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Feedback Stabilization of Nonlinear Systems: Sufficient Conditions and Lyapunov and Input-output Techniques

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1 Introduction.

This lecture is devoted to the survey of some recent results on feedback stabilization of nonlinear systems. This text can be seen as a prolongation of the overview written by E. Sontag in 1990 [83] in several directions where progress has been made. It consists of three parts:
The first part is devoted to sufficient conditions on the stabilization problem by means of discontinuous or time-varying state or output feedback.
In the second part, we present some techniques for explicitly designing these feedbacks by using Lyapunov's method. This introduces us with the notion of assignable Lyapunov function and leads us to concentrate our attention on systems having some special recurrent structure.
The third part presents some techniques for designing feedback based on $\mathcal{L}_\infty$ stability properties. This last section also addresses robustness through a small gain theorem.

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2 Sufficient conditions for state or output feedback stabilization

2.1 Introduction.

It is a classical result, see e.g. [82] Theorem 7 p. 134, that any linear control system which is controllable can be asymptotically stabilized by means of continuous feedback laws. A natural question is if this result still holds for nonlinear control systems. In 1979 Sussmann has shown that the global version of this result does not hold for nonlinear control systems: in [89] he has given an example of a nonlinear analytic control system which is globally controllable but cannot be globally asymptotically stabilized by means of continuous feedback laws. In [4] Brockett has shown that the local version also does not hold. To get around the problem of impossibility to stabilize many controllable systems by means of continuous feedback laws two main strategies have been proposed
(i) Asymptotic stabilization by means of a discontinuous feedback law -see e.g. the pioneer work by H. Sussmann [89]-
(ii) Asymptotic stabilization by means of a continuous periodic time-varying feedback law -see the pioneer work by Sontag and Sussmann [86], [74], and Section 2.3 below.

In Section 2.2 we give a relation between these two strategies. In Section 2.3 we present results showing that, in many cases, controllability implies stabilizability by means of time-varying static feedback laws.

In many practical situations only part of the state – called the output – is measured and therefore state feedback cannot be implemented; only output feedback are allowed. It is well known, see e.g. [82] Section 6.2, that any linear control system which is controllable and observable can be asymptotically stabilized by means of dynamic continuous feedback laws. Again it is natural to look if this result can be extended to the nonlinear case. In the nonlinear case there are many possible definitions for observability. The weakest requirement for observability is that, given two different states, there exists a control $t \to u(t)$ which leads two outputs which are not identical. With this definition of observability, the nonlinear control system

$$\dot{x} = u \in \mathbb{R},\ y = x^2 \in \mathbb{R} \tag{1}$$

where the state is x, the control u, and the output y is observable. This system is also clearly controllable and asymptotically stabilizable by means of (stationary) static feedback laws. But, see [14], this system cannot be asymptotically stabilized by means of stationary dynamic feedback laws. Again the introduction of time-varying feedback laws improve the situation; indeed control system (1) can be asymptotically stabilized by means of time-varying dynamic feedback laws. In Section 2.4 we present a result contained in [14] showing that many locally controllable and observable nonlinear control systems can be locally asymptotically stabilized by means of time-varying dynamic output feedback laws.

2.2 Discontinuous/continuous time-varying stabilizing feedback.

Throughout out all this survey, by (C) we denote the nonlinear control system

$$(C) : \dot{x} = f(x, u), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control. We assume that $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and that

$$f(0, 0) = 0. \tag{3}$$

The goal of this section is to show a relation between stabilizability by means of discontinuous feedback laws and stabilizability by means of continuous time-varying stabilizing feedback laws.

Before stating this relation let us first recall the definition of asymptotically stable for a time-varying dynamic system – we should in fact say uniformly asymptotically stable –

Definition 1. Let X be in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$. One says that 0 is locally asymptotically stable for $\dot{x} = X(x, t)$ if
(i) for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $\tau \in \mathbb{R}$ and for all $t \geq \tau$,

$$(\dot{x} = X(x, t), |x(\tau)| < \eta) \Rightarrow |x(t)| < \varepsilon \tag{4}$$

and if
(ii) there exists $\delta > 0$ such that, for all $\varepsilon > 0$, there exists $M > 0$ such that, for all s in $\mathbb{R}$,

$$\dot{x} = X(x, t) \text{ and } |x(s)| < \delta \tag{5}$$

imply

$$|x(\tau)| < \varepsilon, \forall \tau > s + M. \tag{6}$$

If, moreover, for all $\delta > 0$, there exists $M > 0$ such that (5) implies (6) for all s in $\mathbb{R}$, one says that 0 is globally asymptotically stable for $\dot{x} = X(x, t)$.

Throughout all this paper, and in particular in (4) and (5), by $\dot{x} = X(x, t)$ we denote any maximal solution of this differential equation. Let us emphasize that, since the vector field X is only continuous, the Cauchy problem $\dot{x} = X(x, t), x(t_0) = x_0$, where t_0 and x_0 are given, may have many maximal solutions. Let us recall that Kurzweil in [46] has shown that, even for vector fields which are only continuous, asymptotic stability is equivalent to the existence of a Lyapunov function.

Let us now define "asymptotically stabilizable by means of a continuous periodic feedback law"

Definition 2. System (C) is locally (resp. globally) asymptotically stabilizable by means of a continuous periodic feedback law of period T if there exists $u \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying

$$u(x, t+T) = u(x,t), \quad \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{7}$$

$$u(0,t) = 0, \quad \forall t \in \mathbb{R}, \tag{8}$$

such that, for the system $\dot{x} = f(x, u(x,t))$, 0 is a locally (resp. globally) asymptotically stable point.

We now need to specify the meaning of asymptotic stability for a system $\dot{x} = X(x)$ where X is a discontinuous vector field. Many definitions are possible but, following H. Hermes [31], it seems natural to adopt

Definition 3. Let $X \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Then 0 is a locally asymptotically stable point of $\dot{x} = X(x)$ if (i) and (ii) of Definition 1 hold for any (maximal) solution in the Filippov sense of $\dot{x} = X(x)$. If, moreover, (ii) of Definition 1 holds for any $\delta > 0$ and any (maximal) solution in the Filippov sense of $\dot{x} = X(x)$, then 0 is a globally asymptotically stable point of $\dot{x} = X(x)$.

Let us recall that a solution in the Filippov sense of $\dot{x} = X(x)$ on an interval I is (see [F]) a locally absolutely continuous map from I into $\mathbb{R}^n$ such that

$$\dot{x}(t) \in F(x(t)) \ \text{ for almost all } t \in I \tag{9}$$

with

$$F(x) \ := \bigcap_{\epsilon > 0} \bigcap_{|N|=0} \overline{\text{conv}} X((x + \epsilon B) \backslash N), \tag{10}$$

where B is the unit ball of $\mathbb{R}^n$, and, for a set A, $|A|$ is the Lebesgue measure of A and $\overline{\text{conv}} A$ is the smaller closed convex set containing A. Of course, a maximal solution in the Filippov sense of $\dot{x} = X(x)$ is a solution x in the Filippov sense on some interval I such that there exists no solution in the Filippov sense defined on an interval which contains strictly I and which is equal to x on I.

Note that, if X is continuous,

$$F(y) = \{X(y)\} \tag{11}$$

and therefore in this case our definition of asymptotic stability coincide with the one given in Definition 1.

We now define "asymptotically stabilizable by means of a discontinuous feedback law".

Definition 4. System (C) is locally (resp. globally) asymptotically stabilizable by means of a discontinuous feedback law if there exists $u \in L^\infty_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\text{Essential Sup } \{|u(x)|;\ |x| < \epsilon\} \to 0 \text{ as } \epsilon \to 0, \tag{12}$$

and 0 is a locally (resp. globally) asymptotically stable point of $\dot{x} = f(x, u(x))$.

The reason for considering in Definition 4 solutions in the Filippov sense is, as explained in [31], the following one : the feedback law $u(x(t))$ is determined after making a measurement of the state $x(t)$ at time t; of course this measurement gives only an approximation of $x(t)$: there is an "error" $e(t)$ between $x(t)$ and its measurement. A direct consequence of Lemma 3 in [31] is the following proposition which is proved in [19] -see also [31]-

Proposition 5. *Assume that f is locally Lipschitzian with respect to x. Let $x : [0,T] \to \mathbb{R}^n$ be a Filippov solution of $\dot{x} = f(x, u(x))$ where $u \in L^\infty_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^m)$. Let ϵ be a positive real number. Then there exist $e \in L^\infty((0,T);\mathbb{R}^m)$ and an absolutely continuous function $y : [0,T] \to \mathbb{R}^n$ such that*

$$|e(t)| \leq \epsilon \text{ for all } t \text{ in } (0,T), \tag{13}$$

$$\dot{y}(t) = f(y(t), u(y(t) + e(t))) \text{ for almost all } t \text{ in } (0,T), \tag{14}$$

$$y(0) = x(0), \tag{15}$$

and

$$|y(t) - x(t)| \leq \epsilon \text{ for all } t \text{ in } [0,T]. \tag{16}$$

Proposition 5 justifies our definition of asymptotically stabilization by means of a discontinuous feedback law. With this definitions one has the following theorem, which is proved in [19],

Theorem 6. *Assume that $\dot{x} = f(x,u)$ can be locally (resp. globally) asymptotically stabilized by means of a discontinuous feedback law. Then, for any $T > 0$, $\dot{x} = f(x,u)$ can be locally (resp. globally) stabilized by means of a continuous time-varying feedback law of period T; if, moreover, $\dot{x} = f(x,u)$ is an affine system (i.e. $f(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$), then $\dot{x} = f(x,u)$ can be locally (resp. globally) asymptotically stabilized by means of a continuous feedback law (independent of t : $u = u(x)$).*

Remark 7. There are completely controllable affine systems which are globally asymptotically stabilized by means of a continuous periodic time varying feedback law which cannot be locally asymptotically stabilized by means of a continuous feedback law (e.g. $\dot{x}_1 = u_1$, $\dot{x}_2 = u_2$, $\dot{x}_3 = x_1u_2 - x_2u_1$; see [4], [74], [9], or Section 2.3). By Theorem 6, these systems cannot be locally asymptotically stabilized by means of a discontinuous feedback law; see also [72] for the same conclusion but with a different approach. Note that this is not in contradiction with [89]: indeed our definition of asymptotic stability is different from the one used in [89] (see e.g. the definition of "steers M to p" in [89]; in particular we do not have any "exit rule" E on the singular set of u in our definition of asymptotic stability).

2.3 Time-varying feedback.

This section is divided in two subsections: the first subsection concerns nonlinear control system without drift, the second subsection concerns systems which may have a drift term.

Systems without drift. In this subsection we assume that

$$f(x,u) = \sum_{i=1}^{m} u_i f_i(x). \tag{17}$$

Let us denote by $\mathrm{Lie}\{f_1, \ldots, f_m\}$ the Lie subalgebra generated by the vector fields $f_1, \ldots, f_m$. Then one has

Theorem 8. *Assume that, for all $x \in \mathbb{R}^n \backslash \{0\}$,*

$$\{h(x);\ h \in Lie\{f_1, \ldots, f_m\}\} = \mathbb{R}^n. \tag{18}$$

Then, for all $T > 0$, there exists u in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that

$$u(0,t) = 0,\ \ \forall t \in \mathbb{R}, \tag{19}$$

$$u(x, t+T) = u(x,t),\ \forall x \in \mathbb{R}^n,\ \ \forall t \in \mathbb{R}, \tag{20}$$

and 0 is globally asymptotically stable for

$$\dot{x} = f(x, u(x,t)) = \sum_{i=1}^{m} u_i(x,t) f_i(x). \tag{21}$$

The proof of this theorem is given in [9]; it relies on a method, that we have called the "return method", which can also be used to prove controllability in some cases -see, e.g., [13], [15]- or obtain numerical techniques for the steering of arbitrary systems without drift, see [84].

