z, v)$ transform $\frac{d}{dt}x = f(x, u)$ into $z_1^{(n)} = v$.

Theorem (Jakubczyk and Respondek, 1980):

$$\frac{d}{dt}x = f(x) + g_1(x)u_1 + \ldots + g_m(x)u_m$$

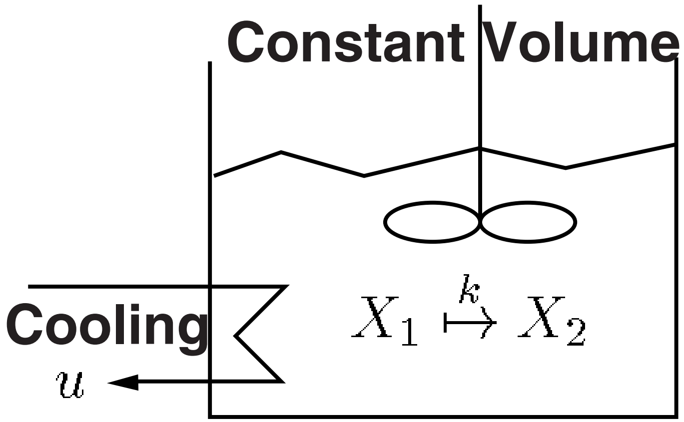
is feedback linearizable around the equilibrium ($x = 0$, $u = 0$) iff, the vector space depending on $x$ (distributions) $E_i(x)$, $i = 1, \cdots, n-1$ defined here below are of constant rank versus $x$, involutive (stable by Lie-Bracket) and the rank of $E_{n-1}(x)$ is $n = \dim(x)$.

$$E_0(x) = \text{span}\{g_1(x), \ldots, g_m(x)\},$$
$$E_i(x) = \text{span}\{E_{i-1}(x), [f, E_{i-1}](x)\} \quad i \geq 1$$

where $[f, g_i] = Df \cdot g_i - Dg_i \cdot f$ is the Lie-Bracket of the two vector fields $f$ and $g_i$.

Example: $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta + u$.

**A simple exothermic batch reactor.**




Dynamics:

$$\frac{d}{dt}x_1 = -k_0 \, \text{exp}(-E/R\theta) \, (x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \, \text{exp}(-E/R\theta) \, (x_1)^\alpha + u.$$

**Modeling: mass and energy balance, Arhenius kinetics.**

$$\frac{d}{dt}(Vx_1) = -Vr(x,\theta)$$
$$\frac{d}{dt}(Vx_2) = Vr(x,\theta)$$
$$\frac{d}{dt}(V\rho C_p\theta) = V\Delta H r(x,\theta) + Q$$
$$r(x,\theta) = k_0 \,\mathrm{exp}(-E/R\theta)\,(x_1)^\alpha$$

Set $k = \frac{\Delta H}{\rho C_p}$ and $u = \frac{Q}{V\rho C_p}$ to obtain

$$\frac{d}{dt}x_1 = -k_0 \,\mathrm{exp}(-E/R\theta)\,(x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \,\mathrm{exp}(-E/R\theta)\,(x_1)^\alpha + u.$$

**Explicit description of batch trajectories**

Instead of fixing the initial condition $x_1(0) = x_1^0$, $\theta(0) = \theta^0$, the control $t \mapsto u(t)$ and integrating

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u(t)$$

take the system in the reverse way and assume that $x_1$ is a known time function

$$t \mapsto x_1 = y(t).$$

Then you bypass integration.

**The inverse system has no dynamics**

Set $x_1 = y(t)$ and compute $\theta$ and $u$ knowing that

$$\frac{d}{dt}x_1 = -k_0 \,\text{exp}(-E/R\theta)\ (x_1)^\alpha$$
$$\frac{d}{dt}\theta = k \,\text{exp}(-E/R\theta)\ (x_1)^\alpha + u.$$

The mass conservation gives the temperature $\theta$,

$$\text{exp}(-E/R\theta) = -\frac{\frac{d}{dt}y}{k_0 y^\alpha} = \text{function of } (y, \dot{y}),$$

and energy balance gives exchanger duty $u$,

$$u = \frac{d}{dt}\theta + \frac{\frac{d}{dt}y}{k} = \text{function of } (y, \dot{y}, \ddot{y}).$$

**Explicit description.**

The system

$$\frac{d}{dt}x_1 = -k_0 \,\text{exp}(-E/R\theta)\ (x_1)^\alpha, \quad \frac{d}{dt}\theta = k\,\text{exp}(-E/R\theta)\ (x_1)^\alpha + u$$

and the system

$$x_1 = y, \quad \theta = \text{function of } (y, \dot{y}), \quad u = \text{function of } (y, \dot{y}, \ddot{y})$$

represent the same object. It is just another presentation of the dynamics with an additional variable $y$ and its derivatives. Dynamics admitting representation similar to the second system are called flat and the additional quantity $y$ is then the flat output.

For the batch reactor, this is the simplest way to use the fact that the system is linearizable via static feedback and change of coordinates.

## Motion planning for the batch reactor

The initial condition

$$p = (x_1^0, \theta^0)$$

and final condition

$$q = (x_1^T, \theta^T)$$

provide initial and final positions and velocities for $y$:

$$y(0) = x_1^0 \qquad \frac{d}{dt}y(0) = -k_0 \exp(-E/R\theta^0)\ (x_1^0)^\alpha$$

$$y(T) = x_1^T \qquad \frac{d}{dt}y(T) = -k_0 \exp(-E/R\theta^T)\ (x_1^T)^\alpha$$

and in between $y(t)$ is free for $t > 0$ and $t < T$.

## Motion planning for the batch reactor

Take $t \mapsto y^r(t)$ with such initial and final constraints. Compute $u^r$ as

$$u^r = \text{function of } (y^r, \dot{y}^r, \ddot{y}^r)$$

Then the solution of the initial value problem

$$\frac{d}{dt}x_1 = -k_0 \exp(-E/R\theta)\ (x_1)^\alpha, \quad x_1(0) = x_1^0$$
$$\frac{d}{dt}\theta = k \exp(-E/R\theta)\ (x_1)^\alpha + u^r(t), \quad \theta(0) = \theta^0$$

is

$$x_1(t) = y^r(t), \quad \theta(t) = \text{function of } (y^r(t), \dot{y}^r(t))$$

and thus reaches $q = (x_1^T, \theta^T)$ at time $T$.

## Tracking for the batch reactor

The reference trajectory defined via $t \mapsto y^r(t)$:

$$x_1^r = y^r, \quad \theta^r = \text{function of } (y^r, \dot{y}^r), \quad u^r = \text{function of } (y^r, \dot{y}^r, \ddot{y}^r)$$

The change of variable:

$$(x_1, \theta) \longleftrightarrow (y, \dot{y}).$$

The linearizing control:

$$u = \text{function of } (y, \dot{y}, v) = u^r + \Delta u$$

The stable closed-loop error dynamics:

$$\ddot{y} = v = \ddot{y}^r - 2\xi\omega_0(\dot{y} - \dot{y}^r) - {\omega_0}^2(y - y^r)$$

with $\xi > 0$ and $\omega_0 > 0$ design parameters.

**Fully actuated mechanical systems**

The computed torque method for

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = \frac{\partial L}{\partial q} + M(q)u$$

consists in setting $t \mapsto q(t)$ to obtain $u$ as a function of $q$, $\dot{q}$ and $\ddot{q}$.

(Fully actuated: $\mathrm{dim}\, q = \mathrm{dim}\, u$ and $M(q)$ invertible).

**Flat systems (Fliess-et-al, 1992,. . . ,1999)**

A basic definition extending remark of Isidori-Moog-DeLuca (CDC86) on dynamic feedback linearization (Charlet-Lévine-Marino (1989)):

$$\frac{d}{dt}x = f(x, u)$$

is flat, iff, exist $m = \dim(u)$ output functions $y = h(x, u, \ldots, u^{(p)})$, $\dim(h) = \dim(u)$, such that the inverse of $u \mapsto y$ has no dynamics, i.e.,

$$x = \Lambda\left(y, \dot{y}, \ldots, y^{(q)}\right), \quad u = \Upsilon\left(y, \dot{y}, \ldots, y^{(q+1)}\right).$$

Behind this: an equivalence relationship exchanging trajectories (absolute equivalence of Cartan and dynamic feedback: Shadwick (1990), Sluis (1992), Nieuwstadt-et-al (1994), . . . ).

**Inversion and structure algorithm on a academic system**

$$\begin{cases} \frac{d}{dt}x_1 & = & x_1x_2 + u_1 \\ \frac{d}{dt}x_2 & = & x_1x_2 + x_3 + u_1 \\ \frac{d}{dt}x_3 & = & x_3 + x_4 + u_2 \\ \frac{d}{dt}x_4 & = & x_3x_4 + \lambda x_4 + u_2 \\ \\ y_1 & = & x_1 \\ y_2 & = & x_2 \end{cases}$$

Assume $t \mapsto y(t)$ is known and compute $u$ and $x$. Similar computations for singular extremal and state constraint in optimal control. Notion of I/O order and more generally of the structure at infinity. Notion of zero dynamics.

## Equivalence and flatness (extrinsic point of view)

Elimination of $u$ from the $n$ state equations $\frac{d}{dt}x = f(x,u)$ provides an under-determinate system of $n-m$ equations with $n$ unknowns

$$F\left(x, \frac{d}{dt}x\right) = 0.$$

An **endogenous transformation** $x \mapsto z$ is defined by

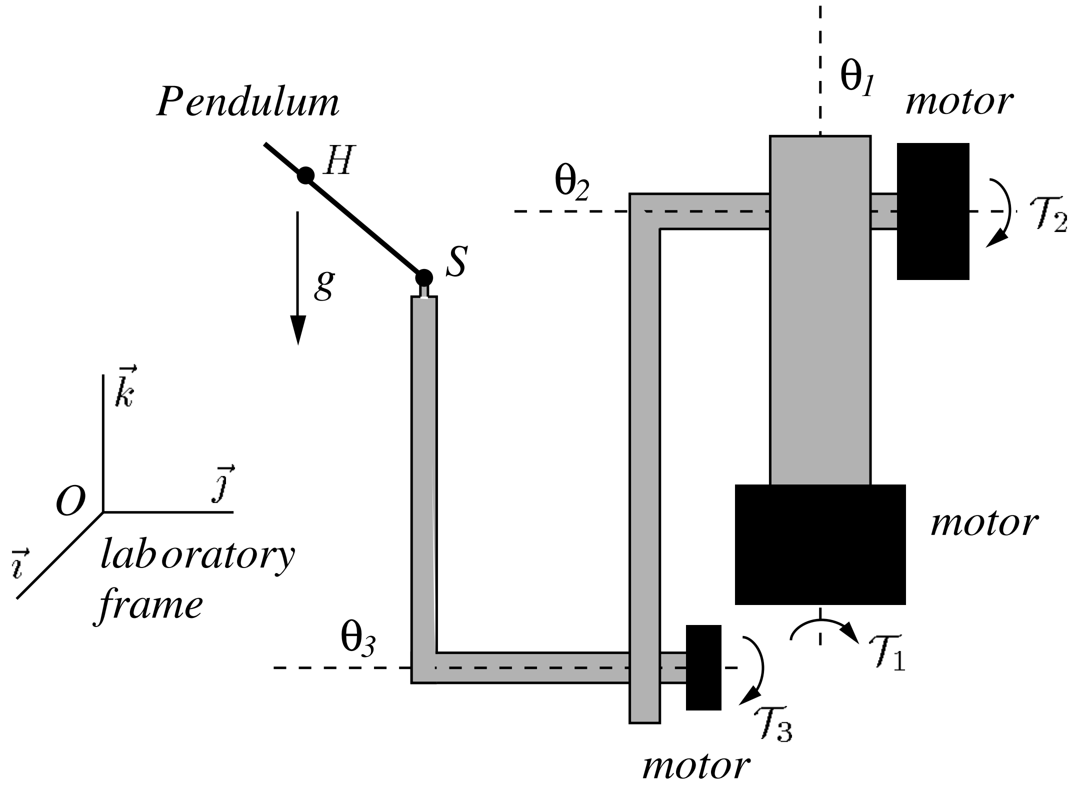
$$z = \Phi(x, \dot{x}, \ldots, x^{(p)}), \quad x = \Psi(z, \dot{z}, \ldots, z^{(q)})$$

(nonlinear analogue of uni-modular matrices, the ”integral free” transformations of Hilbert).

Two systems are equivalents, iff, exists an endogenous transformation exchanging the equations.

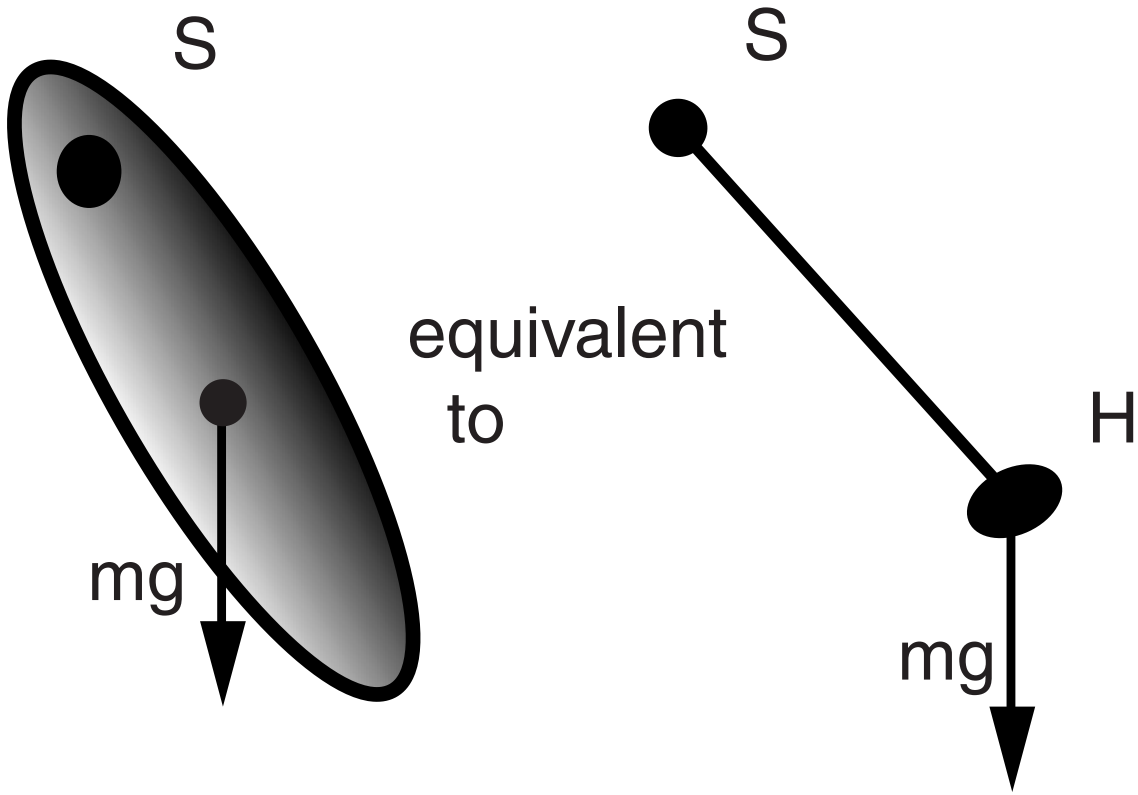
A system equivalent to the trivial equation $z_1 = 0$ with $z = (z_1, z_2)$ is flat with $z_2$ the flat output.

## $2k\pi$ the juggling robot (Lenoir-et-al CDC98) (video)

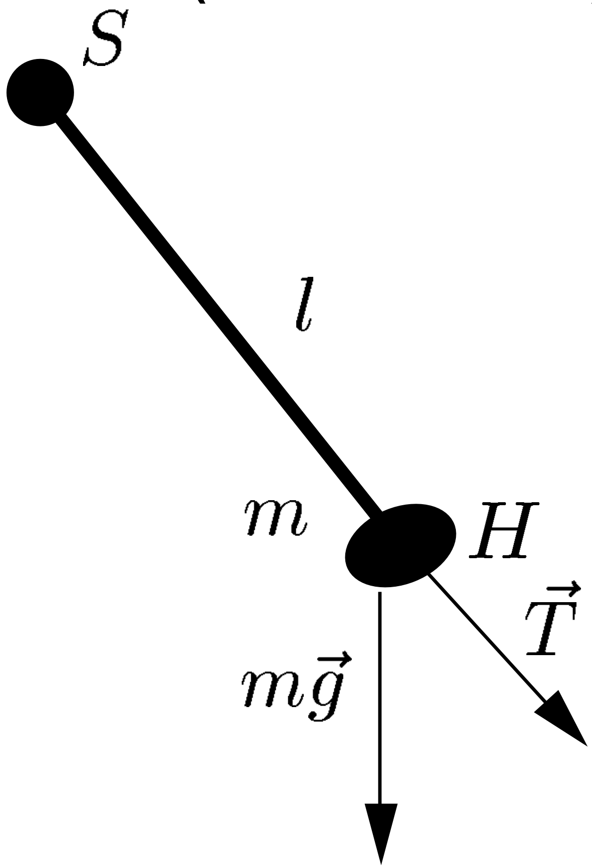



5 degrees of freedom$(\theta_1, \theta_2, \theta_3)$ and the direction $\vec{SH}$. 3 motors.

**Huygens isochronous pendulum**



**The implicit model ($S$ is the control)**




Newton law

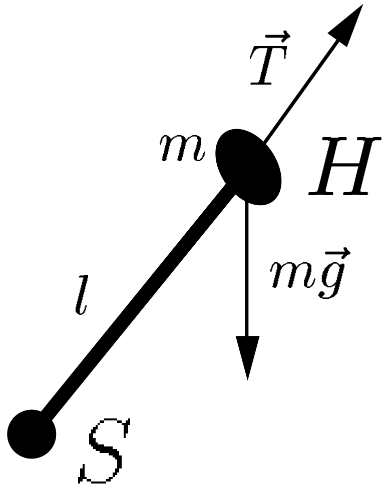
$$m\ddot{H} = \vec{T} + m\vec{g}$$

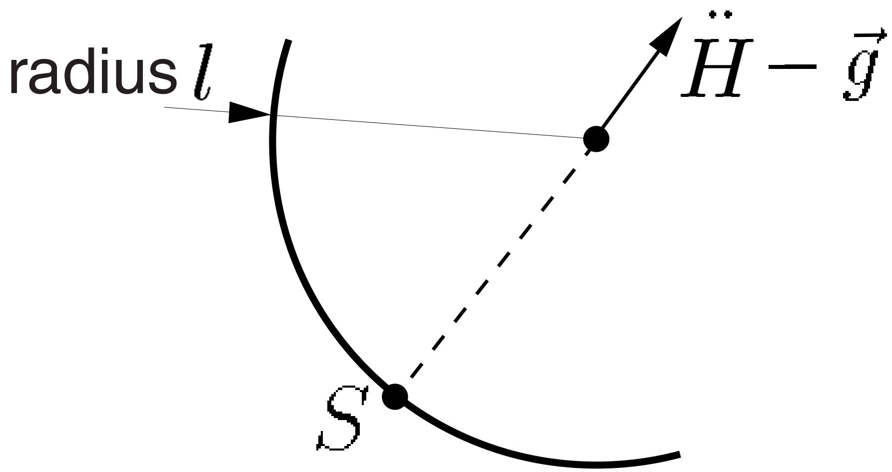
Constraints

$$\vec{T} // \overrightarrow{HS}$$

$$||\overrightarrow{HS}|| = l$$

## $H$ **as flat output**



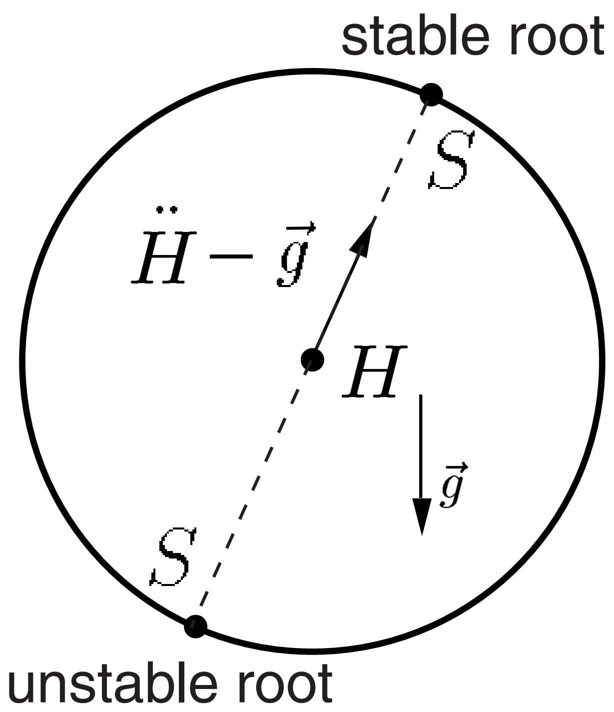

Since

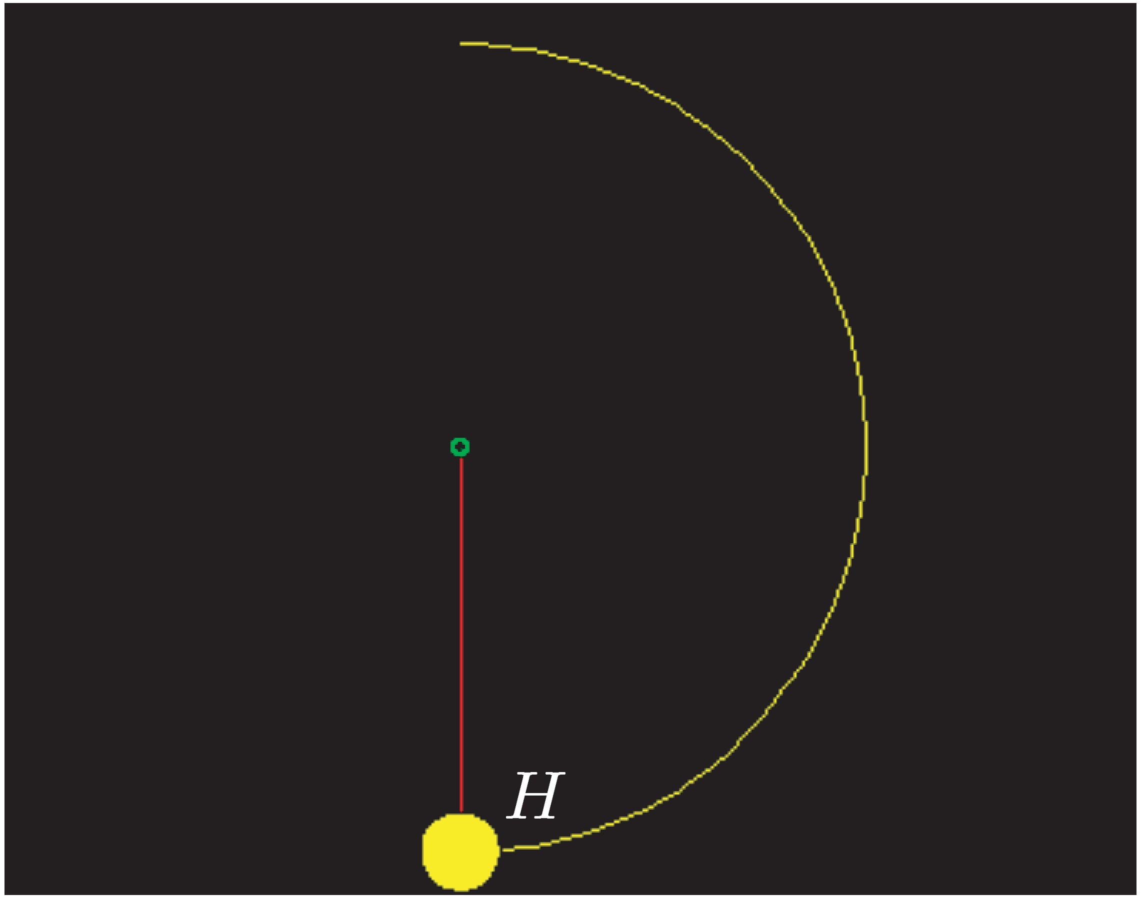
$$\vec{T}/m = \ddot{H} - \vec{g} \quad \text{and} \quad \vec{T} \,//\, \overrightarrow{HS}$$

we have $S$ via

$$\overrightarrow{HS} \,//\, \ddot{H} - \vec{g} \quad \text{and} \quad HS = l.$$

**Planning the inversion trajectory** Any smooth trajectory connecting the stable to the unstable equilibrium is such that $\ddot{H}(t) = \vec{g}$ for at least one time $t$. During the motion there is a switch from the stable root to the unstable root (singularity crossing when $\ddot{H} = \vec{g}$)



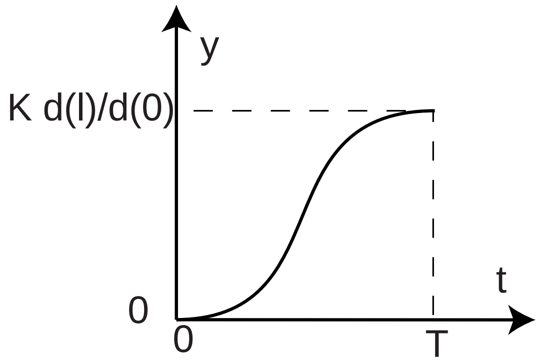

$H$

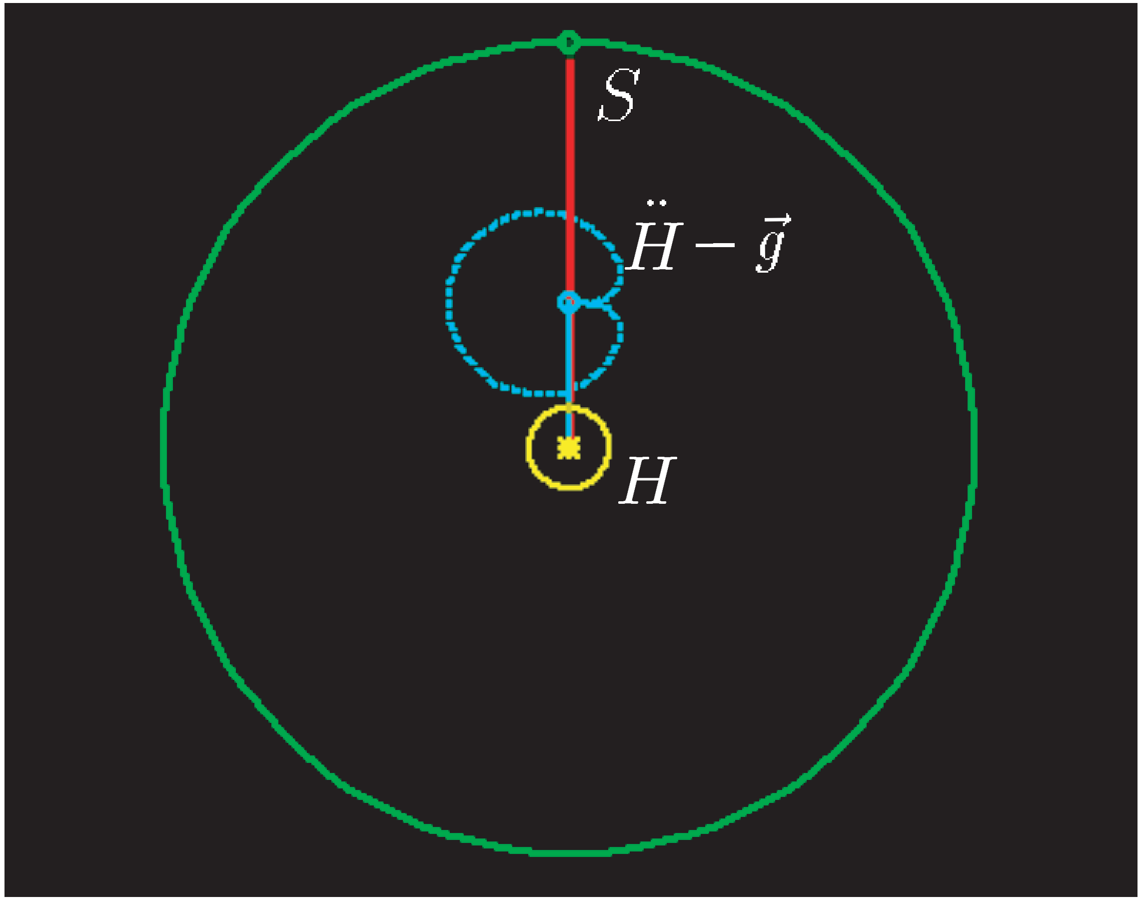
**Crossing smoothly the singularity $\ddot{H} = \vec{g}$**

The geometric path followed by $H$ is a half-circle of radius $l$of center $O$:

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix} \text{ with } \theta(s) = \mu(s)\pi, \quad s = t/T \in [0, 1]$$

where $T$ is the transition time and $\mu(s)$ a sigmoid function of the form:




$S$
$\ddot{H} - \vec{g}$
$H$

**Time scaling and dilation of $\ddot{H} - \vec{g}$**

Denote by $'$ derivation with respect to $s$. From

$$H(t) = 0 + l \begin{bmatrix} \sin\theta(s) \\ -\cos\theta(s) \end{bmatrix}, \quad \theta(s) = \mu(t/T)\pi$$

we have

$$\ddot{H} = H''/T^2.$$

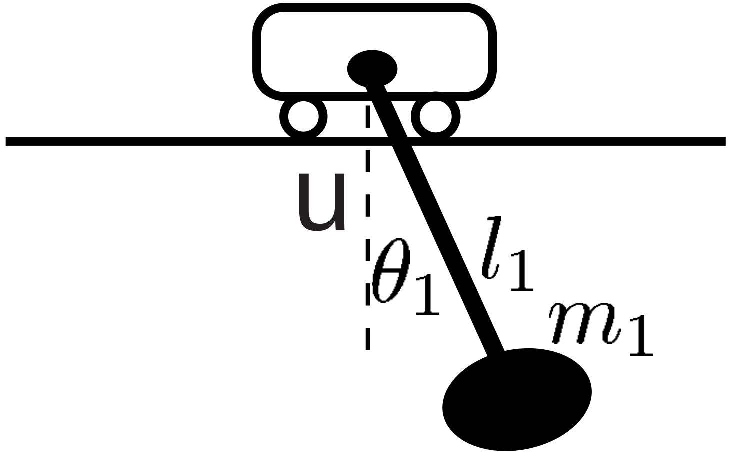
Changing $T$ to $\alpha T$ yields to a dilation of factor $1/\alpha^2$ of the closed geometric path described by $\ddot{H} - \vec{g}$ for $t \in [0, T]$ ($\ddot{H}(0) = \ddot{H}(T) = 0$), the dilation center being $-\vec{g}$.