General systems. Let us first point out that in [86] Sontag and Sussmann have proved that any one dimensional state nonlinear control system which is locally (resp. globally) controllable can be locally (resp. globally) asymptotically stabilized by means of time-varying static feedback laws. Let us also point out that it follows from Sussmann [89] that a result similar to Theorem 8 does not hold for systems with a drift term: more precisely there are analytic control systems (C) controls which are globally controllable for which there is no u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ for which 0 is globally asymptotically stable for $\dot{x} = f(x, u(x,t))$. In fact the proof of [89] requires uniqueness of the trajectories of $\dot{x} = f(x, u(x,t))$. But this can always been assumed; indeed it follows easily from Kurzweil's result [46] that, if there exists u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that 0 is globally asymptotically stable for $\dot{x} = f(x, u(x,t))$, then there exists $\bar{u}$ in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m) \cap C^\infty\left((\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}; \mathbb{R}^m\right)$ such that 0 is globally asymptotically stable for $\dot{x} = f(x, \bar{u}(x,t))$; for such a $\bar{u}$ one has uniqueness of the trajectories of $\dot{x} = f(x, \bar{u}(x,t))$. But we are going to see in this subsection that a local version of Theorem 8 holds for many control systems which are Small Time Locally Controllable (STLC).

Let us again introduce some definitions

Definition 9. The origin (of $\mathbb{R}^n$) is locally continuously reachable (for (C)) in small time if, for all positive real number T, there exist a positive real number ε and u in $C^0\left(\mathbb{R}^n; L^1\left((0,T);\mathbb{R}^m\right)\right)$ such that

$$\operatorname{Sup}\{|u(a)(t)|; t \in (0,T)\} \to 0 \text{ as } a \to 0,$$

$$(\dot{x} = f(x, u(x(0))(t)),\ |x(0)| < \varepsilon) \Rightarrow x(T) = 0.$$

Let us notice that, following a method due to M. Kawski [42] (see also [30]), we have proved in [10, Lemma 3.1 and Section 5] that "many" sufficient conditions for Small Time Locally Controllability imply that the origin is locally continuously reachable in small time. This is in particular the case for the Hermes condition [32] or [91] and its generalization due to H.J. Sussmann [92], Theorem 7.3 ; this is in fact also the case for the Bianchini and Stefani condition [3, Corollary p. 970], which extends [92, Theorem 7.3] .

Our next definition is

Definition 10. System (C) is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws if, for any positive real number T, there exist ε in $(0,+\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ of class C^∞ on $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}$ such that

$$u(0,t) = 0,\ \forall t \in \mathbb{R}, \tag{22}$$

$$u(x, t+T) = u(x,t),\ \forall t \in \mathbb{R}, \tag{23}$$

$$((\dot{x} = f(x, u(x,t)) \text{ and } x(s) = 0) \Rightarrow (x(t) = 0\, \forall t \geq s)),\ \forall s \in \mathbb{R}, \tag{24}$$

$$\begin{aligned}((\dot{x} = f(x, u(x,t)) &\text{ and } |x(s)| \leq \varepsilon) \Rightarrow \\ &(x(t) = 0,\ \forall t \geq s + T))\ \forall s \in \mathbb{R}.\end{aligned} \tag{25}$$

Note that (23), (24), and (25) imply that 0 is locally asymptotically stable for $\dot{x} = f(x, u(x,t))$; see [12, Lemma 2.15] for a proof.

Definition 11. [11]. The strong jet accessibility subspace of (C) at $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is the subspace of $\mathbb{R}^n$, denoted by $a\,(\bar{x}, \bar{u})$, spanned by

$$\{h(\bar{x}); h \in \{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u})\alpha \in \mathbb{N}^m, |\alpha| \geq 1 \cup \operatorname{Br}_2(f, \bar{u})\}\} \tag{26}$$

where $\operatorname{Br}_2(f, \bar{u})$ denotes the set of iterated Lie brackets of length at least 2 of vector fields in $\{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u}); \alpha \in \mathbb{N}^m\}$.

Remark 12. One easily checks that the usual strong accessibility subspace of (C) at $\bar{x}$ (see e.g. [93], p. 101) contains $a(\bar{x},\bar{u})$ for all $\bar{u}$ in $\mathbb{R}^m$ and that, if f is analytic with respect to x and u or is a polynomial with respect to u, these inclusions are all equalities.

Our last definition before stating our main result is

Definition 13. System (C) satisfies the strong jet accessibility rank condition at $(\bar{x},\bar{u})$ if

$$a(\bar{x},\bar{u}) = \mathbb{R}^n. \tag{27}$$

Remark 14. It follows from Remark 12 that if (C) satisfies the strong jet accessibility rank condition at $(\bar{x},\bar{u})$ then it satisfies the usual strong accessibility rank condition at $\bar{x}$ and the converse holds if f is analytic with respect to x and u or is a polynomial with respect to u.

Note that, if (C) is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws, then $0 \in \mathbb{R}^n$ is locally continuously reachable for (C). The main result of this subsection is that the converse holds if $n \notin \{2,3\}$ and if (C) satisfies the strong jet accessibility rank condition at $(0,0)$, i.e.

Theorem 15. *Assume that* 0 *is locally continuously reachable in small time, that* (C) *satisfies the strong jet accessibility rank condition at* $(0,0)$*, and that*

$$n \notin \{2,3\}. \tag{28}$$

Then (C) *is locally stabilizable in small time by means of almost smooth periodic time-varying feedback laws.*

This theorem is proved in [12] when $n \geq 4$ and in [16] when $n = 1$. Let us just sketch the proof of [12].

Let I be an interval of $\mathbb{R}$. By a trajectory of the control system (C) on I we mean $(\gamma,u) \in C^\infty(I;\mathbb{R}^n \times \mathbb{R}^m)$ satisfying $\dot{\gamma}(t) = f(\gamma(t),u(t))$ for all t in I. The linearized control system around (γ,u) is $\dot{\xi} = A(t)\xi + B(t)w$ where the state is $\xi \in \mathbb{R}^n$, the control is $w \in \mathbb{R}^m$, and $A(t) = \partial f/\partial x(\gamma(t),u(t)) \in \mathcal{L}\left(\mathbb{R}^{\backslash};\mathbb{R}^{\backslash}\right)$, $B(t) = \partial f/\partial u(\gamma(t),u(t)) \in \mathcal{L}\left(\mathbb{R}^{\Updownarrow};\mathbb{R}^{\backslash}\right)$ for all t in I. We first introduce the following definition

Definition 16. The trajectory (γ,u) is supple on $S \subset I$ if, for all s in S,

$$\mathrm{Span}\{((d/dt) - A(t))^i B(t)\big|_{t=s} w \ ; \ w \in \mathbb{R}^m, i \geq 0\} = \mathbb{R}^n. \tag{29}$$

In (29) we use the classical convention $(d/dt - A(t))^0 B(t) = B(t)$. Let us recall that L. Silverman and H. Meadows have shown in [75] that (2.1) implies that the linearized control system around (γ,u) is controllable with impulsive controls at time s (in the sense of [40] p. 614). Let T be a positive real number. For u in $C^0(\mathbb{R}^n \times [0,T];\mathbb{R}^m)$ and a in $\mathbb{R}^n$, let $x(a,\cdot;u)$ be the maximal solution of $\partial x/\partial t = f(x,u(a,t))$, $x(a,0;u) = a$. Let, also, C^* be the set of $u \in C^0(\mathbb{R}^n \times [0,T];\mathbb{R}^m)$ of class C^∞ on $(\mathbb{R}^n\backslash\{0\}) \times [0,T]$ and vanishing on $\{0\} \times [0,T]$. For simplicity, in this sketch of proof, we omit some details which are important to take care of the uniqueness property (24) (note that without (24) one does not have stability).

Step 1. Using (1.8), (1.9), and [10] or [11], one proves that there exist ϵ_1 in $(0,+\infty)$ and u_1 in C^*, vanishing on $\mathbb{R}^n \times \{T\}$, such that

$$|a| \leq \epsilon_1 \Rightarrow x(a,T;u_1) = 0, \tag{30}$$

$$0 < |a| \leq \epsilon_1 \Rightarrow (x(a,\cdot;u_1),u_1(a,\cdot)) \text{ is supple on } [0,T]. \tag{31}$$

Step 2. Let Γ be a closed submanifold of $\mathbb{R}^n\backslash\{0\}$ of dimension 1 such that $\Gamma \subset \{x \in \mathbb{R}^n; 0 < |x| < \epsilon_1\}$. Perturbing in a suitable way u_1 one obtains a map u_2 in C^*, vanishing on $\mathbb{R}^n \times \{T\}$, such that

$$|a| \leq \epsilon_1 \Rightarrow x\,(a,T;u_2) = 0, \tag{32}$$

$$0 < |a| \leq \epsilon_1 \Rightarrow (x\,(a,\cdot;u_2)\,,u_2(a,\cdot)) \text{ is supple on } [0,T], \tag{33}$$

and

$$a \in \Gamma \rightarrow x\,(t,a;u_2) \text{ is an embedding of } \Gamma \text{ into } \mathbb{R}^n\backslash\{0\},\ \forall t \in [0,T). \tag{34}$$

Here one uses the assumption $n \geq 4$ and one proceeds as in the classical proof of the Whitney embedding theorem (see e.g. [26] Chapter II, Section 5). Let us emphasize that this is only in this step that we use this assumption.

Step 3. From Step 2 one deduces the existence of u_3^* in C^*, vanishing on $\mathbb{R}^n \times \{T\}$, and of an open neighborhood $\mathcal{N}^*$ of Γ in $\mathbb{R}^n\backslash\{0\}$ such that

$$a \in \mathcal{N}^* \Rightarrow x\,(a,T;u_3^*) = 0, \tag{35}$$

$$a \in \mathcal{N}^* \rightarrow x\,(a,t;u_3^*) \text{ is an embedding of } \mathcal{N}^* \text{ into } \mathbb{R}^n\backslash\{0\},\ \forall t \in [0,T). \tag{36}$$

This embedding property allows to transform the open-loop control u_3^* into a feedback law u_3 on $\{(x\,(a,t;u_3)\,,t)\,; a \in \mathcal{N}, t \in [0,T)\}$. So - see in particular (2.7) and note that u_3^* vanishes on $\mathbb{R}^n \times \{T\}$- there exist u_3 in C^* and an open neighborhood $\mathcal{N}$ of Γ in $\mathbb{R}^n\backslash\{0\}$ such that

$$\left(x(0) \in \mathcal{N} \text{ and } \dot{x} = \{\,(x,u_3(x,t))\right) \Rightarrow (x(T) = 0). \tag{37}$$

One can also impose, for all τ in $[0,T]$,

$$(\dot{x} = f\,(x,u_3(x,t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0 \quad \forall t \in [\tau,T]). \tag{38}$$

Step 4. In this last step one shows the existence of a closed submanifold of $\mathbb{R}^n\backslash\{0\}$ of dimension 1 included in the set $\{x \in \mathbb{R}^n; 0 < |x| < \epsilon_1\}$ such that for any neighborhood $\mathcal{N}$ of Γ in $\mathbb{R}^n \backslash \{0\}$ there exists u_4 in C^* such that, for some ϵ_4 in $(0, +\infty)$,

$$(\dot{x} = f(x, u_4(x,t)) \text{ and } |x(0)| < \epsilon_4) \Rightarrow (x(T) \in \mathcal{N} \cup \{\imath\}), \tag{39}$$

$$((\dot{x} = f(x, u_4(x,t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0 \quad \forall t \in [\tau, T])) \, \forall \tau \in [0, T]. \tag{40}$$

Finally let $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ be equal to u_4 on $\mathbb{R}^n \times [0, T]$, $2T$-periodic with respect to time, and such that $u(x,t) = u_3(x, t-T)$ for all (x,t) in $\mathbb{R}^n \times (T, 2T)$. Then u vanishes on $\{0\} \times \mathbb{R}$, is continuous on $\mathbb{R}^n \times (\mathbb{R} \backslash \mathbb{Z}T)$, of class C^∞ on $(\mathbb{R}^n \backslash \{0\}) \times (\mathbb{R} \backslash \mathbb{Z}T)$, and satisfies

$$(\dot{x} = f(x, u(x,t)) \text{ and } |x(0)| < \epsilon_4) \Rightarrow (x(2T) = 0), \tag{41}$$

$$(\dot{x} = f(x, u(x,t)) \text{ and } x(\tau) = 0) \Rightarrow (x(t) = 0, \quad \forall t \geq \tau) \, \forall \tau \in \mathbb{R}, \tag{42}$$

which implies, see [12], that (25) holds, with $4T$ instead of T and $\epsilon > 0$ small enough, and that 0 is uniformly locally asymptotically stable for the system $\dot{x} = f(x, u(x,t))$. Since T is arbitrary, Theorem 15 is proved (modulo a problem of regularity of u at (x,t) in $\mathbb{R}^n \times \mathbb{Z}T$ that is fixed in [12]).