The inversion time is obtained when this closed path passes through 0. This construction holds true for generic $\mu$.

**Outline (flatness and PDE)**

- Pendulum dynamics.
- Water in a moving box
- Heat equation
- Quantum particle in a moving box
- Conclusion

## One linearized pendulum



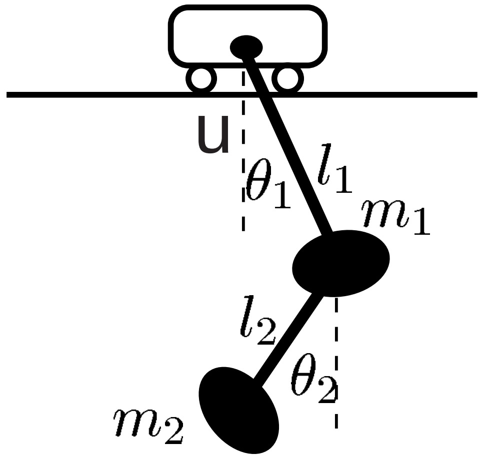

Newton equation with $y = u + l_1\theta_1$:

$$\frac{d^2y}{dt^2} = -g\theta_1 = \frac{g}{l_1}(y - u).$$

Computed torque method:

$$\theta_1 = -\frac{\ddot{y}}{g}, \quad u = y - l_1\theta_1.$$

**Two linearized pendulums in series**

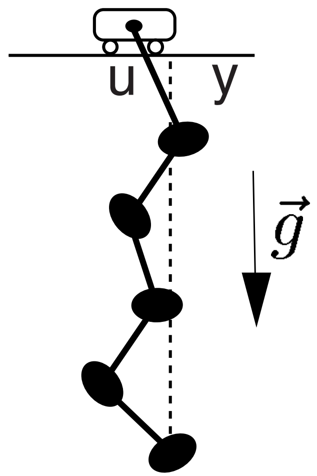



Brunovsky (flat) output $y = u + l_1\theta_1 + l_2\theta_2$:

$$\theta_2 = -\frac{\ddot{y}}{g}, \quad \theta_1 = -\frac{m_1(\overbrace{y - l_2\theta_2}^{\cdot\cdot})}{(m_1+m_2)g} + \frac{m_2}{m_1+m_2}\theta_2$$

and $u = y - l_1\theta_1 - l_2\theta_2$ is a linear combination of $(y, y^{(2)}, y^{(4)})$.

**$n$ pendulums in series**

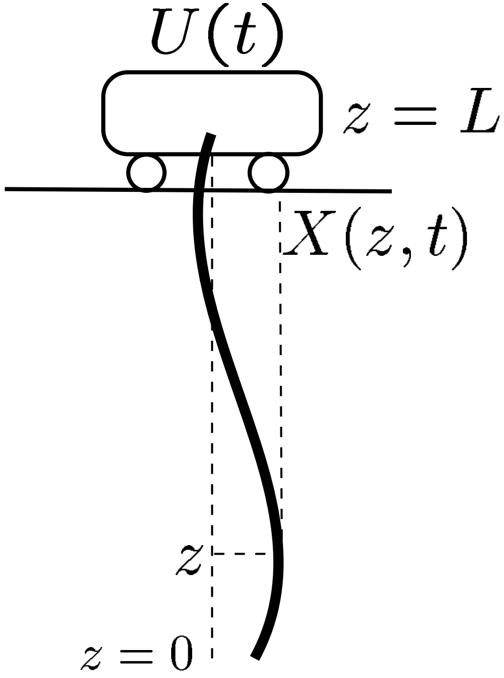



Brunovsky (flat) output $y = u + l_1\theta_1 + \ldots + l_n\theta_n$:

$$u = y + a_1 y^{(2)} + a_2 y^{(4)} + \ldots + a_n y^{(2n)}.$$

When $n$ tends to $\infty$ the system tends to a partial differential equation.

**The heavy chain** (Petit-R 2001).




$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U(t)$$

Flat output $y(t) = X(0,t)$ with

$$U(t) = \frac{1}{2\pi} \int_0^{2\pi} y\left(t - 2\sqrt{L/g}\,\sin\zeta\right)\,d\zeta$$

With the same flat output, for a discrete approximation ($n$ pendulums in series, $n$ large) we have

$$u(t) = y(t) + a_1 \ddot{y}(t) + a_2 y^{(4)}(t) + \ldots + a_n y^{(2n)}(t),$$

for a continuous approximation (the heavy chain) we have

$$U(t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t + 2\sqrt{L/g}\ \sin\zeta\right)\ d\zeta.$$

Why? Because formally

$$y(t + 2\sqrt{L/g}\ \sin\zeta) = y(t) + \ldots + \frac{\left(2\sqrt{L/g}\ \sin\zeta\right)^n}{n!}\, y^{(n)}(t) + \ldots$$

But integral formula is preferable (divergence of the series. . . ).

The general solution of the PDE

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

is

$$X(z,t) = \frac{1}{2\pi} \quad \int_0^{2\pi} y\left(t - 2\sqrt{z/g}\ \sin\zeta\right)\ d\zeta$$

where $t \mapsto y(t)$ is any time function.

Proof: replace $\frac{d}{dt}$ by $s$, the Laplace variable, to obtain a singular second order ODE in $z$ with bounded solutions. Symbolic computations and operational calculus on

$$s^2 X = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right).$$

**Symbolic computations in the Laplace domain**

Thanks to $x = 2\sqrt{\frac{z}{g}}$, we get

$$x \frac{\partial^2 X}{\partial x^2}(x,t) + \frac{\partial X}{\partial x}(x,t) - x \frac{\partial^2 X}{\partial t^2}(x,t) = 0.$$

Use Laplace transform of $X$ with respect to the variable $t$

$$x \frac{\partial^2 \widehat{X}}{\partial x^2}(x,s) + \frac{\partial \widehat{X}}{\partial x}(x,s) - x s^2 \widehat{X}(x,s) = 0.$$

This is a the Bessel equation defining $J_0$ and $Y_0$:

$$\widehat{X}(z,s) = a(s)\ J_0(2\imath s\sqrt{z/g}) + b(s)\ Y_0(2\imath s\sqrt{z/g}).$$

Since we are looking for a bounded solution at $z = 0$ we have $b(s) = 0$ and (remember that $J_0(0) = 1$):

$$\widehat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\widehat{X}(0,s).$$

$$\widehat{X}(z,s) = J_0(2\imath s\sqrt{z/g})\widehat{X}(0,s).$$

Using Poisson's integral representation of $J_0$

$$J_0(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\imath\zeta \sin\theta)\, d\theta, \quad \zeta \in \mathbb{C}$$

we have

$$J_0(2\imath s\sqrt{x/g}) = \frac{1}{2\pi}\int_0^{2\pi} \exp(2s\sqrt{x/g}\sin\theta)\, d\theta.$$

In terms of Laplace transforms, this last expression is a combination of delay operators:

$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\, d\theta$$

with $y(t) = X(0,t)$.

**Explicit parameterization of the heavy chain**

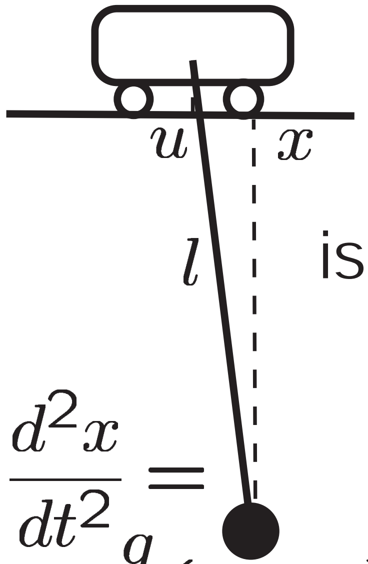
The general solution of

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right), \quad U(t) = X(L,t)$$

reads

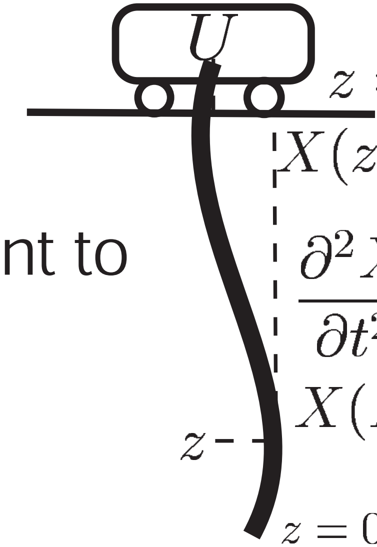
$$X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} y(t + 2\sqrt{z/g}\sin\theta)\, d\theta$$

There is a one to one correspondence between the (smooth) solutions of the PDE and the (smooth) functions $t \mapsto y(t)$.




$$\frac{d^2x}{dt^2} = \frac{g}{l}(u - x)$$

is equivalent to



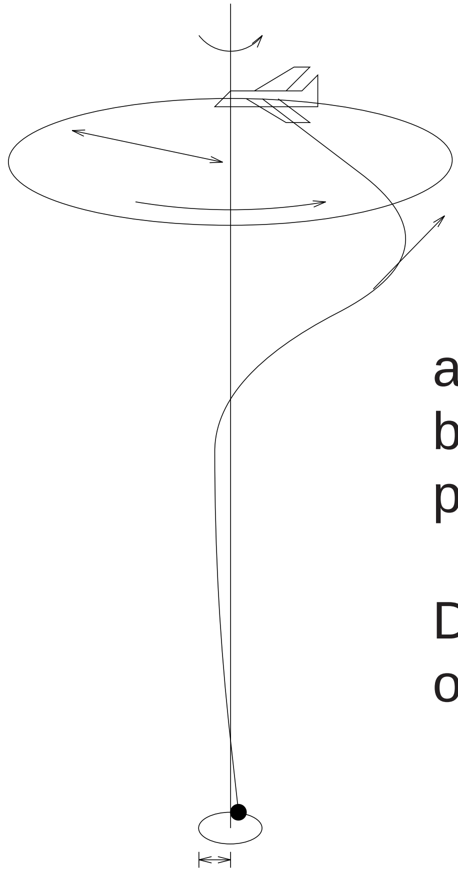

$$\frac{\partial^2 X}{\partial t^2} = \frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)$$

$$X(L,t) = U$$

The following maps exchange the trajectories:

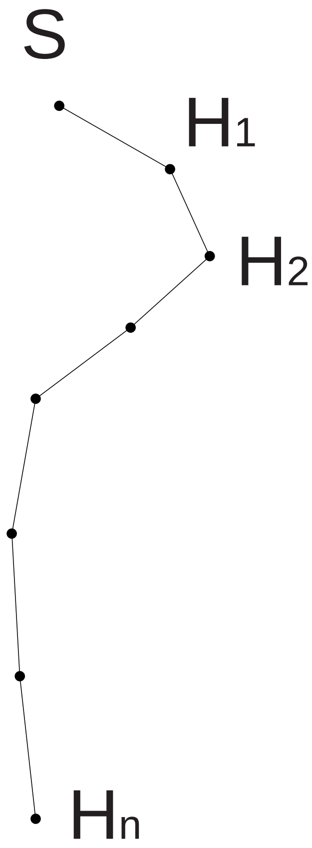
$$\begin{cases} x(t) = X(0,t) \\ u(t) = X(0,t) + \frac{l}{g}\frac{\partial^2 X}{\partial t^2}(0,t) \end{cases} \qquad \begin{cases} X(z,t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{z/g}\,\sin\zeta\right) d\zeta \\ U(t) = \frac{1}{2\pi}\int_0^{2\pi} x\left(t - 2\sqrt{L/g}\,\sin\zeta\right) d\zeta \end{cases}$$

## The Towed Cable Flight Control System, Murray (1996)

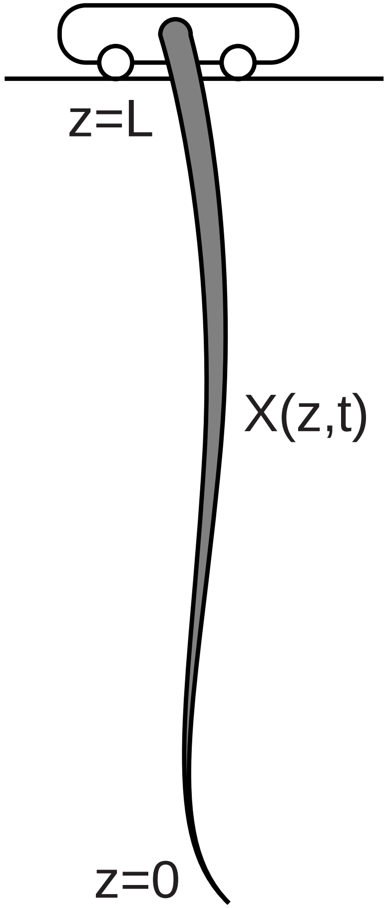


approximated
by a series of small
pendulums

Derivatives
of order 2n



## Heavy chain with a variable section

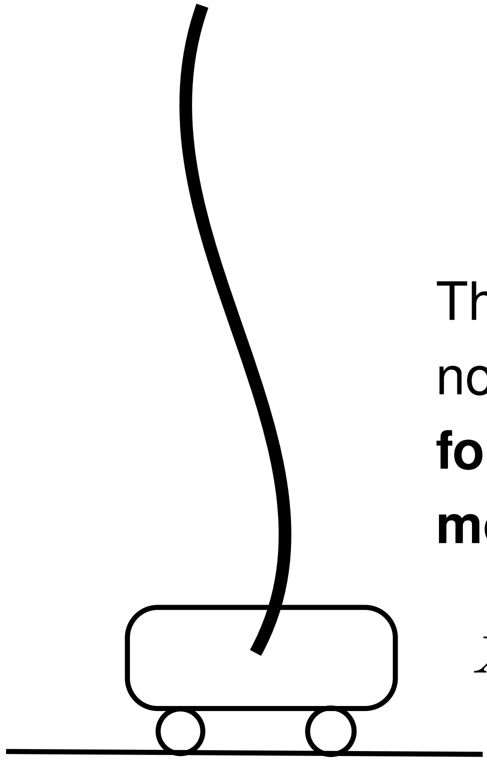



The general solution of

$$\begin{cases} \dfrac{\tau'(z)}{g} \quad \dfrac{\partial^2 X}{\partial t^2} = \dfrac{\partial}{\partial z}\left(\tau(z)\dfrac{\partial X}{\partial z}\right) \\ \\ X(L,t) = u(t) \end{cases}$$

where $\tau(z) \geq 0$ is the tension in the rope, can be parameterized by an arbitrary time function $y(t)$, the position of the free end of the system $y = X(0,t)$, via delay and advance operators with **compact** support (Paley-Wiener theorem).