Remark 17. We conjecture that assumption (28) can be removed in Theorem 15.

2.4 Time-varying output feedback.

In this section only part of the state (called the output) is measured; let us denote by $(\tilde{C})$ we denote the control system

$$(\tilde{C}) : \dot{x} = f(x, u), \; y = h(x), \tag{43}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $y \in \mathbb{R}^p$ is the output. Again $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and satisfies (3); we also assume that $h \in C^\infty(\mathbb{R}^n; \mathbb{R}^p)$ and satisfies

$$h(0) = 0. \tag{44}$$

In order to state the main result of this section we first introduce some definitions

Definition 18. System $(\tilde{C})$ is said to be locally stabilizable in small time by means of continuous static periodic time-varying output feedback laws if, for any positive real number T, there exist ε in $(0, +\infty)$ and u in $C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that (22), (23), (24), (25) hold and such that

$$u(x,t) = \bar{u}(h(x), t) \tag{45}$$

for some $\bar{u}$ in $C^0(\mathbb{R}^p \times \mathbb{R}; \mathbb{R}^n)$.

Our next definition concerns dynamic stabilizability.

Definition 19. System $(\tilde{C})$ is locally stabilizable in small time by means of continuous dynamic periodic time-varying state (resp. output) feedback laws if, for some integer $k \geq 0$, the control system

$$\dot{x} = f(x,u),\ \dot{z} = v,\ \tilde{h}(x,z) = (h(x),z), \tag{46}$$

where the state is $(x,z) \in \mathbb{R}^n \times \mathbb{R}^k$, the control $(u,v) \in \mathbb{R}^m \times \mathbb{R}^k$, and the output $\tilde{h}(x,z) \in \mathbb{R}^p \times \mathbb{R}^k$, is locally stabilizable in small time by means of continuous static periodic time-varying state (resp. output) feedback laws.

In the above definition, System (46) with $k = 0$ is, by convention, system $(\tilde{C})$. Let us also point out that it is proved in [10, Section 3], that if, for system$(\tilde{C})$, 0 is continuously reachable in small time then $(\tilde{C})$ is locally stabilizable in small time by means of continuous dynamic periodic time-varying state feedback; this also follows from Theorem 15 – but the proof given in [10, Section 3], which gives a weaker result, is much simpler than the proof of Theorem 15 –.

For our last definition one needs to introduce some notations. For α in $\mathbb{N}^m$ and $\bar{u}$ in $\mathbb{R}^m$, let $f_{\bar{u}}^{\alpha}$ in $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be defined by

$$f_{\bar{u}}^{\alpha}(x) = \frac{\partial^{|\alpha|} f}{\partial u^\alpha}(x,\bar{u})\ \forall x \in \mathbb{R}^n. \tag{47}$$

Let $\mathcal{O}(\tilde{C})$ be the subspace of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m;\mathbb{R}^p)$ spanned by the maps ω such that, for some integer $r \geq 0$ -depending on ω- and for some sequence $\alpha_1, ..., \alpha_r$ of r multi-indices in $\mathbb{N}^m$,we have, for all $x \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^m$,

$$\omega(x,u) = L_{f_u^{\alpha_1}} ... L_{f_u^{\alpha_r}} h(x), \tag{48}$$

where $L_{f_{\bar{u}}^{\alpha_i}}$ denotes Lie derivatives with respect to $f_{\bar{u}}^{\alpha_i}$ and where, by convention, if $r = 0$ the right hand side of (48) is $h(x)$. With these notations our last definition is

Definition 20. System $(\tilde{C})$ is locally Lie null-observable if there exists a positive real number $\bar{\varepsilon}$ such that
(i) for all a in $\mathbb{R}^n \backslash \{0\}$ such that $|a| < \bar{\varepsilon}$ there exists q in $\mathbb{N}$ such that

$$L_{f_0}^q h(a) \neq 0 \tag{49}$$

with $f_0(x) = f(x,0)$ and the usual convention $L_{f_0}^0 h = h$,
(ii) for all $(a_1, a_2) \in (\mathbb{R}^n \backslash \{0\})^2$ with $a_1 \neq a_2, |a_1| < \bar{\varepsilon}$, and $|a_2| < \bar{\varepsilon}$, and for all u in $\mathbb{R}^m$ with $|u| < \bar{\varepsilon}$, there exists ω in $\mathcal{O}(\tilde{C})$ such that

$$\omega(a_1,u) \neq \omega(a_2,u). \tag{50}$$

Note that (i) implies the following property
(i)* for any $a \neq 0$ in $B_{\bar{\varepsilon}} := \{x \in \mathbb{R}^m, |x| < \bar{\varepsilon}\}$ there exists a positive real number τ such that

$$x(\tau) \text{ exists and } h(x(\tau)) \neq 0 \tag{51}$$

where $x(t)$ is defined by $\dot{x} = f(x,0), x(0) = a$. Moreover if f and g are analytic, (i)* implies (i). The reason of "null" in "null-observable" comes from condition (i) or (i)* : roughly speaking we want to be able to distinguish from 0 any a in $B_{\bar{\varepsilon}} \setminus \{0\}$ by using the control law which vanishes identically.
When f is affine with respect to u, i.e. $f(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ with f_1, ..., f_m in $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$, then a slightly simpler version of (ii) can be given. Let $\tilde{\mathcal{O}}(\tilde{C})$ be the observation space -see e.g. [29] or Remark 5.4.2 in [82]- i.e. the set of maps $\tilde{\omega}$ in $C^\infty(\mathbb{R}^n;\mathbb{R}^p)$ such that for some integer $r \geq 0$ - depending on $\tilde{\omega}$ - and for some sequence $i_1, ..., i_r$ of integers in $[0,m]$

$$\tilde{\omega}(x) = L_{f_{i_1}} ... L_{f_{i_r}} h(x), \ \forall x \in \mathbb{R}^n, \tag{52}$$

with the convention that, if $r = 0$, the right hand side of (52) is $h(x)$. Then (ii) is equivalent to

$$((a_1,a_2) \in B_{\bar{\varepsilon}}^2, \tilde{\omega}(a_1) = \tilde{\omega}(a_2) \ \forall \tilde{\omega} \in \tilde{\mathcal{O}}(\tilde{C})) \Rightarrow (a_1 = a_2). \tag{53}$$

Finally let us remark that if f is a polynomial with respect to u or if f and g are analytic then (ii) is equivalent to
(ii)* for all $(a_1,a_2) \in \mathbb{R}^n \setminus \{0\}$ with $a_1 \neq a_2, |a_1| < \varepsilon$ and $|a_2| < \varepsilon$ there exists u in $\mathbb{R}^m$ and ω in $\mathcal{O}(\tilde{C})$ such that (50) holds.
Indeed in these cases the subspace of $\mathbb{R}^p$ spanned by $\omega(x,u)$; $\omega \in \mathcal{O}(\tilde{C})$ does not depend on u: it is the observation space of $(\tilde{C})$ evaluated at x – as defined for example in [29] –.

With these definitions we have

Theorem 21. *Assume that the origin (of $\mathbb{R}^n$) is locally continuously reachable (for (C)) in small time. Assume that $(\tilde{C})$ is locally Lie null-observable. Then $(\tilde{C})$ is locally stabilizable in small time by means of continuous dynamic periodic time-varying output feedback laws.*

This theorem is proved in [14]. Let us just sketch the proof given in [14].

We assume that the assumptions of Theorem 21 are satisfied. Let T be a positive real number. Our proof of Theorem 21 is divided in three steps.

Step 1. Using the assumption that system (C) is locally Lie null-observable we prove, using [11], that there exist u^* in $C^\infty(\mathbb{R}^p \times [0,T];\mathbb{R}^m)$ and a positive real number ε^* such that

$$u^*(y,T) = u^*(y,0) = 0, \ \forall y \in \mathbb{R}^p, \tag{54}$$

$$u^*(0,t) = 0, \ \forall t \in [0,T], \tag{55}$$

and, for all (a_1, a_2) in $B^2_{\varepsilon^*}$, for all s in $(0,T)$,

$$(h^{(i)}_{a_1}(s) = h^{(i)}_{a_2}(s),\ \forall i \in \mathbb{N}) \Rightarrow (a_1 = a_2) \tag{56}$$

where $h_a(s) = h(x^*(a,s))$ with x^* defined by $\partial x^*/\partial t = f(x^*, u^*(h(x^*),t))$, $x^*(a,0) = a$. Let us note that in [58] a similar u^* was considered, but it was taken depending only on time and so (55), which is important to get stability, was not satisfied - in general -. In this step we do not use any reachability property for (C).

Step 2. Let $q = 2n+1$. In this step, using (56), we prove the existence of $(q+1)$ real numbers $0 < t_0 < t_1 ... < t_q < T$ such that the map $K : B_{\varepsilon^*} \longrightarrow (\mathbb{R}^p)^q$ defined by

$$K(a) = \left(\int_{t_0}^{t_1} (s-t_0)(t_1-s)h_a(s)ds, ..., \int_{t_0}^{t_q} (s-t_0)(t_q-s)h_a(s)ds\right) \tag{57}$$

is one-to-one and so, as we will see, there exists a map $\theta : (\mathbb{R}^p)^q \longrightarrow \mathbb{R}^n$ such that

$$\theta \circ K(a) = x^*(a,T),\ \forall a \in B_{\varepsilon^*/2}. \tag{58}$$

Step 3. In this step we prove the existence of $\bar{u}$ in $C^0(\mathbb{R}^n \times [0,T]; \mathbb{R}^m)$ and $\bar{\varepsilon}$ in $(0,+\infty)$ such that

$$\bar{u} = 0 \text{ on } (\mathbb{R}^n \times \{0,T\}) \cup (\{0\} \times [0,T]), \tag{59}$$

$$(\dot{x} = f(x, \bar{u}(x(0),t)) \text{ and } |x(0)| < \bar{\varepsilon}) \Rightarrow (x(T) = 0). \tag{60}$$

Property (60) means that $\bar{u}$ is a "dead-beat" open-loop control. In this last step we use the reachability assumption on (C), but do not use the Lie null-observability assumption.

Using these three steps let us end the proof of Theorem 21. The dynamic extension of system (C) that we consider is

$$\dot{x} = f(x,u),\quad \dot{z} = v = (v_1, ..., v_q, v_{q+1}) \in \mathbb{R}^p \times ... \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}, \tag{61}$$

with $z_1 = (z_1, ..., z_q, z_{q+1}) \in \mathbb{R}^p \times ... \times \mathbb{R}^p \times \mathbb{R}^n \simeq \mathbb{R}^{pq+n}$. For this system the output is $\tilde{h}(x,z) = (h(x), z) \in \mathbb{R}^p \times \mathbb{R}^{pq+n}$. For $s \in \mathbb{R}$ let $s^+ = \max(s,0)$ and let $\operatorname{sgn}(s) = 1$ if $s > 0$, 0 if $s = 0$, -1 if $s < 0$. Finally, for r in $\mathbb{N}\backslash\{0\}$ and $b = (b_1, ..., b_r)$ in $\mathbb{R}^r$, let

$$b^{1/3} = (|b_1|^{1/3}\operatorname{sgn}(b_1), ..., |b_r|^{1/3}\operatorname{sgn}(b_r)). \tag{62}$$

We now define $u : \mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \longrightarrow \mathbb{R}^m$ and $v : \mathbb{R}^p \times \mathbb{R}^{pq+n} \times \mathbb{R} \longrightarrow \mathbb{R}^{pq+n}$ by requiring for (y,z) in $\mathbb{R}^p \times \mathbb{R}^{pq+n}$ and for all i in $[1,q]$

$$u(y,z,t) = u^*(y,t),\ \forall t \in [0,T), \tag{63}$$

$$v_i(y,z,t) = -t(t_0 - t)^+ z_i^{1/3} + (t-t_0)^+(t_i - t)^+ y,\ \forall t \in [0,T), \tag{64}$$

$$v_{q+1}(y,z,t) = -t(t_q - t)^+ z_{q+1}^{1/3} + 6\frac{(T-t)^+(t-t_q)^+}{(T-t_q)^3}\theta(z_1, ..., z_q), \tag{65}$$

$$u(y,z,t) = \bar{u}(z_{q+1}, t-T), \ \forall t \in [T, 2T), \tag{66}$$

$$v(y,z,t) = 0, \ \forall t \in [T, 2T), \tag{67}$$

$$u(y,z,t) = u(y,z,t+2T), \ \forall t \in \mathbb{R}, \tag{68}$$

$$v(y,z,t) = v(y,z,t+2T), \ \forall t \in \mathbb{R}. \tag{69}$$

Roughly speaking the strategy is the following one
(i) During the interval of time $[0,T]$, one "excites" system (C) by means of $u^*(y,t)$ in order to be able to deduce from the observation during this interval of time what is the state at time T: at time T we have $z_{q+1} = x$
(ii) During the interval of time $[T, 2T]$, z_{q+1} does not move and one uses the dead-beat open-loop $\bar{u}$ but transforms it into an output feedback by using in its argument z_q instead of the value of x at time T – this method has been used previously in the proof of Theorem 1.7 of [10] –.
In a context of adaptive control, a similar strategy has been used later on by Kreisselmeier and Lozano in [43].