**The Indian rope.**

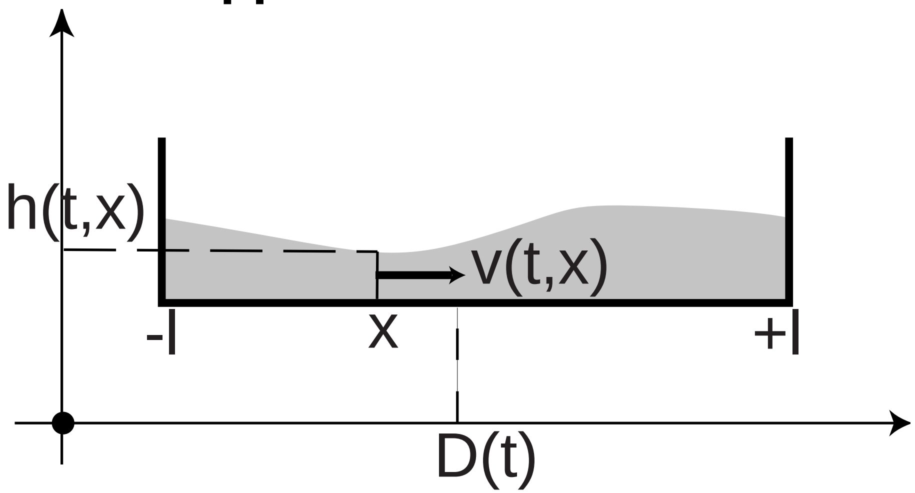


$$\frac{\partial}{\partial z}\left(gz\frac{\partial X}{\partial z}\right)+\frac{\partial^2 X}{\partial t^2} = 0$$
$$X(L,t) = U(t)$$

The equation becomes elliptic and the Cauchy problem is not well posed in the sense of Hadamard. Nevertheless **formulas are still valid with a complex time and $y$ holomorphic**

$$X(z,t)=\frac{1}{2\pi}\int_0^{2\pi} y\left(t-(2\sqrt{z/g}\,\sin\zeta)\,\sqrt{-1}\right)\,d\zeta.$$

**1D Tank: shallow water approximation**

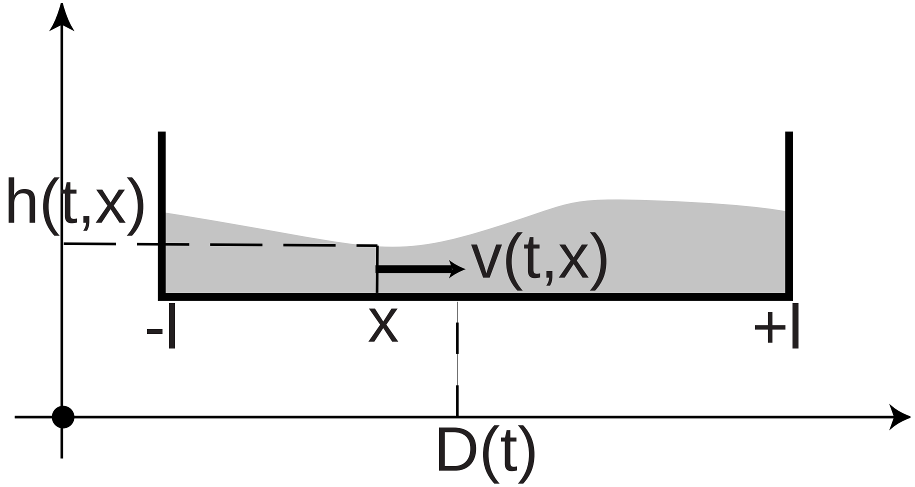



$$\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \ddot{D} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

with $v(t, -l) = v(t, l) = 0$.

The nonlinear dynamics is controllable (Coron 2002) but the tangent linearization is not controllable (Petit-R, 2002).

**1D tank: tangent linearization.**




Assumptions: $h = \bar{h} + H$, $|H| \ll \bar{h}$; $|\ddot{D}| \ll g$, $|v| \ll c = \sqrt{g\bar{h}}$.

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\frac{\partial^2 H}{\partial x^2}, \qquad \frac{\partial H}{\partial x}(t,-l) = \frac{\partial H}{\partial x}(t,l) = -\frac{1}{g}\,\ddot{D}(t)$$

**Non controllable system**

Since $H = \phi(t + x/c) + \psi(t - x/c)$, with $\phi$ and $\psi$ arbitrary, one gets

$$\begin{cases} \phi'(t+\Delta) - \psi'(t-\Delta) = -c\ddot{D}(t)/g \\ \phi'(t-\Delta) - \psi'(t+\Delta) = -c\ddot{D}(t)/g \end{cases}$$

with $2\Delta = l/c$. Elimination of $D$ yields

$$\phi'(t+\Delta) + \psi'(t+\Delta) = \phi'(t-\Delta) + \psi'(t-\Delta).$$

So the quantity $\pi(t) = \phi(t) + \psi(t)$ satisfies an autonomous equation (torsion element of the underlying module, Fliess, Mounier, ...)

$$\pi(t + 2\Delta) = \pi(t).$$

The system is not controllable.

**Trajectories passing through a steady-state**

Since $\pi(t) = \phi(t) + \psi(t) \equiv 0$ we have

$$\phi'(t + \Delta) + \phi'(t - \Delta) = -c\ddot{D}(t)/g$$

thus

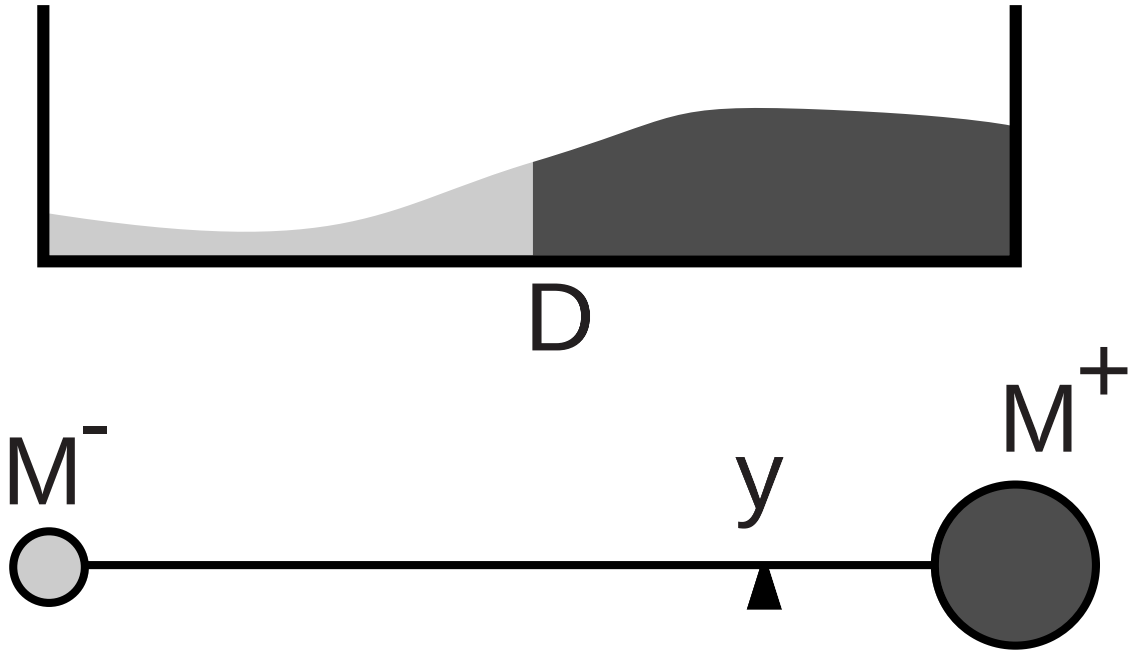
$$\phi(t) = -\left(\frac{c}{2g}\right) y'(t), \quad D(t) = (y(t + \Delta) + y(t - \Delta))/2$$

and

$$\begin{cases} H(t,x) = \dfrac{1}{2}\sqrt{\dfrac{\bar{h}}{g}}\left[y'(t + x/c) - y'(t - x/c)\right] \\ D(t) = \dfrac{1}{2}\left[y(t + \Delta) + y(t - \Delta)\right] \end{cases}$$

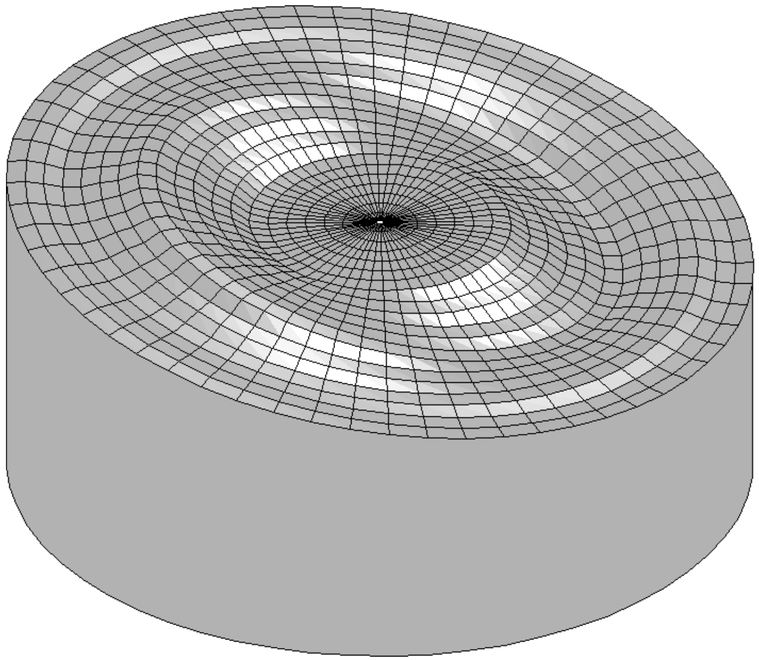
with $t \mapsto y(t)$ an arbitrary time function.

**Physical interpretation of $y$**




$$M^{-} = \int_{-l}^{0} h(t,x)\,dx \qquad M^{+} = \int_{0}^{l} h(t,x)\,dx$$

## The tumbler in movement: 2D cylindrical tank



**Modelling the 2D tank**

The liquid occupies a cylinder with vertical edges with the 2D domain $\Omega$ as horizontal section. The tangent linear equations are:

$$\frac{\partial^2 H}{\partial t^2} = g\bar{h}\Delta H \quad \text{in } \Omega$$

$$\nabla H \cdot \vec{n} = -\frac{\ddot{D}(t)}{g} \cdot \vec{n} \quad \text{on } \partial\Omega$$

with $D = (D_1, D_2)$, $\vec{n}$ the normal to $\partial\Omega$.

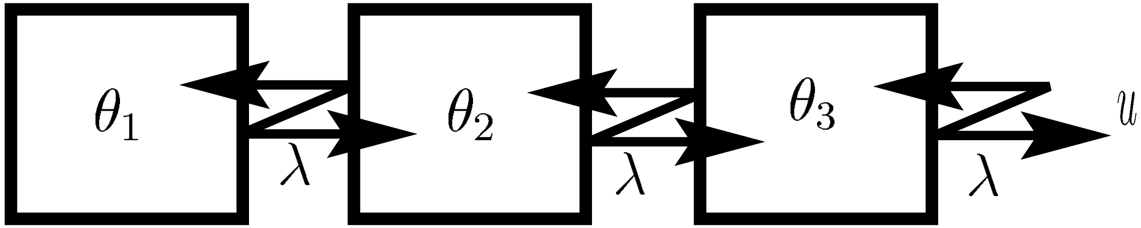
**2D Tank, circular shape.**

Steady-state motion planning results from a symbolic computations in polar coordinates:

$$H(t, x_1, x_2) = \frac{1}{\pi}\sqrt{\bar{h}/g} \int_0^{2\pi} \left[ \cos\zeta \, y_1' \left( t - \frac{x_1 \cos\zeta + x_2 \sin\zeta}{c} \right) \right.$$
$$\left. + \sin\zeta \, y_2' \left( t - \frac{x_1 \cos\zeta + x_2 \sin\zeta}{c} \right) \right] d\zeta$$
$$D_1(t) = \frac{1}{\pi} \int_0^{2\pi} \left[ \cos^2\zeta \, y_1 \left( t - \frac{l \cos\zeta}{c} \right) \right] d\zeta$$
$$D_2(t) = \frac{1}{\pi} \int_0^{2\pi} \left[ \sin^2\zeta \, y_2 \left( t - \frac{l \sin\zeta}{c} \right) \right] d\zeta$$

with $t \mapsto y_1(t)$ and $t \mapsto y_2(t)$ as you want.

**Compartmental approximation of the heat equation**




Energy balance equations

$$\begin{cases} \frac{d}{dt}\theta_1 = (\theta_2 - \theta_1) \\ \frac{d}{dt}\theta_2 = (\theta_1 - \theta_2) + (\theta_3 - \theta_2) \\ \frac{d}{dt}\theta_3 = (\theta_2 - \theta_3) + (u - \theta_3). \end{cases}$$

Linear system controllable with $y = \theta_1$ as Brunovsky or flat output: it can be transformed via linear change of coordinates and linear static feedback into $y^{(3)} = v$.