One easily sees that u and v are continuous and vanishes on $\{(0,0)\} \times \mathbb{R}$. Let (x,z) be any maximal solution of the closed loop system

$$\dot{x} = f(x, u(\tilde{h}(x,z),t)) \ , \quad \dot{z} = v(\tilde{h}(x,z),t); \tag{70}$$

then one easily checks that, if $|x(0)| + |z(0)|$ is small enough,

$$z_i(t_0) = 0, \ \forall i \in [1,q], \tag{71}$$

$$(z_1(t), ..., z_q(t)) = K(x(0)), \ \forall t \in [t_q, T], \tag{72}$$

$$z_{q+1}(t_q) = 0, \tag{73}$$

$$z_{q+1}(T) = \theta \circ K(x(0)) = x(T), \tag{74}$$

$$x(t) = 0, \ \forall t \in [2T, 3T], \tag{75}$$

$$z(2T + t_q) = 0. \tag{76}$$

Equalities (71) (resp. (73) are proved by computing explicitly, for $i \in [1,q]$, z_i on $[0,t_0]$ (resp. z_{q+1} on $[0,t_q]$) and by seeing that this explicit solution reaches 0 before time t_0 (resp. t_q) and by pointing out that if, for some s in $[0,t_0]$ (resp.

$[0,t_q])$, $z_i(s) = 0$ (resp. $z_{q+1}(s) = 0$) then $z_i = 0$ on $[s,t_0]$ (resp. $z_{q+1} = 0$ on $[s,t_q]$)-note that $z_i \dot{z}_i \leq 0$ on $[0,t_0]$ (resp. $z_{q+1}\dot{z}_{q+1} \leq 0$ on $[0,t_q]$)-.

Moreover one has also, for all s in $\mathbb{R}$ and all $t \geq s$,

$$((x(s),z(s)) = (0,0)) \Rightarrow ((x(t),z(t)) = (0,0)). \tag{77}$$

Indeed, first note that without loss of generality we may assume $s \in [0,2T]$ and $t \in [0,2T]$. If $s \in [0,T]$, then, since u^* is of class C^∞ we get, using (55), that $x(t) = 0$, $\forall t \in [s,T]$ and then, using (44) and (64), get that, for all $i \in [1,q]$, $z_i \dot{z}_i \leq 0$ on $[s,T]$ and so z_i also vanishes on $[s,T]$; this, with (65) and $\theta(0) = 0$-see(57) and (58)-, implies that $z_{q+1} = 0$ also on $[s,T]$. Hence we may assume that $s \in [T,2T]$. But, in this case, using (67), we get that $z = 0$ on $[s,2T]$ and, from (59) and (66), we get that $x = 0$ also on $[s,2T]$.

From (75), (76), and (77) we get – see Lemma 2.15 in [12] – the existence of ε in $(0,+\infty)$ such that, for any s in $\mathbb{R}$ and any maximal solution (x,z) of $\dot{x} = f(x,u(\tilde{h}(x,z),t))$, $\dot{z} = v(\tilde{h}(x,z),t)$, we have

$$(|x(s)| + |y(s)| \leq \varepsilon) \Rightarrow ((x(t),z(t)) = (0,0),\ \forall t \geq s + 5T). \tag{78}$$

Since T is arbitrary Theorem 21 is proved.

Remark 22. Concerning the proof, let us emphasize that we use an idea due to Lozano [53], Mazenc and Praly [58]: as in [53] and [58] we will first recover the state from the output. A related idea is also used in Section 3 of [10], where we first recover initial data from the state. Moreover as in [58] our proof relies on the existence -see [90] for analytic systems and [11] for C^∞ systems- of an output feedback which distinguishes every pair of distinct states. In [58] it is established that distinguishability with a universal time-varying control, global stabilizability by state feedback, and observability of blow-up are sufficient conditions for the existence of a time-varying dynamic (of infinite dimension and in a sense more general than the one considered in Definition 19) output feedback guaranteeing boundedness and convergence of all the solutions defined at time $t = 0$; the methods developed in [58] can be applied directly to our situation; in this case Theorem 21 gives two improvements: we get that 0 is asymptotically stable for the closed loop system, instead of only attractor for time 0, and our dynamic extension is of finite dimension, instead of infinite dimension.

If $(\tilde{C})$ is locally stabilizable in small time by means of continuous dynamic periodic time-varying output feedback laws, then the origin (of $\mathbb{R}^n$) is locally continuously reachable (for $(\tilde{C})$) in small time (use Lemma 3.5 in [14]) and, if moreover f and h are analytic, then $(\tilde{C})$ is locally Lie null-observable - see [14, Proposition 4.3].

Let us remark that it follows from our proof of Theorem 21 that it suffices to consider dynamic extension of dimension $n+(2n+1)p$, i.e. under the assumption of Theorem 21, System (46) with $k = n+(2n+1)p$ is locally stabilizable in small time by means of continuous static periodic time-varying output feedback laws. We conjecture that, as in the linear case, this result still holds for $k = n-1$. Note that this conjecture is true if $n = 1$, i.e. we have

Proposition 23. *Assume that $n = 1$ and that the origin (of $\mathbb{R}$) is locally continuously reachable (for $(\tilde{C})$) in small time. Assume that $(\tilde{C})$ is locally Lie null-observable. Then $(\tilde{C})$ is locally stabilizable in small time by means of continuous periodic time-varying output feedback laws.*

Let us prove this theorem. Since $(\tilde{C})$ is locally Lie null-observable, there exist $i \in [1, m]$ and a positive integer l such that

$$h_i^{(l)}(0) \neq 0. \tag{79}$$

Hence there are two maps w_+ and w_- in $C^0(\mathbb{R}; \mathbb{R})$, of class C^∞ on $\mathbb{R} \backslash \{0\}$, and a positive real number ϵ_0 such that

$$w_+(h_i(x)) = x, \ \forall x \in [0, \epsilon_0], \tag{80}$$

$$w_-(h_i(x)) = x, \ \forall x \in [-\epsilon_0, 0]. \tag{81}$$

Modifying, if necessary, h outside a neighborhood of $0 \in \mathbb{R}$, we may assume, without loss of generality, that

$$h(x) \neq 0, \ \forall x \in \mathbb{R} \backslash \{0\}. \tag{82}$$

Similarly, without loss of generality, we may assume that

$$|f(x, u)| \leq 1, \ \forall (x, u) \in \mathbb{R} \times \mathbb{R}^m \tag{83}$$

Let T be a positive real number. One has the following lemma [16, Lemma 2.12]

Lemma 24. *There exist $\bar{u}_+$ and $\bar{u}_-$ in $C^\infty([0, T); \mathbb{R}^m) \cup C^0([0, T]; \mathbb{R}^m)$, vanishing for $t = T$, such that, if we denote by $\bar{x}_+$ and by $\bar{x}_-$ the solutions of*

$$\dot{\bar{x}}_+ = f(\bar{x}_+, \bar{u}_+(t)), \ \bar{x}_+(T) = 0 \tag{84}$$

$$\dot{\bar{x}}_- = f(\bar{x}_-, \bar{u}_-(t)), \ \bar{x}_-(T) = 0, \tag{85}$$

then

$$\bar{x}_+(t) > 0, \ \forall t \in [0, T), \tag{86}$$

$$\bar{x}_-(t) < 0, \ \forall t \in [0, T). \tag{87}$$

Straightforward arguments relying on partition of unity -proceed for example as in the proof of Lemma 2.11 of [12]- we get the existence of $\bar{u}$ in $C^0(\mathbb{R} \times \mathbb{R}; \mathbb{R}^m)$ of class C^∞ on $(\mathbb{R} \backslash \{0\}) \times \mathbb{R}$ satisfying (22) and (23) such that

$$\bar{u}(\bar{x}_+(t), t) = \bar{u}_+(t), \ \forall t \in [T/2, T], \tag{88}$$

$$\bar{u}(\bar{x}_-(t), t) = \bar{u}_-(t), \ \forall t \in [T/2, T], \tag{89}$$

$$\bar{u} \text{ vanishes on a neighborhood of } \{0\} \times [0,T) \text{ in } \mathbb{R} \times [0,T). \tag{90}$$

$$\bar{u}(x,t) = 0\, \forall t \geq T,\ \forall x \in \mathbb{R} \tag{91}$$

$$\bar{u}(x,t) = 0,\ \forall t \leq 0,\ \forall x \in \mathbb{R} \tag{92}$$

For y in $\mathbb{R}^p$, let us denote by y_i the i-th component of y. Finally, let us define $\tilde{u} : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^m$ by

$$\tilde{u}(y,t) = \bar{u}(w_+(y_i),t),\ \forall (y,t) \in \mathbb{R}^p \times [0,T), \tag{93}$$

$$\tilde{u}(y,t) = \bar{u}(w_-(y_i),t),\ \forall (y,t) \in \mathbb{R}^p \times [T,2t), \tag{94}$$

$$\tilde{u}(y,t) = \tilde{u}(y,t+2T),\ \forall (y,t) \in \mathbb{R}^p \times \mathbb{R}. \tag{95}$$

Clearly $\tilde{u}$ is continuous on $\mathbb{R}^p \times \mathbb{R}$. Let us prove that this time-varying output feedback stabilizes $(\tilde{C})$ in finite time. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ be defined by

$$u(x,t) = \tilde{u}(h(x),t),\ \forall (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{96}$$

Then u is continuous on $\mathbb{R} \times \mathbb{R}$, of class C^∞ on $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}$, is $2T$-periodic with respect to time and vanishes on $\{0\} \times \mathbb{R}$. Note, see in particular (90), that (24) holds. Let us point out that there exists $\tau \in [0,T)$ such that

$$\dot{\bar{x}}_+ = f(\bar{x}_+, u(\bar{x}_+,t)),\ \forall t \in [\tau, T] \tag{97}$$

$$\dot{\bar{x}}_-(t-T) = f(\bar{x}_-(t-T), u(\bar{x}_-(t-T),t)),\ \forall t \in [\tau+T, 2T]. \tag{98}$$

Let $x_+ : [0,T] \to \mathbb{R}$ be defined by

$$\dot{x}_+ = f(x_+, u(x_+,t)),\ \forall t \in [0,T], \tag{99}$$

$$x_+(t) = \bar{x}_+(t),\ \forall t \in [\tau.T]. \tag{100}$$

Then

$$x_+(t) > 0,\ \forall t \in [0,T), \tag{101}$$

$$x_+(T) = 0. \tag{102}$$

Moreover, for any solution of $\dot{x} = f(x, u(x,t))$, one has, for all $t \in [0,T]$,

$$x(0) \in [0, x_+(0)] \Rightarrow (x(t) \in [0, x(t)]). \tag{103}$$

So, for any solution of $\dot{x} = f(x, u(x,t))$, one has

$$x(0) \in [0, x_+(0)] \Rightarrow x(T) = 0. \tag{104}$$

Similarly, let $x_- : [0, 2T] \to \mathbb{R}$ be defined by

$$\dot{x}_- = f(x_-, u(x_-, t)),\ \forall t \in [0, 2T], \tag{105}$$

$$x_-(t) = \bar{x}_-(t - T),\ \forall t \in [T + \tau, 2T]. \tag{106}$$

Then $x_-(0) < 0$ and, for any solution of $\dot{x} = f(x, u(x,t))$, one has

$$x(0) \in [x_-(0), 0] \Rightarrow x(2T) = 0. \tag{107}$$

Hence, using (104), (107), and [12, Lemma 2.15], we get that for $\varepsilon > 0$ small enough

$$\begin{aligned}((\dot{x} = f(x, u(x,t)) \text{ and } |x(s)| \le \varepsilon) \Rightarrow \\ (x(t) = 0\ \forall t \ge s + 4T))\ \forall s \in \mathbb{R}.\end{aligned} \tag{108}$$

This ends the proof of Proposition 23.

Remark 25. There are linear control system which are controllable and observable which cannot be locally asymptotically stabilized by means of a continuous time-varying static feedback law. This is for example the case for the controllable and observable linear system, with $n = 2$, $m = 1$, and $p = 1$,

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u,\ y = x_1 \tag{109}$$

Assume that this system can be locally asymptotically stabilized by means of a continuous time-varying static output feedback law $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Hence there exist $r > 0$ and $\tau > 0$ such that, if $\dot{x}_1 = x_2$, $\dot{x}_2 = u(x_1, t)$,

$$x_1(0)^2 + x_2(0)^2 \le r^2 \Rightarrow x_1(\tau)^2 + x_2(\tau)^2 \le r^2/5 \tag{110}$$

Let $(u^n : n \in \mathbb{N})$ be a sequence of functions from $\mathbb{R}$ into $\mathbb{R}$ of class C^∞ which converges uniformly to u on each compact subset of $\mathbb{R} \times \mathbb{R}$. Then, for n large enough, we have, if $\dot{x}_1 = x_2$, $\dot{x}_2 = u^n(x_1, t)$,

$$x_1(0)^2 + x_2(0)^2 \le r^2 \Rightarrow x_1(\tau)^2 + x_2(\tau)^2 \le r^2/4. \tag{111}$$

But, since the time-varying vector field X on $\mathbb{R}^2$ defined by

$$X_1(x_1, x_2, t) = x_1,\ X_2(x_1, x_2, t) = u^n(x_1, t), \tag{112}$$

has a divergence equal to 0, the flow associated to X preserves area, which is in contradiction with (111).

3 Lyapunov design of stabilizing state or output feedback.

3.1 Introduction.

For the study of the (global) uniform asymptotic stability of $X = 0$, an equilibrium point in $\mathbb{R}^N$ of the time-varying dynamic system

$$\dot{X} = F(t, X) , \tag{113}$$

it is well established that, a very efficient tool is provided by Lyapunov function theory. But it is also well known that finding an appropriate Lyapunov function is in general a very difficult task. Here, as in the previous part, instead of the uncontrolled system (113), we are concerned with the following controlled dynamic system

$$\dot{x} = f(x, u) \tag{114}$$

where x is in $\mathbb{R}^n$, u is in $\mathbb{R}^m$, f is continuous on a neighborhood of $(0, 0)$ and

$$f(0,0) = 0. \tag{115}$$

We want to design an asymptotically stabilizing feedback law. Precisely, we want to find an integer p and two continuous functions φ and ϕ so that:

1. The control u is given by the dynamic system

$$\dot{\chi} = \varphi(t, h(x), \chi) \quad , \qquad u = \phi(t, h(x), \chi) \tag{116}$$

   with χ living in $\mathbb{R}^p$ and h is the imposed output function.
2. The point $(x = 0, \chi = 0)$ is a (globally) uniformly asymptotically stable equilibrium of the closed-loop system (114),(116).

The key difference between (113) and (114) is that the former is a given dynamic system whereas the latter is not completely defined. If, to solve the stabilizability problem for (114), we first find, by some way, the system (116) and then check that indeed we have stability, then we are back to the study of stability for a system like (113). But another approach consists in first finding a Lyapunov function and then complete the definition of (114) by choosing (116) so that this Lyapunov function will be appropriate. This approach is called Lyapunov design. This idea has been studied since at least the beginning of the sixties (see for example [27, 60]). To give a better grasp on the idea we wish to convey, we could say that in the first technique the problem is: *given a dynamic system, find a Lyapunov function,* whereas, in the second technique, the problem is: *given a Lyapunov function, find a dynamic system.* Of course this latter problem is not as simple as this, since part of the searched system is already given in the form of (114). This introduces implicitly a constraint on the Lyapunov function.

We shall be presenting various aspects of this Lyapunov design. But to limit the size of this presentation, we have chosen not to address two important topics:

1. *Lyapunov design of time-varying feedback laws for driftless systems.* There has been several publications on this topic, see [62, 17].

2. *Lyapunov design of output feedback laws.* This topic has received a lot of attention. Let us mention [101, 24] where results in the spirit of Theorem 27 below are presented. The case where the unmeasured components appear linearly has been exploited in [41, 55, 56, 64, 63, 25]. A class of systems where unmeasured components appear linearly is those linearly parameterized with unknown parameters. Then, we can have both unmeasured states and unknown parameters. Lyapunov design has a very long history in this topic and the literature is extremely rich. Let us mention two surveys [66, 45].

3.2 Assignable Lyapunov functions.

Using the analogy with the fact that poles can be assigned to a controllable linear system, we introduce the notion of assignable Lyapunov function

Definition 26. A function V is called an (resp. globally) assignable Lyapunov function for the system (114) if

1. it is in $C^1([0,+\infty) \times \mathcal{V}; [0,+\infty))$ (resp. $C^1([0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^p; [0,+\infty))$), where $\mathcal{V}$ is a neighborhood of $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^p$,
2. it is (resp. radially unbounded) positive definite and decrescent[4],
3. there exist two continuous functions φ and ϕ such that the feedback law defined by (116) makes non positive the time derivative of V along all the trajectories issued from points in $\mathcal{V}$ and solutions of (114),(116), i.e., for all (t,x,χ) in $[0,+\infty) \times \mathcal{V}$ (resp. $[0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^p$), we have

$$\begin{aligned} -W(x,\chi) \geq & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (117) \\ & \tfrac{\partial V}{\partial x}(t,x,\chi)\, f(x,\phi(t,h(x),\chi)) + \tfrac{\partial V}{\partial \chi}(t,x,\chi)\, \varphi(t,h(x),\chi) + \tfrac{\partial V}{\partial t}(t,x,\chi) \end{aligned}$$

   where W is a non negative continuous function. If W is in fact positive definite then V is called a strictly assignable Lyapunov function.

If we have a Lyapunov function which is assignable but not strictly assignable, we are guaranteed of having uniform stability but not uniform asymptotic stability. To prove the latter, we shall need to invoke an Invariance Theorem (see [28, Theorem 55.1] for instance) and for this, it will be more appropriate to restrict the functions φ, ϕ and V to be periodic in t if not time-invariant.

For the case where we restrict ourselves with time invariant feedback, as for uncontrolled dynamic system where we have equivalence between existence of Lyapunov functions and asymptotic stability of an equilibrium, Artstein has exhibited, in [1], a property in terms of a Lyapunov function which is equivalent to the existence of an asymptotically stabilizing feedback law[5] ϕ. This can be stated as

[4] See [28, p.194-195] for the definitions of these terms.

[5] In this context the dynamic extension χ is collected with the system state x so that the pair $[\dot{x} = f(x,u)\,,\, u]$ represents in fact the pair $[(\dot{x} = f(x,u),\ \dot{\chi} = v)\,,(u,v)]$. Of course this is possible only once the dimension p has been chosen.

Theorem 27. *Let the set of admissible control be $\mathcal{U}$, a convex subset of $\mathbb{R}^m$. If the system (114) can be (resp. globally) asymptotically stabilized by means of a discontinuous feedback law (see Definition 4), then there exists a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^n$ (resp. $\mathcal{V} = \mathbb{R}^n$) and a function V defined on $\mathcal{V}$ which is "a control Lyapunov function", i.e. it is positive definite (resp. radially unbounded), in $C^1(\mathcal{V};[0,+\infty))$ (resp. $C^1(\mathbb{R}^n;[0,+\infty))$ and such that*[6]

$$x \in \mathcal{V}\backslash\{0\} \qquad \Longrightarrow \qquad \exists\, u \in \mathcal{U} \quad s.t. \quad L_{f(x,u)}V(x) < 0, \tag{118}$$

and the small control property holds. Namely, we have also that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\{x \in \mathcal{V},\ 0 < |x| < \delta\} \quad \Longrightarrow \quad \exists\, u \in \mathcal{U} \quad s.t. \quad \left\{|u| < \varepsilon,\ L_{f(x,u)}V(x) < 0\right\}. \tag{119}$$

Conversely, if such neighborhood $\mathcal{V}$ (resp. $\mathcal{V} = \mathbb{R}^n$) and function V exist, then (114) can be (resp. globally) asymptotically stabilized by means of a discontinuous feedback law or a time-varying continuous feedback law with period T (see Definition 2) where T is an arbitrary strictly positive real number. If, moreover, f is affine in u, then (114) can be (resp. globally) asymptotically stabilized by means of a time-invariant continuous feedback law.

In fact, the notions of control Lyapunov function and of small control property – with u a relaxed control instead of a vector in $\mathcal{U}$ – are already present in the work of Sontag [77] where the problem of asymptotic controllability is addressed. In [1], Artstein studies stabilization but using relaxed controls instead of discontinuous or time-varying continuous controls, the latter being established by Coron and Rosier in [19].

With this theorem, we know that a strictly assignable Lyapunov function is a control Lyapunov function which satisfies the small control property.

Remark 28. A direct consequence of [81, Lemma 3.2], which has been exploited by Freeman and Kokotovic in [21] in the context of section 3.4, is the following: If V can be strictly assigned by a continuous feedback law ϕ, then there exists a positive positive definite function[7] r which can be as many times continuously

[6] For a "matrix field" indexed by u, $g(x,u) = (g_1(x,u),\dots,g_m(x,u))$, for each u, we denote by $L_g(x,u)V(x)$ the row vector $(L_{g_1}V,\dots,L_{g_m}V)$ where $L_{g_i}V$ is the derivative of V along the vector field g_i obtained by fixing u.

[7] A direct construction for r would be:
From [46, Remark p.74 and Theorem 7], there exists a C^∞ positive definite and proper function on the domain of attraction of $x = 0$ for the system $\dot{x} = f(x,\phi(x))$. This allows us to define two sequences of strictly positive real numbers

$$r_i^+ = \min\left\{R^+\,,\ \inf_{\left\{(x,u)|\,i+1\leq V(x)\leq i+2\,,\,L_{f(x,u)}V(x)\,\geq\,0\right\}} |u-\phi(x)|\right\},$$

$$r_i^- = \min\left\{R^-\,,\ \inf_{\left\{(x,u)|\,\frac{1}{i+2}\leq V(x)\leq \frac{1}{i+1}\,,\,L_{f(x,u)}V(x)\,\geq\,0\right\}} |u-\phi(x)|\right\}.$$

differentiable as we want and such that[8]

$$L_{f(x,u)}V(x) \;<\; 0 \qquad \forall\,(x,u)\;\in\;\mathcal{V}\backslash\{0\}\times\overline{B}(\phi(x),r(x)). \tag{120}$$

Also, the continuity of ϕ and the positive definiteness of r imply that, conversely, there exists a continuous positive definite function $\overline{r}$ such that, for all x in $\mathcal{V}\backslash\{0\}$,

$$L_{f(x,\phi(x_m))}V(x) \;<\; 0 \qquad \forall\,x_m\;\in\;\overline{B}(x,\overline{r}(x)). \tag{121}$$

This proves that state measurement is allowed. Indeed, if the actual state is x but its measurement is x_m, the Lyapunov function will still be decaying along the actual solution using the control $\phi(x_m)$ provided the norm of the measurement error $|x-x_m|$ is smaller than $\overline{r}(x)$. Unfortunately, typically $\overline{r}(x)$ tends to zero as x tends to 0 or to the boundary of $\mathcal{V}$. Freeman has displayed this problem with a counter-example in [20]. But in [22], Freeman and Kokotovic have exhibited a class of systems for which a feedback law can be constructed in such a way that the problem at infinity is rounded.