## Compartmental approximation of the heat equation (end)

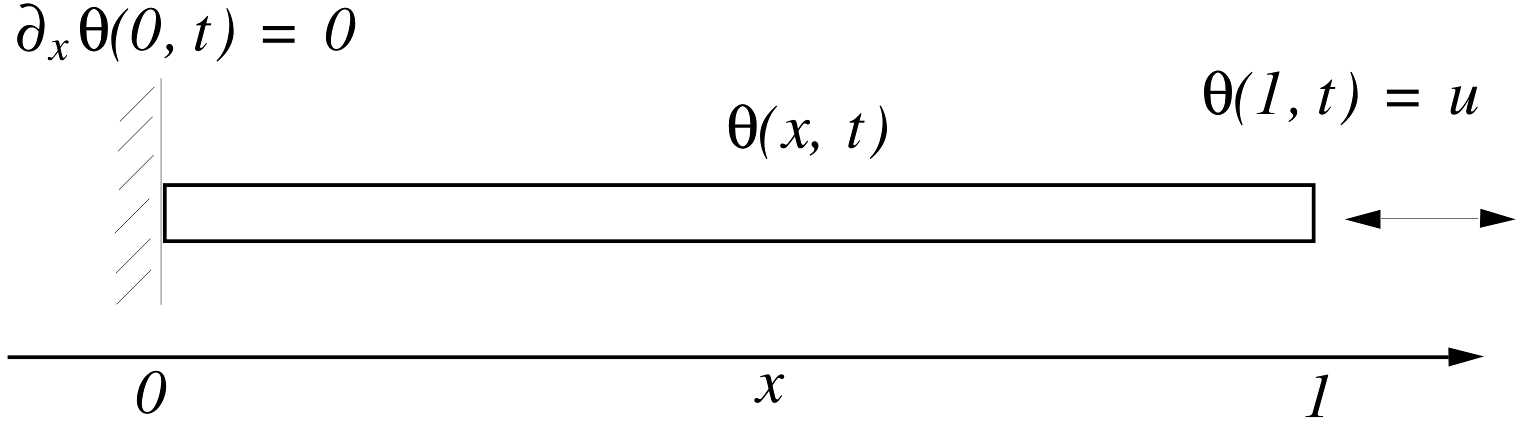
An arbitrary number $n$ of compartments yields

$$\begin{cases} \dot\theta_1 = (\theta_2 - \theta_1) \\ \dot\theta_2 = (\theta_1 - \theta_2) + (\theta_3 - \theta_2) \\ \quad\vdots \\ \dot\theta_i = (\theta_{i-1} - \theta_i) + (\theta_{i+1} - \theta_i) \\ \quad\vdots \\ \dot\theta_{n-1} = (\theta_{n-2} - \theta_{n-1}) + (\theta_n - \theta_{n-1}) \\ \dot\theta_n = (\theta_{n-1} - \theta_n) + (u - \theta_n). \end{cases}$$

$y = \theta_1$ remains the Brunovsky output: via linear change of coordinates and linear static feedback we have $y^{(n)} = v$.

When $n$ tends to infinity we recover $\partial_t \theta = \partial_x^2 \theta$

## Heat equation




$$\partial_t \theta(x,t) = \partial_x^2 \theta(x,t), \quad x \in [0,1]$$
$$\partial_x \theta(0,t) = 0 \qquad \theta(1,t) = u(t).$$

**Series solutions**

Set, formally

$$\theta = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}, \quad \partial_t\theta = \sum_{i=0}^{\infty} \frac{da_i}{dt}\left(\frac{x^i}{i!}\right), \quad \partial_x^2\theta = \sum_{i=0}^{\infty} a_{i+2}\left(\frac{x^i}{i!}\right)$$

and $\partial_t\theta = \partial_x^2\theta$ reads $da_i/dt = a_{i+2}$. Since $a_1 = \partial_x\theta(0,t) = 0$ and $a_0 = \theta(0,t)$ we have

$$a_{2i+1} = 0, \quad a_{2i} = a_0^{(i)}$$

Set $y := a_0 = \theta(0,t)$ we have

$$\theta(x,t) = \sum_{i=0}^{\infty} y^{(i)}(t)\left(\frac{x^{2i}}{(2i)!}\right)$$

**Symbolic computations:** $s := d/dt$**,** $s \in \mathbb{C}$

The general solution of $\theta'' = s\theta$ reads ($' := d/dx$)

$$\theta = \cosh(x\sqrt{s})\; a(s) + \frac{\sinh(x\sqrt{s})}{\sqrt{s}}\; b(s)$$

The boundary condition $\theta(1) = u$ and $\theta'(0) = 0$ reads

$$u = \cosh(\sqrt{s})\; a(s) + \frac{\sinh(\sqrt{s})}{\sqrt{s}}\; b(s), \quad b = 0$$

Since $y = \theta(0) = a$ we have

$$\theta(x,s) = \cosh(x\sqrt{s})\; y(s) = \left( \sum_{i \geq 0} \frac{x^{2i}}{(2i)!} s^i \right)\; y(s).$$

The general solution parameterized via $t \mapsto y(t) \in \mathbb{R}$, $C^\infty$ ($y(t) := \theta(0,t)$)

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\, x^{2i}$$

$$u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

Convergence issue.

## Gevrey function of order $\alpha$

A $C^\infty$ time function $[0,T] \ni t \mapsto y(t)$ is of Gevrey order $\alpha$ when,

$$\exists\, C, D > 0, \quad \forall t \in [0,T], \forall i \geq 0, \quad |y^{(i)}(t)| \leq C D^i \Gamma(1 + (\alpha+1)i)$$

where $\Gamma$ is the classical gamma function with $n! = \Gamma(n+1), \forall n \in \mathbb{N}$.

Analytic functions correspond to Gevrey functions of order $\leq 0$. When $\alpha > 0$, the class of $\alpha$-order functions contains non-zero functions with compact supports. Prototype of such functions:

$$t \mapsto y(t) = \begin{cases} e^{-\left(\frac{1}{t(1-t)}\right)^{\frac{1}{\alpha}}} & \text{if } t \in ]0,1[ \\ 0 & \text{otherwise.} \end{cases}$$

**Operators $P(s)$ as entire functions of $s$, order at infinity**

$\mathbb{C} \ni s \mapsto P(s) = \sum_{i\geq 0} a_i s^i$ is an entire function when the radius of convergence is infinite. If its order at infinity is $\sigma > 0$ and its type is finite, i.e., $\exists M, K > 0$ such that $\forall s \in \mathbb{C}$, $|P(s)| \leq M \exp(K|s|^\sigma)$, then

$$\exists A, B > 0 \mid \forall i \geq 0, \quad |a_i| \leq A \frac{B^i}{\Gamma(i/\sigma + 1)}.$$

$\cosh(\sqrt{s})$ and $\sinh(\sqrt{s})/\sqrt{s}$ are entire functions (order $\sigma = 1/2$, type 1).

Take $P(s)$ of order $\sigma$ with $s = d/dt$. Then $P(s)y(s)$ corresponds to series with a strictly positive convergence radius

$$P(s)y(s) \equiv \sum_{i=0}^{\infty} a_i \, y^{(i)}(t)$$

when $t \mapsto y(t)$ is a Gevrey function of order $\alpha < 1/\sigma - 1$.

**Motion planning of the heat equation**

Take $\sum_{i\geq 0} a_i \frac{\xi^i}{i!}$ and $\sum_{i\geq 0} b_i \frac{\xi^i}{i!}$ entire functions of $\xi$. With $\sigma > 1$

$$y(t) = \left(\sum_{i\geq 0} a_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{(T-t)^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right) + \left(\sum_{i\geq 0} b_i \frac{t^i}{i!}\right)\left(\frac{e^{\frac{-T^\sigma}{t^\sigma}}}{e^{\frac{-T^\sigma}{t^\sigma}} + e^{\frac{-T^\sigma}{(T-t)^\sigma}}}\right)$$

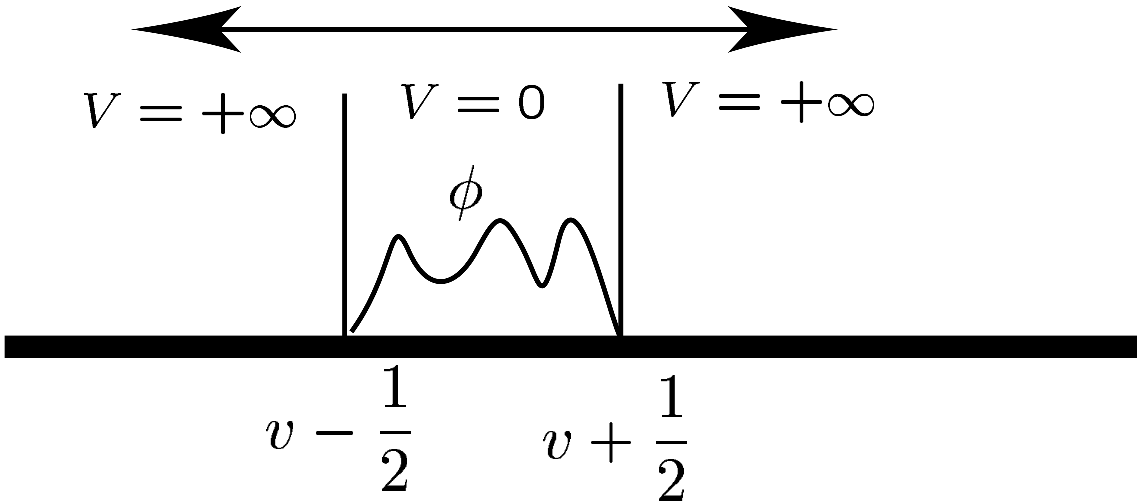
the series

$$\theta(x,t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}\, x^{2i}, \quad u(t) = \sum_{i=1}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}.$$

are convergent and provide a trajectory from

$$\theta(x,0) = \sum_{i\geq 0} a_i \frac{x^{2i}}{(2i)!} \text{ to } \theta(x,T) = \sum_{i\geq 0} b_i \frac{x^{2i}}{(2i)!}$$

**A quantum analogue of the water-tank problem: the quantum box problem (R. 2002) (Beauchard 2005) )**




In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

**Particle in a moving box of position $v$**

In a Galilean frame

$$\imath \frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial z^2}, \quad z \in [v - \frac{1}{2}, v + \frac{1}{2}],$$
$$\phi(v - \frac{1}{2}, t) = \phi(v + \frac{1}{2}, t) = 0$$

where $v$ is the position of the box and $z$ is an absolute position .

In the box frame:

$$\imath \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial q^2} + \ddot{v} q \psi, \quad q \in [-\frac{1}{2}, \frac{1}{2}],$$
$$\psi(-\frac{1}{2}, t) = \psi(\frac{1}{2}, t) = 0$$

**Tangent linearization around eigen-state $\bar{\psi}$ of energy $\bar{\omega}$**

$$\psi(t,q) = \exp(-\imath\bar{\omega}t)(\bar{\psi}(q) + \Psi(q,t))$$

and $\Psi$ satisfies

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q(\bar{\psi} + \Psi)$$
$$0 = \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t).$$

Assume $\Psi$ and $\ddot{v}$ small and neglecte the second order term $\ddot{v}q\Psi$:

$$\imath\frac{\partial\Psi}{\partial t} + \bar{\omega}\Psi = -\frac{1}{2}\frac{\partial^2\Psi}{\partial q^2} + \ddot{v}q\bar{\psi}, \quad \Psi(-\frac{1}{2},t) = \Psi(\frac{1}{2},t) = 0.$$

**Operational computations** $s = d/dt$

The general solution of

$$(\imath s + \bar{\omega})\Psi = -\frac{1}{2}\Psi'' + s^2 v q \bar{\psi}$$

is

$$\Psi = A(s,q)a(s) + B(s,q)b(s) + C(s,q)v(s)$$

where

$$A(s,q) = \cos\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)$$
$$B(s,q) = \frac{\sin\left(q\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}$$
$$C(s,q) = (-\imath s q \bar{\psi}(q) + \bar{\psi}'(q)).$$

**Case $q \mapsto \bar{\phi}(q)$ even**

The boundary conditions imply

$$A(s,1/2)a(s) = 0, \quad B(s,1/2)b(s) = -\psi'(1/2)v(s).$$

$a(s)$ is a torsion element: the system is not controllable.

Nevertheless, for steady-state controllability, we have

$$b(s) = -\bar{\psi}'(1/2)\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}}\, y(s)$$
$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}}\frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s)$$
$$\Psi(s,q) = B(s,q)b(s) + C(s,q)v(s)$$

**Series and convergence**

$$v(s) = \frac{\sin\left(\frac{1}{2}\sqrt{2\imath s + 2\bar{\omega}}\right)}{\sqrt{2\imath s + 2\bar{\omega}}} \frac{\sin\left(\frac{1}{2}\sqrt{-2\imath s + 2\bar{\omega}}\right)}{\sqrt{-2\imath s + 2\bar{\omega}}} y(s) = F(s)y(s)$$

where the entire function $s \mapsto F(s)$ is of order $1/2$,

$$\exists K, M > 0, \forall s \in \mathbb{C}, \quad |F(s)| \leq K \exp(M|s|^{1/2}).$$

Set $F(s) = \sum_{n \geq 0} a_n s^n$ where $|a_n| \leq K^n/\Gamma(1+2n)$ with $K > 0$ independent of $n$. Then $F(s)y(s)$ corresponds in the time domain to

$$\sum_{n \geq 0} a_n y^{(n)}(t)$$

that is convergent when $t \mapsto y(t)$ is a $C^\infty$ time function of Gevrey order $\alpha < 1$: i.e. $\exists M > 0$ such that $|y^{(n)}(t)| \leq M^n \Gamma(1 + (\alpha+1)n)$

**Steady state controllability**

Steering from $\Psi = 0$, $v = 0$ at time $t = 0$, to $\Psi = 0$, $v = D$ at $t = T$ is possible with the following Gevrey function of order $\sigma$:

$$[0, T] \ni t \mapsto y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \bar{D} \dfrac{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right)}{\exp\left(-\left(\frac{T}{t}\right)^{\frac{1}{\sigma}}\right) + \exp\left(-\left(\frac{T}{T-t}\right)^{\frac{1}{\sigma}}\right)} & \text{for } 0 < t < T \\ \bar{D} & \text{for } t \geq T \end{cases}$$

with $\bar{D} = \frac{2\bar{\omega}D}{\sin^2(\sqrt{\bar{\omega}/2})}$. The fact that this function is of Gevrey order $\sigma$ results from its exponential decay of order $\sigma$ around $0$ and $1$.

**Practical computations via Cauchy formula**

$$y^{(n)}(t) = \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi$$

where $\gamma$ is a closed path around zero, $\sum_{n\geq 0} a_n y^{(n)}(t)$ becomes

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{2\imath\pi} \oint_\gamma \frac{y(t+\xi)}{\xi^{n+1}}\, d\xi = \frac{1}{2\imath\pi} \oint_\gamma \left( \sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} \right) y(t+\xi)\, d\xi.$$

But

$$\sum_{n\geq 0} a_n \frac{\Gamma(n+1)}{\xi^{n+1}} = \int_{D_\delta} F(s) \exp(-s\xi) ds = B_1(F)(\xi)$$

is the Borel transform of $F$.