For the case where f is affine in u, Lin and Sontag have proposed an explicit expression for the feedback law ϕ which strictly assigns the Lyapunov function V (see [51] for more general sets $\mathcal{U}$)

Theorem 29 [79, 50]. *If the set of admissible control is*

$$\mathcal{U}=\left\{u\in\mathbb{R}^m\;\middle|\;|u|\leq\frac{1}{k}\right\} \tag{122}$$

where k is in $[0,+\infty]$ and if V is a C^1 (resp. global) control Lyapunov function satisfying the small control property with[9]

$$f(x,u) \;=\; a(x) \;+\; b(x)\,u \tag{123}$$

then a (resp. globally) stabilizing time invariant continuous feedback law is given by

$$\phi(x) \;=\; \begin{cases} -\frac{L_aV+\sqrt{(L_aV)^2+|L_bV|^4}}{|L_bV|^2\left(1+k\sqrt{1+|L_bV|^2}\right)}\,L_bV^\top & \textit{if}\quad L_bV\neq 0,\\ 0 & \textit{if}\quad L_bV=0,\end{cases} \tag{124}$$

The feedback law ϕ given by this Theorem has the following property

$$L_bV(x)\;\neq\;0 \qquad\Longrightarrow\qquad L_bV(x)\,\phi(x)\;<\;0. \tag{125}$$

Such a control which also assigns strictly V is said to be sign optimal.

where R^+ and R^- are two strictly positive real numbers. Then the function $r(x)$ can be obtained by interpolation between the $\frac{1}{2}r_i$'s according to the time needed to reach the closest level sets of V to x, following the solution $\dot{X}=\frac{\partial V}{\partial x}(X)$, $X(0)=x$.

[8] $\overline{B}(x,\delta)$ denotes the closed ball with center x and radius δ.

[9] Here and in the following of this section, $a(x)$ is a vector and $b(x)$ is a matrix.

Let us mention a first interesting application of Theorem 29:
Assume that we know the feedback law ϕ only implicitly. Precisely assume the existence of a strictly assignable Lyapunov function V and of a C^1 function $\Phi : \mathcal{V} \times \mathcal{U} \to \mathbb{R}^m$ such that $\frac{\partial \Phi}{\partial u}(x,u)$ is invertible on a neighborhood of $(0,0)$,

$$\Phi(0,0) \ = \ 0 \tag{126}$$

and the solution

$$u \ = \ \phi(x) \tag{127}$$

of

$$\Phi(x,u) \ = \ 0 \tag{128}$$

satisfies (118) and (119). The properties of Φ implies that we can find explicitly[10] C^1 functions $\mathcal{F}$ satisfying, at least on a neighborhood of $(0,0)$,

$$\left|\frac{\partial \mathcal{F}}{\partial u}(x,u)\right| \ \leq \ 1 \ - \ \varepsilon, \tag{129}$$

with ε in $(0,1)$, and such that ϕ is also solution of

$$u \ = \ \mathcal{F}(x,u). \tag{130}$$

Our idea to get an explicit expression for the feedback law is to solve (130) "on-line". This will lead to a dynamic controller of the form (116)

$$\dot{\chi} \ = \ \varphi(x,\chi) \quad , \qquad u \ = \ \chi \tag{131}$$

where φ is to be chosen. To design this function, we simply follow Theorem 27. We look for a positive definite function $U(x,\chi)$ such that (118) holds, i.e. for $(x,\chi) \neq 0$,

$$\frac{\partial U}{\partial \chi}(x,\chi) \ = \ 0 \tag{132}$$

implies

$$\frac{\partial U}{\partial \chi}(x,\chi)f(x,\chi) \ < \ 0. \tag{133}$$

In view of what we know, the following choice is appropriate

$$U(x,\chi) \ = \ V(x) \ + \ \tfrac{1}{2}\left|\chi - \mathcal{F}(x,\chi)\right|^2. \tag{134}$$

Also we can check that if f is C^1 and V is C^2 then the small control property (119) holds. It follows that (124) gives an explicit expression for φ. This technique has been used in [64].

[10] For instance, $\mathcal{F}(x,u) \ = \ u \ - \ \frac{\partial \Phi}{\partial u}(x,u)^{-1}\Phi(x,u)$

3.3 Robustness and Lyapunov redesign.

For uncontrolled dynamic systems, asymptotic stability implies total stability but the bound on the perturbations is fixed. In the case of controlled systems, we can take advantage of the freedom given by the control to increase the level of allowed perturbations in some "directions". Let us consider the following perturbation of (114) for the affine case (123)

$$\dot{x} \;=\; a(x) \;+\; b(x)\,u \;+\; c(x,d) \tag{135}$$

where c is a continuous function and d represents an exogenous signal supposed to be in $L^\infty([0,+\infty);\mathbb{R}^q)$. We assume that we know a strictly assignable Lyapunov function V and that we have already implemented a corresponding sign optimal feedback law (see (125)) so that, for $x \neq 0$,

$$L_aV(x) \;<\; 0. \tag{136}$$

If we do not modify this nominal feedback law, we are guaranteed that the state x of the system will be attracted in finite time and then remain in $\overline{B}(0,\delta)$, if d takes values in the set

$$\mathcal{D}_{\delta,V} \;=\; \left\{d \in \mathbb{R}^q \,\middle|\, x \notin \overline{B}(0,\delta) \quad\Rightarrow\quad L_{c(x,d)}V(x) < -L_aV(x)\right\}\,. \tag{137}$$

This set is in general only a subset of the actual set of "admissible" perturbations d keeping the ball $\overline{B}(0,\delta)$ attractive and invariant. To get a better approximation of this set, we may have to reshape[11] the Lyapunov function V in order to minimize the ratio $\frac{L_{c(x,d)}V(x)}{-L_aV(x)}$. This shaping has been done for linear systems for example in the context of the so called quadratic stability when some a priori knowledge on the uncertainties is available (see [71, 8] for instance). For the nonlinear case, some aspects of this problem are addressed by Freeman and Kokotovic in [23]. Let us remark also that the technique used in section 3.4 solves this question for a particular class of systems.

For the time being, let us concentrate our attention on finding a feedback law u in order to increase $\mathcal{D}_{\delta,V}$. This problem has received a lot of attention. The surveys by Khalil [39, Chapter 5.5] and Corless [7]) give a good idea on the state of the art. To give the reader a flavor of the available results, let us state the following technical Lemma

Lemma 30 [68, Lemma 1]. *Assume we know a positive definite function r such that there exists a continuous function Δ on $\mathbb{R}^m$ satisfying*[12]

$$L_{c(x,d)}V(x) \;\leq\; -\frac{1}{1+r(x)}\,L_aV(x) \;+\; L_bV(x)\,\Delta(x,d). \tag{138}$$

[11] modify the level sets of V.

[12] Note that this inequality is invariant under any transformation of V which does not change its level sets.

Under this condition and (136), for any function μ in $C^0([0,+\infty);[0,+\infty))$, non decreasing and with $\mu(0)=0$, there exists[13] *m bounded functions $\theta_i : \mathcal{V} \to [0,1]$ which are continuous on $\mathcal{V}\backslash\{0\}$ and such that by choosing the ith component of the feedback law as*

$$\phi(x)_i \;=\; -\theta_i(x)\,\mathrm{sign}(L_{b_i}(x))\,\mu(V(x)), \tag{140}$$

we get

$$L_{a(x)+b(x)\phi(x)+c(x,d)}V(x) \leq \frac{r(x)}{2(1+r(x))}\,L_aV(x) - |L_bV(x)|\,[\mu(V(x)) - |\Delta(x,d)|]\,. \tag{141}$$

The condition (138) can be interpreted as saying that the remainder of the "division" of $L_{c(x,d)}V(x)$ by $L_bV(x)$ is strictly smaller than $-L_aV(x)$. The case where $r=\infty$ has been considered for example by Qu in [69]. This Lemma, which is at the basis of the proof of Lemma 49 in the third part of this survey, shows in particular that, for each solution of (135),(140) which satisfies

$$|\Delta(x(t),d(t))| \in \mu([0,+\infty)) \qquad \forall t \in [0,+\infty), \tag{142}$$

we have

$$\limsup_{t\to+\infty} V(x(t)) \leq \limsup_{t\to+\infty} \mu^{-1}\left(|\Delta(x(t),d(t))|\right), \tag{143}$$

$$\limsup_{t\to+\infty} |\phi(x(t))| \leq \limsup_{t\to+\infty} |\Delta(x(t),d(t))|\,. \tag{144}$$

It follows that by choosing μ appropriately, we can make the ball $\overline{B}(0,\delta)$ attractive.

Remark 31. By choosing the function μ as a continuous, strictly increasing function mapping $[0,+\infty)$ onto itself and such that[14]

$$\mu(V(x)) \;>\; \sup_{\{d\,|\,|d|\leq|x|\}} \{|\Delta(x,d)|\}\,, \tag{145}$$

it follows from [88] that there exists a C^1 positive definite, (resp. radially unbounded) function which we still denote by V and continuous and strictly increasing functions σ and α which are 0 at 0 and with α onto $[0,+\infty)$ such that

$$L_{a(x)+b(x)\phi(x)+c(x,d)}V(x) \leq -\alpha(V(x)) \;+\; \sigma(|d|). \tag{146}$$

[13] For example, we can take

$$\theta_i = \begin{cases} \mathrm{sat}\left(\dfrac{\sqrt{\left(\frac{r(x)L_aV(x)}{2m(1+r(x))}\right)^2 + 3\left(L_{b_i}V(x)\mu(V(x))\right)^2} + \frac{r(x)L_aV(x)}{2m(1+r(x))}}{|L_{b_i}V(x)|\mu(V(x))}\right) & \text{if } |L_{b_i}V(x)|\mu(V(x)) \neq 0, \\ 0 & \text{if } |L_{b_i}V(x)|\mu(V(x)) = 0, \end{cases} \tag{139}$$

where sat : $(-\infty,+\infty) \to [-1,1]$ is the standard saturation function.

[14] which is always possible since V is positive definite (and radially unbounded in the global case).

3.4 Adding Integrators.

The undisturbed case. We consider the system

$$\dot{x}_1 = f_1(x_1, u_1), \tag{147}$$

where x_1 is in $\mathbb{R}^{n_1}$, u_1 is in $\mathbb{R}^{n_2}$, f_1 is C^1 and we assume

Assumption A1: *We know a C^1 Lyapunov function V_1 which can be strictly assigned by the time-invariant continuous feedback law ϕ_1 to the system (147).*

Can we design a time-invariant asymptotically stabilizing continuous feedback law for the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2, u_2), \end{cases} \tag{148}$$

with u_2 in $\mathbb{R}^m$? This problem is called adding one integrator. Its solution allows us to prove for example that systems admitting the following special recurrent structure, called feedback form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2)\,, \\ \dot{x}_2 = x_3 + f_2(x_1, x_2)\,, \\ \quad \vdots \\ \dot{x}_n = u + f_n(x_1, \ldots\ldots, x_n)\,. \end{cases} \tag{149}$$

are (resp. globally) asymptotically stabilizable by means of a time invariant continuous feedback law if the functions f_i's are C^{n-i} and $\dot{x} = f_1(x, u)$ is (resp. globally) asymptotically stabilizable by means of a time invariant C^{n-1} feedback law.