**Practical computations via Cauchy formula (end)**

In the time domain $F(s)y(s)$ corresponds to

$$\frac{1}{2\imath\pi}\oint_\gamma B_1(F)(\xi)y(t+\xi)\,d\xi$$

where $\gamma$ is a closed path around zero. Such integral representation is very useful when $y$ is defined by convolution with a real signal $Y$,

$$y(\zeta)=\frac{1}{\varepsilon\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp(-(\zeta-t)^2/2\varepsilon^2)Y(t)\,dt$$

where $\mathbb{R}\ni t\mapsto Y(t)\in\mathbb{R}$ is any measurable and bounded function:

$$v(t)=\int_{-\infty}^{+\infty}\left[\frac{1}{\imath\varepsilon(2\pi)^{\frac{3}{2}}}\oint_\gamma B_1(F)(\xi)\exp(-(\xi-\tau)^2/2\varepsilon^2)\,d\xi\right]Y(t-\tau)\,d\tau.$$

## Conclusion

- 1-D wave equation: eigenvalue asymptotics $|\lambda_n| \sim n$:

$$\text{Prototype:} \quad \prod_{n=1}^{+\infty} \left(1 + \frac{s^2}{n^2}\right) = \frac{\sinh(\pi s)}{\pi s}$$

  entire function of exponential type, i.e, order one and finite type (OK).

- 1-D Heat equation: eigenvalue asymptotics $|\lambda_n| \sim n^2$:

$$\text{Prototype:} \quad \prod_{n=1}^{+\infty} \left(1 - \frac{s}{n^2}\right) = \frac{\sinh(\pi\sqrt{s})}{\pi\sqrt{s}}$$

  entire function of order $1/2$ (OK).

## Conclusion (continued)

Systems described by 2-D partial differential equation on $\Omega$ with 0-D control $u(t)$ on the boundary. An Example

$$\begin{aligned} \frac{\partial H}{\partial t} &= \Delta H \text{ on } \Omega \\ H &= u(t) \text{ on } \Gamma_1 \\ \frac{\partial H}{\partial n} &= 0 \text{ on } \Gamma_2 \end{aligned}$$

where the control is not distributed on $\Gamma_1$ ($\partial\Omega = \Gamma_1 \bigcup \Gamma_2$).

Steady-state controllability: steering in finite time from one steady-state to another steady-state not possible in general (Chitour et al. 2005).

## Conclusion (continued)

- 2D-heat equation: eigenvalue asymptotics $w_n \sim -n$

  Prototype: $$\prod_{n=1}^{+\infty}\left(1+\frac{s}{n}\right)\exp(-s/n) = \frac{\exp(-\gamma s)}{s\Gamma(s)}$$

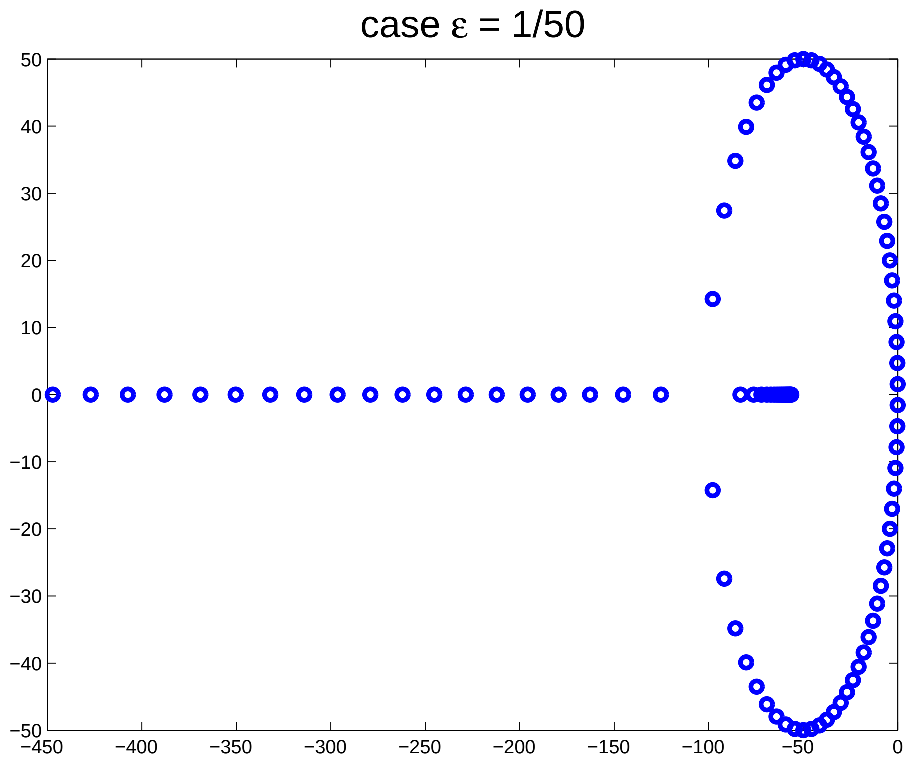
  entire function of order $1$ but of infinite type (**?**).

- 2D-wave equation: eigenvalue asymptotics $|w_n| \sim \sqrt{n}$

  Prototype: $$\prod_{n=1}^{+\infty}\left(1-\frac{s^2}{n}\right)\exp(s^2/n) = \frac{-\exp(\gamma s^2)}{s^2\Gamma(-s^2)}$$

  entire function of order $2$ but of infinite type (**?**).

## Conclusion (end)




1-D wave equation with internal damping:

$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 H}{\partial x^2} + \epsilon \frac{\partial^3 H}{\partial x^2 \partial t}$$

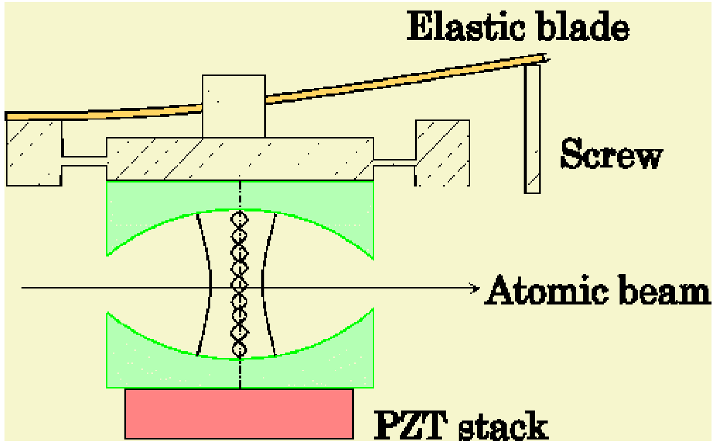
$$H(0,t) = 0, \quad H(1,t) = u(t)$$

where the eigenvalues are the zeros of analytic function

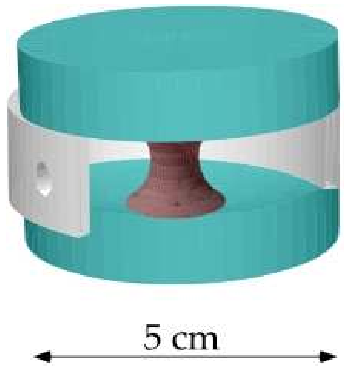
$$P(s) = \cosh\left(\frac{s}{\sqrt{\epsilon s + 1}}\right).$$

Approximate controllability depends on the choice made for functional space (Rosier-R 2006).

## *La Cavité Fabry-Perot supraconductrice*




*Mode gaussien avec un diamètre de 6mm*
*Grand champ par photon (1,5mV/m)*
*Grande durée de vie du champ (1ms) allongée par l 'anneau autour des miroirs*
*Accord en fréquence facile*
*Faible champ thermique ( < 0,1 photon )*



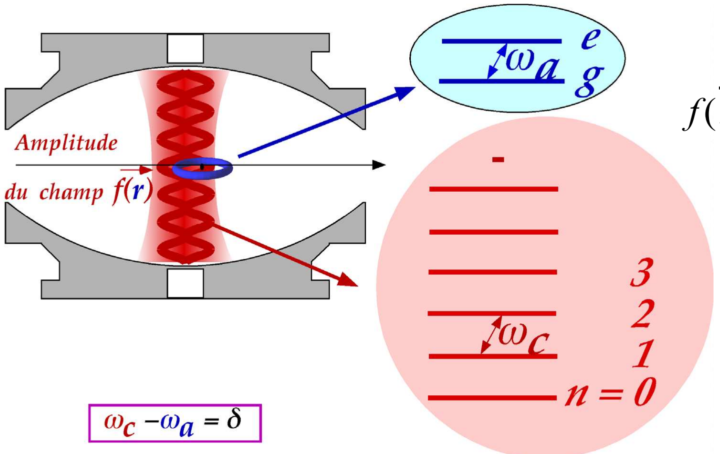

*



*Source: S. Haroche, cours au collège de France.

## *Le système atome-cavité*

***Spin 1/2* couplé à un *oscillateur harmonique***




$$f(\vec{r}) \approx e^{-\frac{x^2+y^2}{w^2}} \cos\left(\frac{2\pi z}{\lambda}\right)$$

$\omega_c - \omega_a = \delta$

**Hamiltonien:** $H = H_a + H_c + V_{a-c}$

$$H_a = \frac{\hbar\,\omega_a}{2} \left( | e \rangle \langle e | - | g \rangle \langle g | \right)$$

$$H_c = \frac{\hbar\,\omega_c}{2} \left( a^\dagger a + a\, a^\dagger \right) = \hbar\omega_c (a^\dagger a + 1/2)$$

$$V_{a-c} = - \vec{D}_a \cdot \vec{E}(r)$$

*Dipôle électrique*

*1000 a.u.*

$$\vec{D}_a = \vec{d}_{eg} | e \rangle \langle g | + \vec{d}^*_{eg} | g \rangle \langle e |$$

$$\vec{E}(r) = i E_0 \left( \vec{f}(\mathbf{r})\, a - \vec{f}^*(\mathbf{r})\, a^\dagger \right)$$

$$E_0 = \sqrt{\frac{\hbar\,\omega_c}{2\varepsilon_0 V_{cav}}}$$ *Champ du vide*

$$V_{cav} = \int |\vec{f}(\mathbf{r})|^2 d^3\mathbf{r}$$ *Volume du mode*

*



*Source: S. Haroche, cours au collège de France.

**A PDE formulation of Jaynes-Cummings controlled model**

$$\imath \frac{d}{dt}\psi = \frac{\Omega}{2}(a\,|e\rangle\,\langle g| + a^\dagger\,|g\rangle\,\langle e|)\psi - (va^\dagger + v^* a)\psi$$

Since $\psi \in L^2(\mathbb{R},\mathbb{C}) \bigotimes \mathbb{C}^2$ and $L^2(\mathbb{R},\mathbb{C}) \bigotimes \mathbb{C}^2 \sim (L^2(\mathbb{R},\mathbb{C}))^2$ we represent $\psi$ by two components $\psi_g$ and $\psi_e$ elements of $L^2(\mathbb{R},\mathbb{C})$. Up-to some scaling:

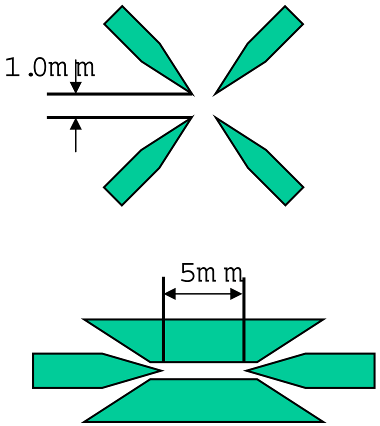
$$\imath \frac{\partial \psi_g}{\partial t} = \left(v_1 x + \imath v_2 \frac{\partial}{\partial x}\right)\psi_g + \left(x + \frac{\partial}{\partial x}\right)\psi_e$$

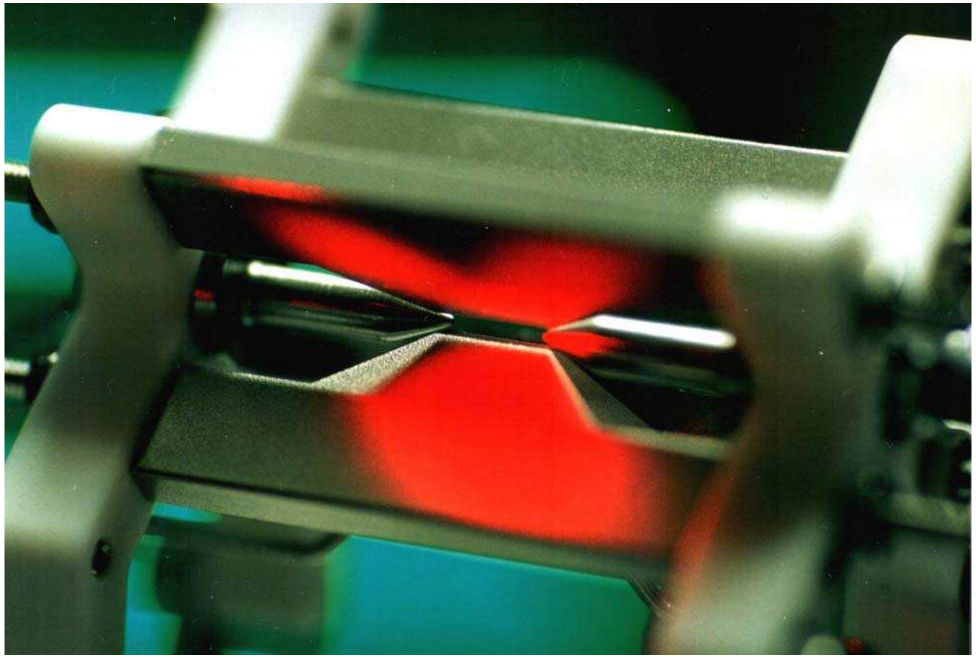
$$\imath \frac{\partial \psi_e}{\partial t} = \left(x - \frac{\partial}{\partial x}\right)\psi_g + \left(v_1 x + \imath v_2 \frac{\partial}{\partial x}\right)\psi_e$$

where $v = v_1 + \imath v_2 \in \mathbb{C}$ is the control. Remember that

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

## Innsbruck linear ion trap



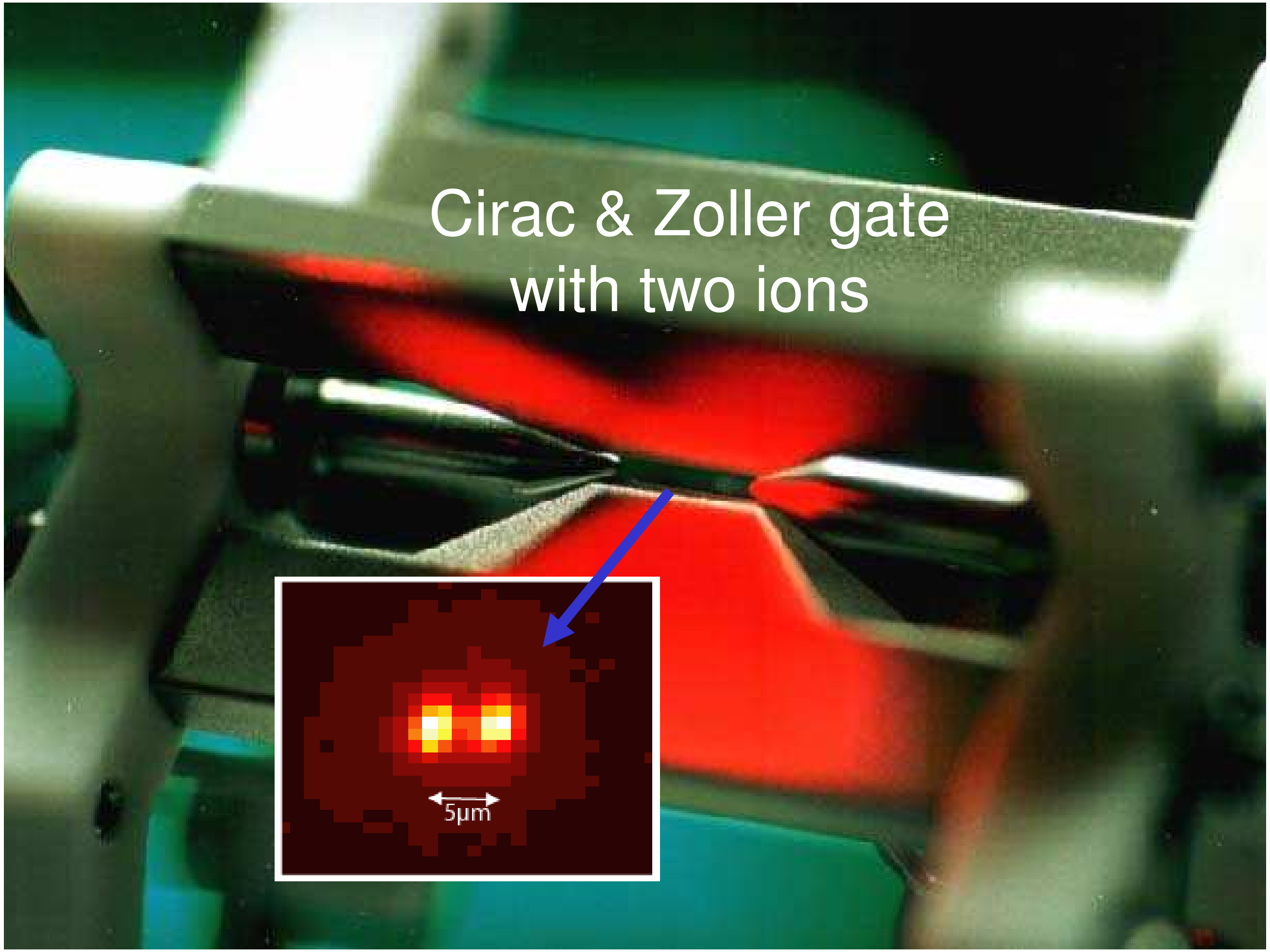



$\omega_{axial} \approx 0.7-2\,MHz$ $\omega_{radial} \approx 5\,MHz$

*



*Source: F. Schmidt-Kaler, séminaire au Collège de France en 2004.

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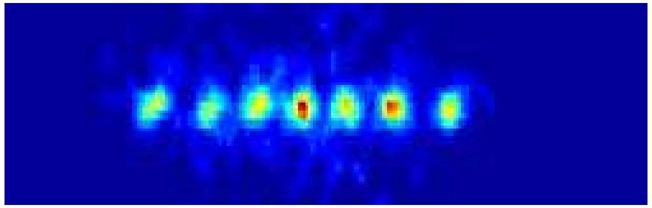
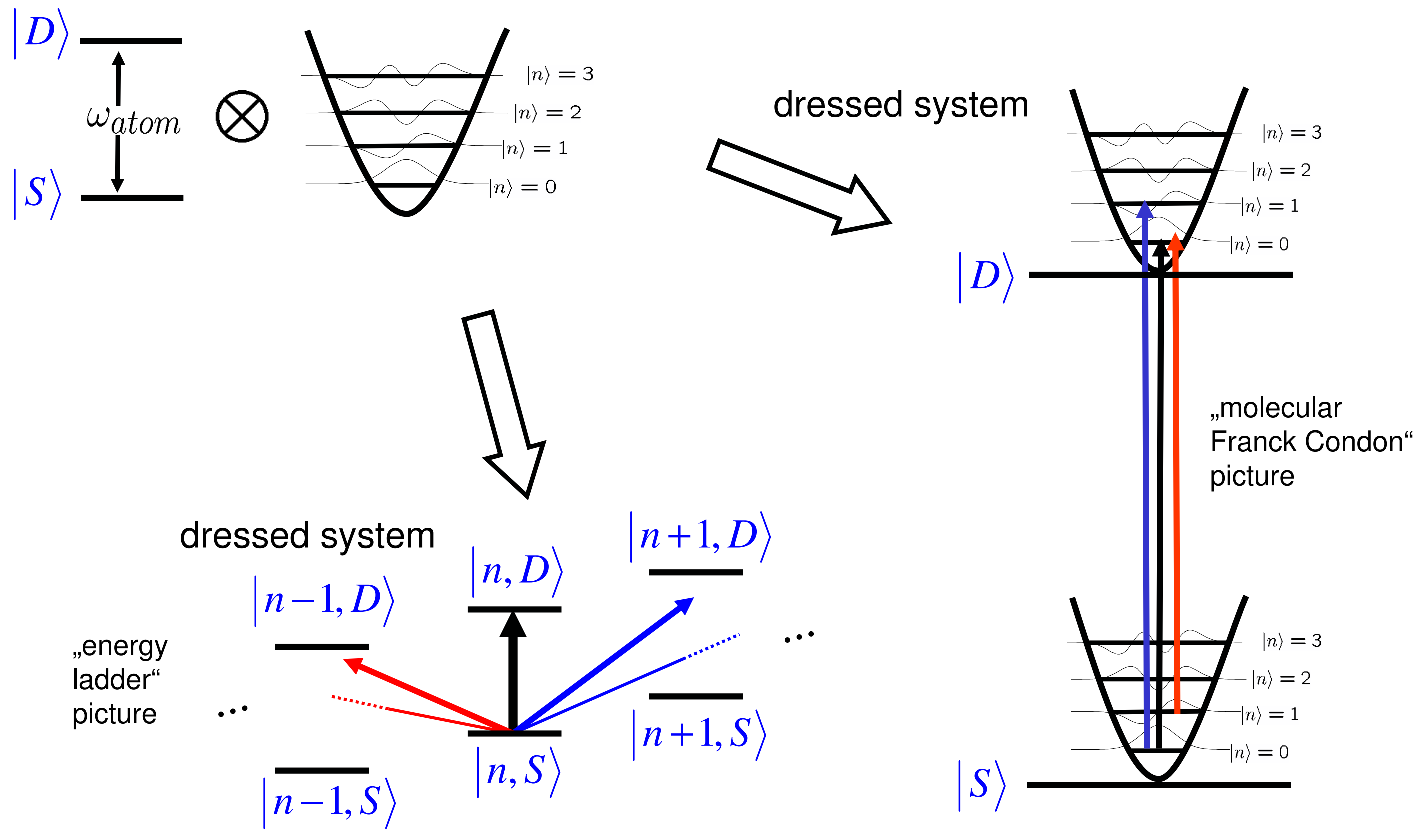
*Source: F. Schmidt-Kaler, séminaire au Collège de France en 2004.

# Laser coupling






*



*Source: F. Schmidt-Kaler, séminaire au Collège de France en 2004.

**A single trapped ion controlled via a laser**

$$\begin{aligned} H =& \Omega(a^\dagger a + 1/2) + \frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \\ & + \left[u e^{\imath(\omega t - kX)} + u^* e^{-\imath(\omega t - kX)}\right](|e\rangle\langle g| + |g\rangle\langle e|) \end{aligned}$$

with $kX = \eta(a + a^\dagger)$, $\eta \ll 1$ Lamb-Dicke parameter, $\omega \approx \omega_0$ and the vibration frequency $\Omega \ll \omega$, $u \in \mathbb{C}$ the control (amplitude and phase modulations of the laser of frequency $\omega$).

Assume $\omega = \omega_0$. In $\imath \frac{d}{dt}\psi = H\psi$, set $\psi = \exp(-\imath\omega t \sigma_z/2)\phi$. Then the Hamiltonian becomes

$$\begin{aligned} & \Omega(a^\dagger a + 1/2) + \\ & \quad \left[u e^{\imath(\omega t - \eta(a+a^\dagger))} + u^* e^{-\imath(\omega t - \eta(a+a^\dagger))}\right]\left(e^{-\imath\omega t}|g\rangle\langle e| + e^{\imath\omega t}|e\rangle\langle g|\right) \end{aligned}$$

## Rotating Wave Approximation and PDE formulation

We neglect highly oscillating terms with $e^{\pm 2\imath\omega t}$ and obtain the averaged Hamiltonian:

$$\tilde{H} = \Omega(a^\dagger a + 1/2) + u e^{-\imath\eta(a+a^\dagger)} |g\rangle\langle e| + u^* e^{\imath\eta(a+a^\dagger)} |e\rangle\langle g|$$

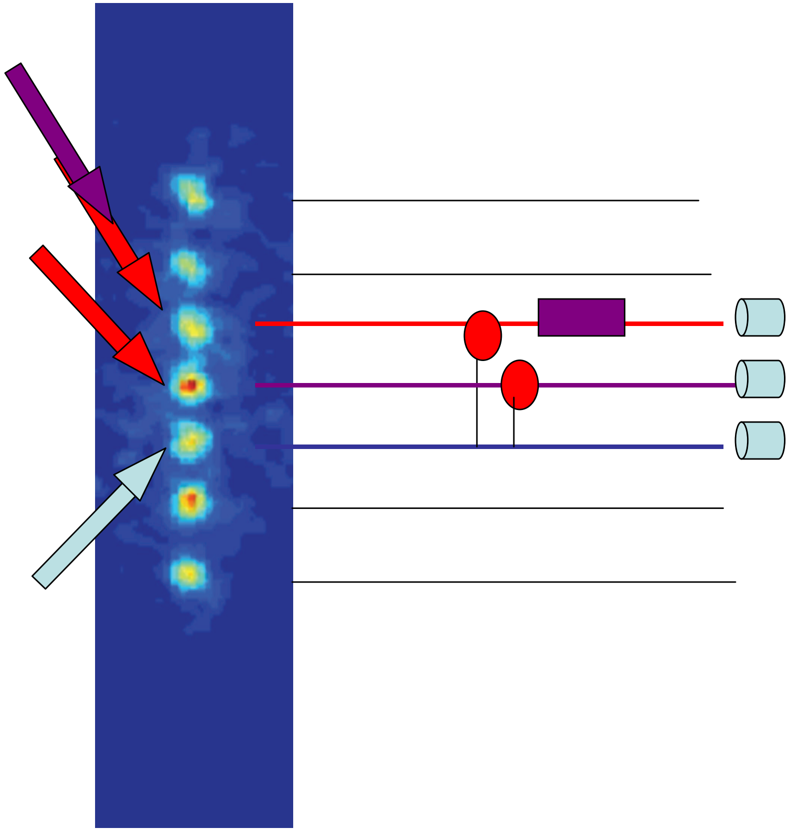
This corresponds to ($\eta \mapsto \eta\sqrt{2}$):

$$\imath\frac{\partial\psi_g}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_g + u e^{-\imath\eta x}\psi_e$$
$$\imath\frac{\partial\psi_e}{\partial t} = u^* e^{\imath\eta x}\psi_g + \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_e$$

where $u \in \mathbb{C}$ is the control $\left|\frac{d}{dt}u\right| \ll \omega|u|$ and $\eta \ll 1$, $\Omega \ll \omega$.

Controllability of this system ?

# Logique quantique avec une chaîne d'ions



*Des impulsions laser appliqués séquentiellement aux ions de la chaîne réalisent des portes à un bit et des portes à deux bits. La détection par fluorescence (éventuellement précédée par une rotation du bit) extrait l'information du système.*

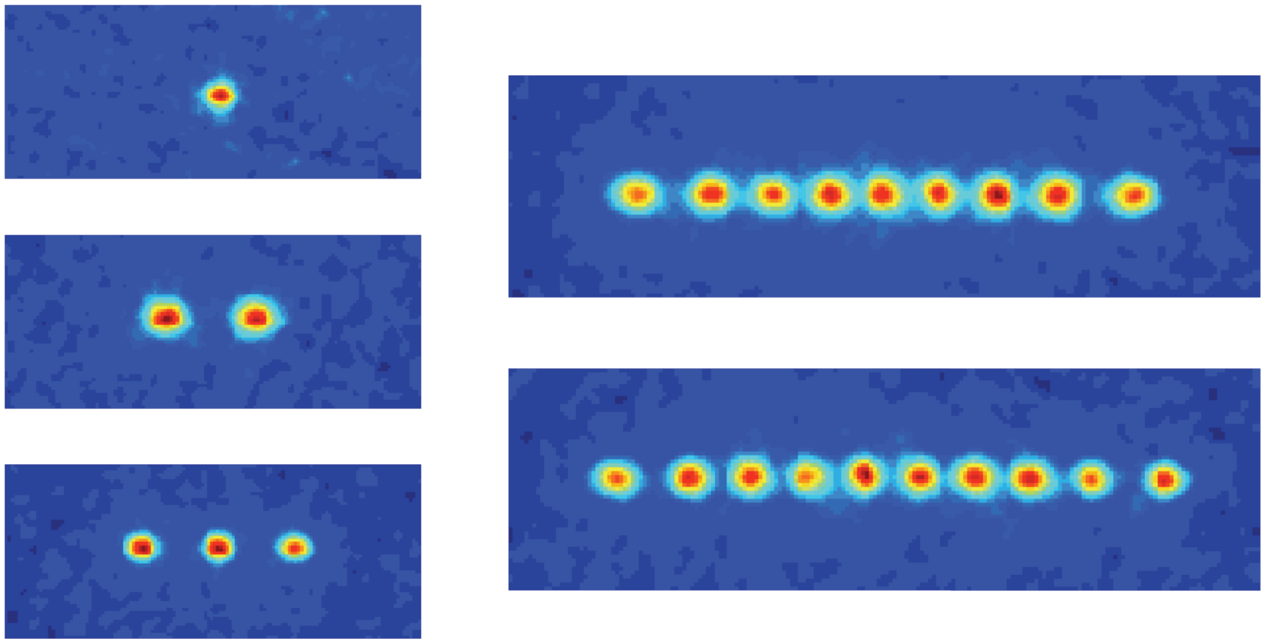
Beaucoup de problèmes à résoudre pour réaliser un tel dispositif…..