This problem of adding one integrator has received many answers and most of them can be obtained by following Theorem 27. For this, we first remark that if we can solve the asymptotic stabilization problem for

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = u, \end{cases} \tag{150}$$

then the problem is also solved for the system (148) if

Assumption A2: *There exists a continuous function $\mathcal{K}$ satisfying*

$$f_2(x_1, x_2, \mathcal{K}(x_1, x_2, u)) = u. \tag{151}$$

So we concentrate our attention on the system (150). We look for a positive definite function $V_2(x_1, x_2)$ such that (118) holds, i.e. for $(x_1, x_2) \neq 0$,

$$\frac{\partial V_2}{\partial x_2}(x_1, x_2) = 0 \quad \Longrightarrow \quad \frac{\partial V_2}{\partial x_1}(x_1, x_2)\, f_1(x_1, x_2) < 0. \tag{152}$$

From Remark 28, we know the existence of a sufficiently smooth positive definite function r (resp. we choose $r \equiv 0$) such that it is sufficient to choose V_2 satisfying

$$\frac{\partial V_2}{\partial x_2}(x_1,x_2) = 0 \quad \Rightarrow \quad \left\{x_2 \in \overline{B}(\phi_1(x_1), r(x_1))\,,\ \frac{\partial V_2}{\partial x_1}(x_1,x_2) = l(x_1)\,\frac{\partial V_1}{\partial x_1}(x_1)\right\} \tag{153}$$

where l is any positive definite function. We conclude that a function V_2 of the following form should be appropriate:

$$V_2(x_1,x_2) \;=\; k(V_1(x_1)) \;+\; \int_0^1 (x_2-\phi_1(x_1))^\top\, \Theta(x_1,\phi_1(x_1)+s(x_2-\phi_1(x_1))\,ds \tag{154}$$

where k is any C^1 function, with $k(0)=0$ and positive definite derivative, and the vector Θ is to be chosen such that V_2 is C^1, positive definite (resp. radially unbounded) and

$$\Theta(x_1,x_2) = 0 \qquad \Longrightarrow \qquad x_2 \in \overline{B}(\phi_1(x_1), r(x_1)). \tag{155}$$

For example, by taking

$$\Theta(x_1,x_2) \;=\; (x_2-\phi_1(x_1))\,\max\left\{1 - \frac{r(x_1)}{|x_2-\phi_1(x_1)|}\,,\,0\right\}, \tag{156}$$

we get

$$V_2(x_1,x_2) \;=\; k(V_1(x_1)) \;+\; \frac{1}{2}\,\max\left\{|x_2-\phi_1(x_1)| - r(x_1)\,,\,0\right\}^2. \tag{157}$$

Assumption A3: *ϕ_1 is a C^1 function.*

In this case, V_2, in (157), is a control Lyapunov function which satisfies the small control property, it is therefore a strictly assignable Lyapunov function. So we have

Theorem 32. *Under assumptions A1 to A3, the system (148) can be (resp. globally) asymptotically stabilized by means of a time-invariant continuous feedback law.*

Moreover, since the system (150) is affine in the control, Theorem 29 applies. But one can also check that an asymptotically stabilizing continuous feedback law is

$$\phi_2(x_1,x_2) \;= \tag{158}$$

$$-\left(k'(V_1(x_1))\,\frac{\partial V_1}{\partial x_1}(x_1)\int_{\min\left\{1,\frac{r(x_1)}{|x_2-\phi_1(x_1)|}\right\}}^1 \frac{\partial f_1}{\partial x_2}(x_1\,,\,\phi_1(x_1)+s(x_2-\phi(x_1))\,ds\right)^\top$$
$$+\,\theta(x_1,x_2)\left\{\left[\frac{\partial \phi_1}{\partial x_1}(x_1) + \frac{x_2-\phi_1(x_1)}{|x_2-\phi_1(x_1)|}\,\frac{\partial r}{\partial x_1}(x_1)\right]\,f_1(x_1,x_2) - S(x_1,x_2)\right\}$$

where $\mathcal{S}$ on $\mathbb{R}^p$ and θ on $[0,1]$ are continuous functions satisfying[15]

$$|x_2 - \phi_1(x_1)| > r(x_1) \quad \Longrightarrow \quad \begin{cases} \theta(x_1,x_2) = 1, \\ (x_2 - \phi_1(x_1))^\top \mathcal{S}(x_1,x_2) > 0, \end{cases} \tag{159}$$

and

$$r \not\equiv 0\,,\ |x_2 - \phi_1(x_1)| = 0 \quad \Longrightarrow \quad \theta(x_1,x_2) = 0. \tag{160}$$

The Lyapunov function (157) and the feedback law (158), and therefore Theorem 34, have been obtained by Byrnes and Isidori [5] and Tsinias [100] (see also [82, Lemma 4.8.3]) for the case where the function $r \equiv 0$. The idea of introducing a non zero function r (see Remark 28) has been proposed by Freeman and Kokotovic in order to "flatten" the Lyapunov function in the neighborhood of the manifold $\{(x_1,x_2)|\, x_2 = \phi_1(x_1)\}$ which is a desirable property, this manifold not being a naturally invariant manifold of the closed loop system (see [21, 22] for more details). The interest of the functions k and Θ is in particular appreciated when ϕ_1 is not C^1 (see [65, 18]).

Assumption A1 can be weakened to the case where V_1 is assignable but not strictly

Assumption A1': *We know a C^1 Lyapunov function V_1 which can be assigned by the time-invariant feedback law ϕ_1 to the system (147) and such that $x_1 = 0$ is the only solution of*

$$\dot{x}_1 = f_1(x_1,\phi_1(x_1)) \quad , \qquad \frac{\partial V_1}{\partial x_1}(x_1)\, f_1(x_1,\phi_1(x_1)) = 0. \tag{161}$$

Indeed, we remark that, by choosing $r \equiv 0$ in (157) the derivative of V_2 along the solutions of (148),(158) is zero if and only if $x_2 = \phi_1(x_1)$. So, we have

Theorem 33 [17, Lemma 1]. *Under assumptions A1', A2 and A3, the system (148) can be (resp. globally) asymptotically stabilized by means of a time-invariant continuous feedback law.*

The disturbed case. Let us now address the problem where a disturbance is present in the dynamics of the x_1 subsystem of (150), i.e.

$$\begin{cases} \dot{x}_1 = f_1(x_1,x_2) + c(x_1,d), \\ \dot{x}_2 = u_2, \end{cases} \tag{162}$$

where c is continuous, x_2 and u_2 are in $\mathbb{R}$ – to simplify the notations – and, as in section 3.3, d is the disturbing signal supposed to be in $L^\infty([0,+\infty);\mathbb{R}^q)$. We assume that we are in the context of Remarks 28 and 31, i.e.

[15] For example $\mathcal{S}(x_1,x_2) = \Theta(x_1,x_2)$ and $\theta(x_1,x_2) = \mathrm{sat}\left(\frac{|x_2-\psi_1(x_1)|}{r(x_1)}\right)$.

We know a C^1 (resp. radially unbounded) positive definite and decrescent function V_1, a C^1 feedback law ϕ_1 and a C^1 positive definite function r (resp. $r \equiv 0$) such that

$$L_{f_1(x_1,u_1)+c(x_1,d)} V_1(x_1) \leq -\alpha_1(V(x_1)) + \sigma_1(|d|) \quad \forall u_1 \in \overline{B}\left(\phi_1(x_1), r(x_1)\right) \tag{163}$$

where σ_1 and α_1 are continuous and strictly increasing functions which are 0 at 0, with α onto $[0, +\infty)$.

Following the previous section, we shall try to assign the function V_2 defined in (157). For this, we let

$$u_2 = \phi_2(x_1, x_2) + u \tag{164}$$

with ϕ_2 given by (158). We get, along the solutions of (147),

$$\begin{aligned} \dot{V}_2 \leq & -k'(V_1(x_1))\left[\alpha_1(V(x_1)) - \sigma_1(|d|)\right] - \Theta(x_1,x_2)\theta(x_1,x_2)\,\mathcal{S}(x_1,x_2) \\ & +\Theta(x_1,x_2)\left[u - \left(\frac{\partial \phi_1}{\partial x_1}(x_1) + \frac{x_2 - \phi_1(x_1)}{|x_2 - \phi_1(x_1)|}\frac{\partial r}{\partial x_1}(x_1)\right)c(x_1,d)\right] \end{aligned} \tag{165}$$

where Θ, θ and $\mathcal{S}$ satisfy (156), (159) and (160) and so in particular

$$\Theta(x_1,x_2)\,\theta(x_1,x_2) = \Theta(x_1,x_2). \tag{166}$$

Then, let us omit the arguments and choose the control

$$u = -\theta\, n_1 \left|\frac{\partial \phi_1}{\partial x_1} + \frac{x_2 - \phi_1}{|x_2 - \phi_1|}\frac{\partial r}{\partial x_1}\right| \gamma\left(\left|\frac{\partial \phi_1}{\partial x_1} + \frac{x_2 - \phi_1}{|x_2 - \phi_1|}\frac{\partial r}{\partial x_1}\right| \Theta\right), \tag{167}$$

where γ is an arbitrary continuous, strictly increasing function mapping $[0, +\infty)$ onto itself and extended on $\mathbb{R}$ by symmetry. By using the fact that θ is smaller than 1 and the following inequality,

$$x\,y \leq |x|\gamma_1(|x|) + |y|\gamma_1^{-1}(|y|), \tag{168}$$

we obtain finally an inequality similar to (141)

$$\dot{V}_2 \leq -\left[k'\alpha_1(V(x_1)) + \Theta\,\mathcal{S}\right] + k'\sigma_1(|d|) + n_1\,|c|\gamma^{-1}(|c|). \tag{169}$$

It remains several degrees of freedom in this inequality. They are determined by the choice of the feedback laws (157) and (167).

To appreciate the interest of this result, let us choose simply a bounded function for r and the identity function for k and γ and $\mathcal{S} = \Theta$. We introduce the following bounding functions

$$C_1(x_1) = \sup_{\{d \,|\, |d| \leq |x_1|\}} \left\{|c(x_1,d)|\,\gamma^{-1}\left(|c(x_1,d)|\right)\right\}, \tag{170}$$

$$C_d(d) = \sup_{\{x_1 \,|\, |x_1| \leq |d|\}} \left\{|c(x_1,d)|\,\gamma^{-1}\left(|c(x_1,d)|\right)\right\}. \tag{171}$$

We get

$$\dot{V}_2 \leq -[\alpha_1(V(x_1)) - n_1 C_1(x_1)] - \Theta(x_1, x_2)^2 + [\sigma_1(|d|) + C_d(d)]. \tag{172}$$

Assume the feedback law ϕ_1 has been chosen such that the term between brackets is a positive definite (resp. radially unbounded function) of x_1. Then, since Θ, in (156), is radially unbounded in x_2 for all x_1, (172) allows us to get an inequality similar to (146). In particular, in this case, the solutions are bounded if the function d is bounded.

The technique presented here and many other improvements have appeared in the literature. We refer the reader in particular to [61, 21, 57, 70, 76].

3.5 Case of dissipative uncontrolled part.

A drawback of the technique of Lyapunov function assignment is the lack of information about the construction, in the general case, of assignable Lyapunov function. On the other hand many dynamic equations representing the dynamics of practical systems are obtained via a variational formulation. In such cases, the "total" energy function provides typically positive definite (resp. radially unbounded) function and the variational approach is based on the fact that without control this function is non increasing along the solutions. This latter property implies that we are almost but not exactly in the context of Theorem 27. To be exactly in that context, we would need a strict decrease along the solutions.

This new context has been the subject of many studies which started with the contributions of Jacobson [34, Theorems 2.5.1 and 2.5.2] and Jurdjevic and Quinn in [38].

To be more explicit let us consider the system

$$\dot{x} \;=\; a(x) \;+\; b(x,u)\, u \tag{173}$$

where x is in $\mathbb{R}^n$, u in $\mathbb{R}^m$, a, b are C^2 functions, $a(0) = 0$ and we denote $\widehat{b}(x) = b(x, 0)$. We introduce the following assumptions:

Assumption B1: *There exists a positive definite and radially unbounded C^2 function V so that, for all $x \in \mathbb{R}^n$,*

$$\frac{\partial V}{\partial x}(x)\, a(x) \;=\; -W(x) \;\leq\; 0. \tag{174}$$

Assumption B2: $x = 0$ *is the only solution of*

$$\dot{x} \;=\; a(x) \quad , \quad \frac{\partial V}{\partial x}(x)\, \widehat{b}(x) \;=\; 0 \quad , \quad \frac{\partial V}{\partial x}(x)\, \widehat{b}(x) \;=\; 0. \tag{175}$$

Assumption B1 expresses the fact that the uncontrolled system is dissipative. Assumption B2 is related to a controllability assumption. It is difficult to check it directly in practice but many sufficient conditions under which it holds have been proposed in the literature (see for instance [48, 49]). For example, following [38], it holds if

There exists an integer r such that

$$E_r = \{x : L_a^k L_{ad_a^i \widehat{b}} V(x) = 0 , \forall k \leq r , i \leq n-1\} = \{0\} . \tag{176}$$

We have:

Theorem 34 [11, Corollary 1.6]. *Under Assumptions B1 and B2, for any $\overline{u}$ in $(0, +\infty]$, the origin can be made a globally asymptotically stable solution of the system (173) by means of a time-invariant continuous feedback law bounded by $\overline{u}$.*

This Theorem relies on the fact that the function V of Assumption B1 is an assignable Lyapunov function and the control is given by any continuous function ϕ – guaranteed to exist – satisfying

1. The function $|\phi(x)|$ is bounded by $\overline{u}$.
2. For all x, the scalar $\frac{\partial V}{\partial y}(y)\, b(x, \phi(x))\, \phi(x)$ is non positive and zero if and only if the vector $\frac{\partial V}{\partial y}(y)\, b(x, 0)$ is zero.