*



*Source: S. Haroche, CdF 2006.

## Quelques chaînes d'ions (Innsbruck)



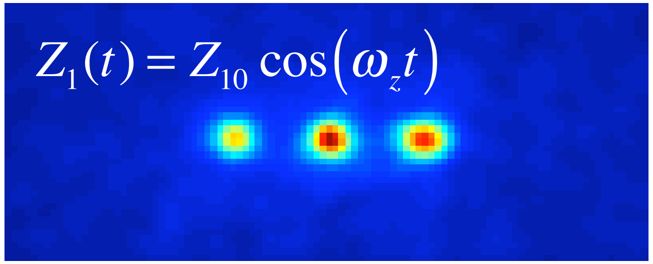
Fluorescence spatialement résolue. Détection en quelques millisecondes (voir première leçon)



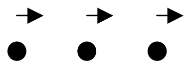
*

*Source: S. Haroche, CdF 2006.

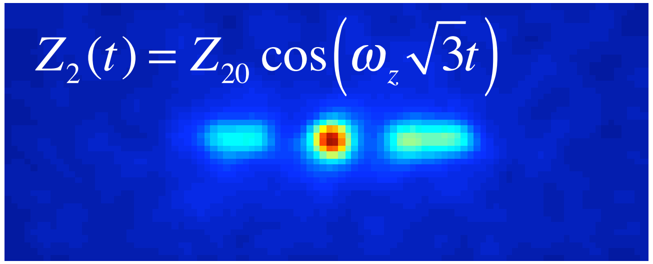
# Visualisation des modes ( N=3)



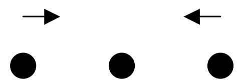

***Mode du CM***



$\omega_z$



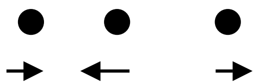

***Mode «accordéon»***



$\omega_z\sqrt{3}$

$$Z_3(t) = Z_{30}\cos\left(\omega_z\sqrt{29/5}\ t\right)$$

***Mode « ciseau »***



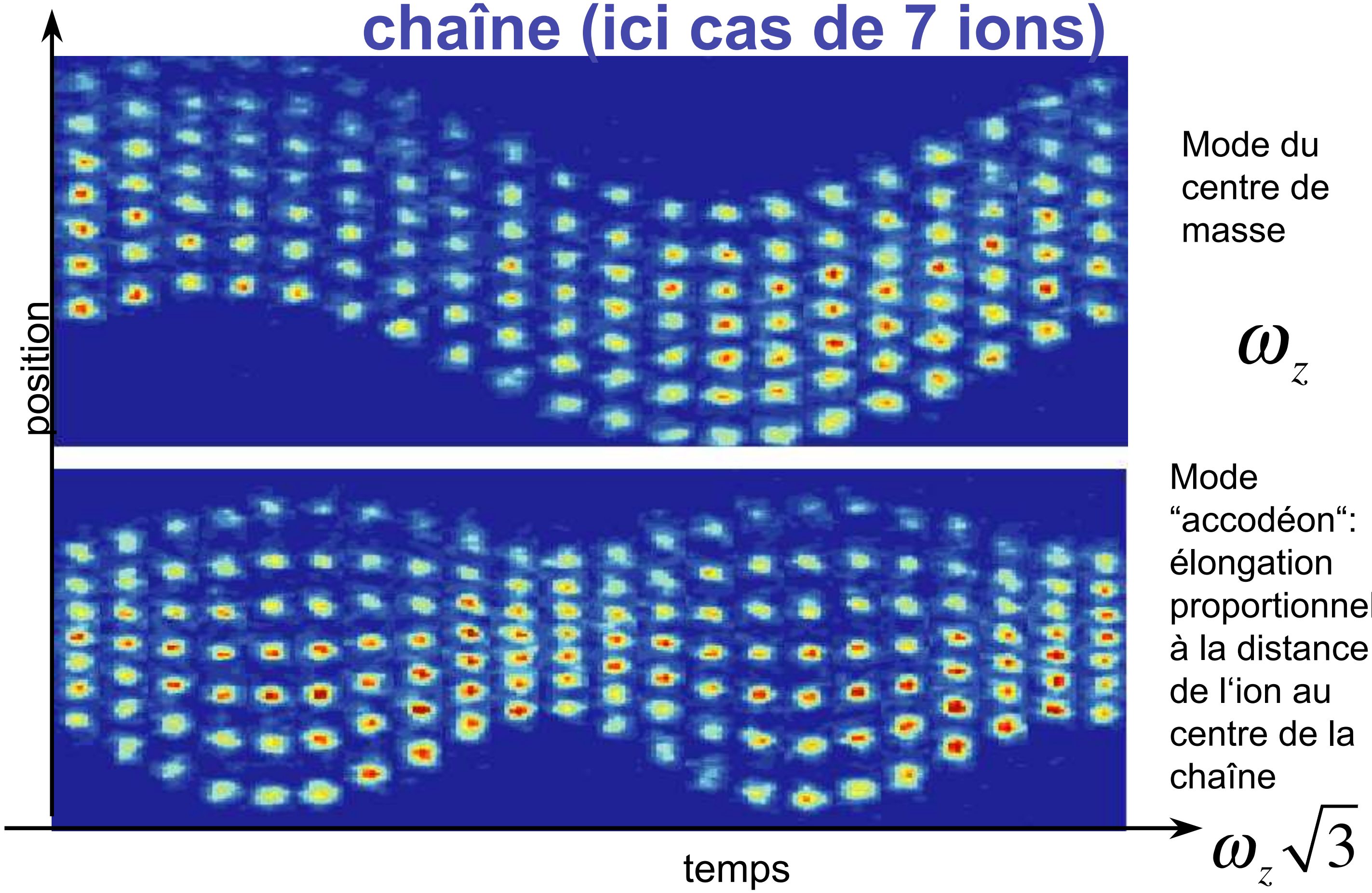
Les deux premiers modes (centre de masse et accordéon) sont pour tout $N$ ceux de fréquences les plus basses. Leurs fréquences sont indépendantes de $N$. Ce n'est plus vrai pour les modes plus élevés.



*

*Source: S. Haroche, CdF 2006.

## Les deux premiers modes de vibration sont indépendants du nombre d'ions dans la chaîne (ici cas de 7 ions)



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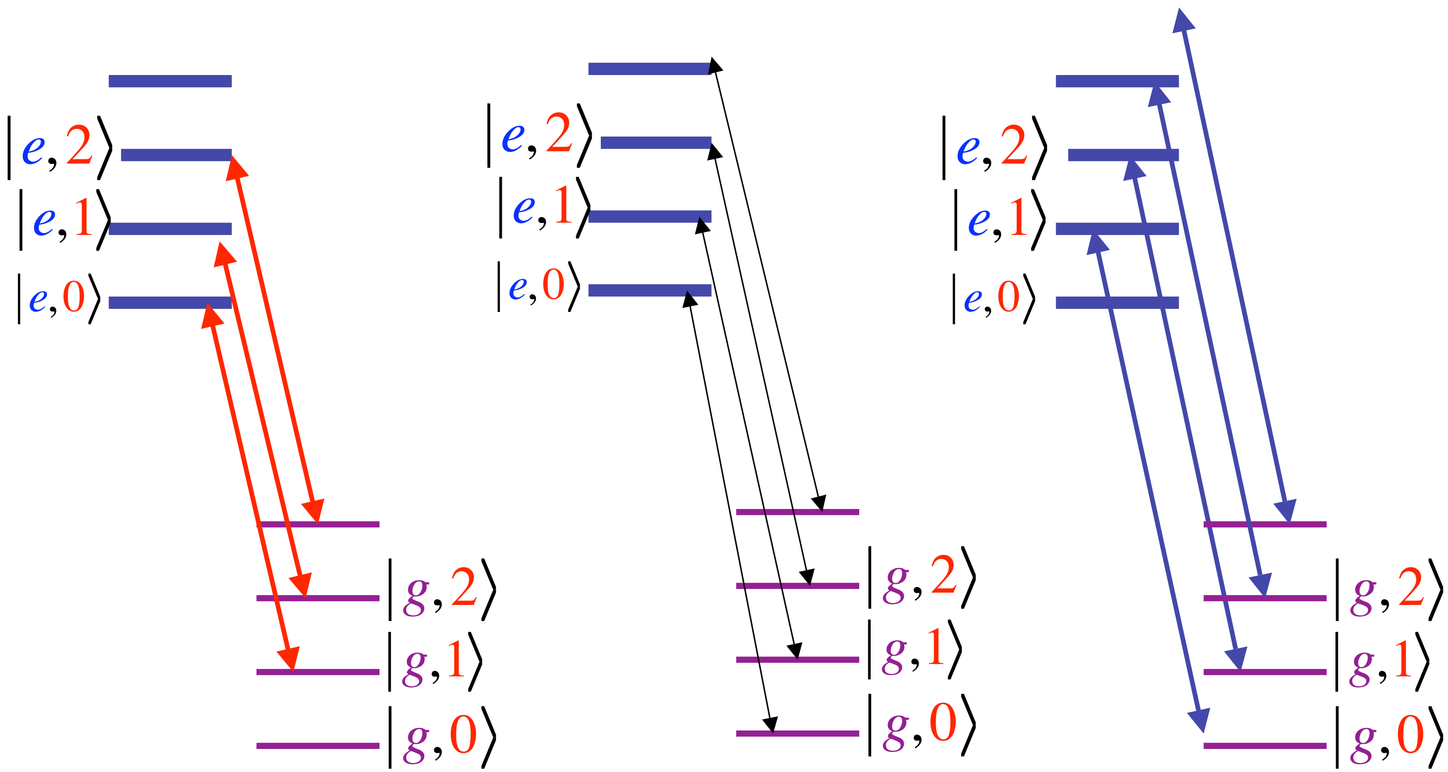
*Source: S. Haroche, CdF 2006.

# Spectre résolu de l'ion interprété en terme de création/annihilation de phonons *($\Gamma < \omega_z$)*




1ère bande latérale rouge:
$\omega_L = \omega_{eg} - \omega\,;\,\Delta n = -1$

Porteuse:
$\omega_L = \omega_{eg} \quad \Delta n = 0$

1ère bande latérale bleue:
$\omega_L = \omega_{eg} + \omega\,;\,\Delta n = +1$



*

*Source: S. Haroche, CdF 2006.

**Two trapped ions controlled via two lasers $\omega$ ( phonons $\Omega$ of the center of mass mode only)**

The instantaneous Hamiltionian:

$$
\begin{aligned}
H &= \Omega(a^\dagger a + 1/2) \\
&+ \frac{\omega_0}{2}(|e_1\rangle\langle e_1| - |g_1\rangle\langle g_1|) \\
&+ \left[u_1 e^{\imath(\omega t - k(a+a^\dagger))} + u_1^* e^{-\imath(\omega t - k(a+a^\dagger))}\right](|e_1\rangle\langle g_1| + |g_1\rangle\langle e_1|) \\
&+ \frac{\omega_0}{2}(|e_2\rangle\langle e_2| - |g_2\rangle\langle g_2|) \\
&+ \left[u_2 e^{\imath(\omega t - k(a+a^\dagger))} + u_2^* e^{-\imath(\omega t - k(a+a^\dagger))}\right](|e_2\rangle\langle g_2| + |g_2\rangle\langle e_2|)
\end{aligned}
$$

**Two trapped ions controlled via two lasers $\omega$**

The averaged Interaction Hamiltonian (RWA)

$$\begin{aligned}\tilde{H} &= \Omega(a^\dagger a + 1/2)\\ &+ u_1 e^{-\imath\eta(a+a^\dagger)} |g_1\rangle\langle e_1| + u_1^* e^{\imath\eta(a+a^\dagger)} |e_1\rangle\langle g_1|\\ &+ u_2 e^{-\imath\eta(a+a^\dagger)} |g_2\rangle\langle e_2| + u_2^* e^{\imath\eta(a+a^\dagger)} |e_2\rangle\langle g_2|\end{aligned}$$

with

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$$

and the wave function (probability amplitude $\psi_{\mu\nu} \in L^2(\mathbb{R},\mathbb{C})$, $\mu,\nu = e, g$):

$$|\psi\rangle = \psi_{gg}(x,t)\,|g_1 g_2\rangle + \psi_{ge}(x,t)\,|g_1 e_2\rangle + \psi_{eg}(x,t)\,|e_1 g_2\rangle + \psi_{ee}(x,t)\,|e_1 e_2\rangle$$

Qbit notations: $|1\rangle$ corresponds to the ground state $|g\rangle$ and $|0\rangle$ to the excited state $|e\rangle$.

**Two trapped ions controlled via two lasers $\omega$**

The PDE satisfied by $\psi(x,t) = (\psi_{gg}, \psi_{eg}, \psi_{ge}, \psi_{ee})$:

$$\imath \frac{\partial \psi_{gg}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{gg} + u_1 e^{-\imath \eta x}\psi_{eg} + u_2 e^{-\imath \eta x}\psi_{ge}$$

$$\imath \frac{\partial \psi_{ge}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{ge} + u_1 e^{-\imath \eta x}\psi_{ee} + u_2^* e^{-\imath \eta x}\psi_{gg}$$

$$\imath \frac{\partial \psi_{eg}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{eg} + u_1^* e^{\imath \eta x}\psi_{gg} + u_2 e^{-\imath \eta x}\psi_{ee}$$

$$\imath \frac{\partial \psi_{ee}}{\partial t} = \frac{\Omega}{2}\left(x^2 - \frac{\partial^2}{\partial x^2}\right)\psi_{ee} + u_1^* e^{\imath \eta x}\psi_{ge} + u_2^* e^{\imath \eta x}\psi_{eg}$$

where $u_1, u_2 \in \mathbb{C}$ are the two controls, laser amplitudes for ion no 1 and ion no 2 $\left|\frac{d}{dt}u_{1,2}\right| \ll \omega |u_{1,2}|$ and $\eta \ll 1$, $\Omega \ll \omega$.

Controllability of this system ?

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**CDG2:** C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg. Atom-Photon interaction: Basic Processes and Applications. Wiley, 1992. (also in French, CNRS editions)

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See also the contributions on the controlled Schrödinger equation of Claude Le Bris, Gabriel Turinicci, Mazyar Mirrahimi, Eric Cancès, Jean-Michel Coron, Karine Beauchard, Jean-Pierre Puel, Enriques Zuazua, . . .