3.6 Adding Integration.

The problem we address now is to design a feedback law for the system

$$\begin{cases} \dot{x} = h(y, u), \\ \dot{y} = f(y, u), \end{cases} \tag{177}$$

where f and h are C^1 and assuming that we know a time-invariant globally asymptotically stabilizing feedback law for the system

$$\dot{y} = f(y, u) \tag{178}$$

As opposed to (148), this time, we add state components which "integrate" functions of the other components. In particular this implies that the x-part of the system is weakly dissipative. This remark explains the strong links between, the results of this section with those of section 3.5.

The knowledge of a solution for this problem, called "adding one integration", allows us to deal with another recurrent structure, called feedforward form, where each state component acts on those following it, in the chain of integration starting from the control, i.e.[16]

$$\left\{\begin{array}{l} \dot{x}_n = f_n(x_1, \dots, x_{n-1}, u) \ , \\ \quad \vdots \\ \dot{x}_2 = f_2(x_1, u) \ , \\ \dot{x}_1 = f_1(x_1, u) \ . \end{array}\right. \tag{179}$$

In particular, a repeated application of Theorem 35 stated below proves that, stabilizability of the system linearized at the origin being assumed, global asymptotic stabilizability by means of a C^1 feedback law holds if $\dot{x} = f_1(x, u)$ is globally asymptotically stabilizable by means of a C^1 feedback law with local exponential stability.

More generally than (177), we consider the following system

$$\left\{\begin{array}{l} \dot{x}_1 = h_0(x_1) + h_1(x_1, x_2, y)y + h_2(x_1, x_2, y, u)u \\ \dot{x}_2 = e_0(x_2) + e_1(x_1, x_2, y)y + e_2(x_1, x_2, y, u)u \\ \dot{y} \ = f_0(y) + f_1(x_1, x_2, y)y + f_2(x_1, x_2, y, u)u \end{array}\right. \tag{180}$$

where y is in $\mathbb{R}^n$, x_1 in $\mathbb{R}^{n_1}$, x_2 in $\mathbb{R}^{n_2}$, u in $\mathbb{R}^m$, all the functions are C^2, $h_0(0) = 0$, $e_0(0) = 0$, $f_0(0) = 0$ and we denote $\widehat{h}_2(x_1) = h_2(x_1, 0, 0)$. Mazenc and Praly have proposed in [59], to study this system under the following assumptions:

Assumption C1: *There exist three positive definite and radially unbounded C^2 functions Q, S and V so that*

$$\frac{\partial Q}{\partial x_1}(x_1)\, h_0(x_1) = -R(x_1) \ \leq \ 0 \qquad \forall x_1 \ , \tag{181}$$

$$\frac{\partial S}{\partial x_2}(x_2)\, e_0(x_2) = -T(x_2) \ < \ 0 \qquad \forall x_2 \neq 0 \ , \tag{182}$$

$$\frac{\partial V}{\partial y}(y)\, f_0(y) = -W(y) \ < \ 0 \qquad \forall y \neq 0 \ . \tag{183}$$

Assumption C2: $x_1 = 0$ *is the only solution of*

$$\dot{x}_1 \ = \ h_0(x_1) \ , \quad \frac{\partial Q}{\partial x_1}(x_1)\, h_2(x_1, 0, 0) \ = \ 0 \ , \quad \frac{\partial Q}{\partial x_1}(x_1)\, h_0(x_1) \ = \ 0 \ . \tag{184}$$

Assumption C3: *There exist positive Lipschitz continuous functions κ and ρ which are zero at zero and such that*

[16] Systems in the form (179) are generically not feedback linearizable. In particular this is the case when, controllability of the system linearized at the origin being assumed, $\frac{\partial^2 f_2}{\partial u^2}\frac{\partial f_1}{\partial u} - \frac{\partial f_2}{\partial u}\frac{\partial^2 f_1}{\partial u^2}$ is not identically equal to zero on an open neighborhood of the origin.

$$\left|\frac{\partial Q}{\partial x_1}(x_1)\,h_1(x_1,x_2,y)\,y\right| + \left|\frac{\partial S}{\partial x_2}(x_2)\,e_1(x_1,x_2,y)\,y\right| \tag{185}$$
$$\leq \sqrt{\kappa(y)}\,(1+\rho(Q(x_1)+S(x_2)))\left[\sqrt{\kappa(y)}\,(1+\rho(Q(x_1)+S(x_2)))+\sqrt{T(x_2)}\right]$$

$$\frac{1}{1+\rho} \notin L^2([0,+\infty))\,, \tag{186}$$

$$\limsup_{y\longrightarrow 0}\left|\frac{\kappa(y)}{W(y)}\right| < +\infty. \tag{187}$$

and we have

$$\left|\frac{\partial V}{\partial y}(y)\,f_1(x_1,x_2,y)\,y\right| \leq \frac{1}{4}W(y). \tag{188}$$

Assumptions C1 and C2 are nothing but B1 and B2 of section 3.5. The new assumption here is C3. It concerns the coupling terms h_1, e_1, f_1. Clearly (188) implies the term f_1 cannot change the asymptotic stability of $y = 0$ whatever $x_1(t), x_2(t)$ are, as long as they are measurable and locally essentially bounded. Inequality (185) with (187) and (186) puts a restriction on the growth in y and (x_1, x_2) on h_1 and e_1. More specifically, (186) is a constraint at infinity for (x_1, x_2) whereas (187) is a constraint on a neighborhood of the origin for y. Unfortunately, with the smoothness of V and f_0, this latter constraint implies, for all x_1

$$\limsup_{y\longrightarrow 0}\frac{\left|\frac{\partial Q}{\partial x_1}(x_1)\,h_1(x_1,0,y)\right|}{|y|} < +\infty. \tag{189}$$

And therefore, we must have $h_1(x_1, 0, 0) = 0$. This shows that a preparatory step may be needed before trying to check if C3 holds. Indeed, in [59], it is shown that, under extra assumptions on the linearization at the origin of (180), the condition (187) is met after a change of variables. To go around this problem, Jankovic, Sepulchre and Kokotovic have proposed an alternative to Assumption C3 in [35]

Assumption C3': *There exist positive Lipschitz continuous functions κ and ρ and $\mathcal{H}$ which are zero at zero and such that (185),(186) hold,*

$$v \mapsto \max_{v\leq V(y)\leq 1}\left\{\frac{\kappa(y)}{W(y)}\right\} \in L^1((0,1];[0,+\infty)), \tag{190}$$

the set $\{(z,x_1,x_2,y)\,|\,z=\mathcal{H}(x_1,x_2,y)\}$ is a C^1 invariant manifold of[17]

$$\begin{cases} \dot z &= -\frac{\partial Q}{\partial x_1}(x_1)\,h_1(x_1,x_2,y)\,y - \frac{\partial S}{\partial x_2}(x_2)\,e_1(x_1,x_2,y)\,y \\ \dot x_1 &= h_0(x_1) + h_1(x_1,x_2,y)\,y \\ \dot x_2 &= e_0(x_2) + e_1(x_1,x_2,y)\,y \\ \dot y &= f_0(y) + f_1(x_1,x_2,y)\,y \end{cases} \tag{191}$$

[17] Sufficient condition for existence of this manifold are given in [35].

and (188) holds.

We have

Theorem 35 [59, 35]. *If Assumptions C1, C2 and C3 or C3' hold, then for any $\overline{u}$ in $(0, +\infty]$, the origin of the system (180) is globally asymptotically stabilizable by means of a time-invariant continuous feedback law bounded by $\overline{u}$.*

In the case of Assumption C3, an assignable Lyapunov function is

$$U(x_1, x_2, y) \;=\; l(Q(x_1) + S(x_2)) + k(V(y)) \; . \tag{192}$$

where l and k are any C^1, positive and radially unbounded functions with strictly positive derivative – guaranteed to exist – satisfying

$$\frac{1}{6} k'(V(y))\, W(y) \;\geq\; \kappa(y) \quad , \qquad l(r) \;=\; \int_0^r l'(s) ds \tag{193}$$

where l' is any function which does not belong to L^1 and satisfies

$$l'(0) \;=\; 1 \quad , 0 \;<\; l' \;\leq\; \frac{1}{(1+\rho)^2} . \tag{194}$$

In the case of Assumption C3', an assignable Lyapunov function is

$$U(x_1, x_2, y) \;=\; Q(x_1) \;+\; S(x_2) \;+\; \mathcal{H}(x_1, x_2, y) \;+\; V(y). \tag{195}$$

In both case, the control is given by any continuous function ϕ – guaranteed to exist – satisfying

1. The function $|\phi(x)|$ is bounded by $\overline{u}$.
2. For all x, the scalar $B(x_1, x_2, y, u)\, \phi(x)$ is non positive and zero if and only if the vector $B(x_1, 0, 0, 0)$ is zero where, in the case of C3,

$$B(x_1, x_2, y, u) \;= \tag{196}$$
$$\left(k'(V(y)) \tfrac{\partial V}{\partial y}(y) \, , \, l'(Q(x_1) + S(x_2)) \tfrac{\partial Q}{\partial x_1}(x_1) \, , \, \tfrac{\partial S}{\partial x_2}(x_2)\right) \begin{pmatrix} f_2(x_1, x_2, y, u) \\ h_2(x_1, x_2, y, u) \\ e_2(x_1, x_2, y, u) \end{pmatrix} .$$

and, in the case of C3',

$$B(x_1, x_2, y, u) \;= \tag{197}$$
$$\left(\tfrac{\partial V}{\partial y}(y) + \tfrac{\partial \mathcal{H}}{\partial y}(x_1, x_2, y) \, , \, \tfrac{\partial Q}{\partial x_1}(x_1) + \tfrac{\partial \mathcal{H}}{\partial x_1}(x_1, x_2, y) \, , \, \tfrac{\partial S}{\partial x_2}(x_2) + \tfrac{\partial \mathcal{H}}{\partial x_2}(x_1, x_2, y)\right)$$
$$\times \begin{pmatrix} f_2(x_1, x_2, y, u) \\ h_2(x_1, x_2, y, u) \\ e_2(x_1, x_2, y, u) \end{pmatrix}$$

The implementation of such feedback laws requires the explicit knowledge of the function V if not of $\mathcal{H}$. If this information is not available, a feedback law can still be designed as follows:
Let

$$\mathcal{B}(x_1, x_2, y) = \qquad (198)$$
$$l'(Q(x_1) + S(x_2)) \left[\frac{\partial Q}{\partial x_1}(x_1) h_2(x_1, x_2, y, 0) + \frac{\partial S}{\partial x_2}(x_2) e_2(x_1, x_2, y, 0)\right].$$

Note that $\mathcal{B}$ does not depend on V or k but depends on l'. However l' can be determined, via (194), from the data of the (x_1, x_2)-subsystem only. Let also R and $\overline{u}$ be two arbitrary strictly positive real numbers and let us introduce two functions independent of V:

1. Let φ_R be a smooth non positive function onto $[0, 1]$ such that

$$\Psi_R(0) = 1 \quad , \qquad \Psi_R(|y|^2) = 0 \quad \forall y \, : \, |y| \geq R \, . \qquad (199)$$

2. Let $\psi_{R,\overline{u}}$ be a smooth function satisfying

$$\psi_{R,\overline{u}}(x_1, x_2) \geq \max \left\{ 1, \sup_{\substack{|u| \leq \overline{u} \\ |y| \leq R}} \left\{ \widehat{\psi}(x_1, x_2, y, u) \right\} \right\} \qquad (200)$$

with

$$\widehat{\psi}(x_1, x_2, y, u) = \left| \frac{f_2(x_1, x_2, y, u) - f_2(x_1, x_2, y, 0)}{u} \right| \qquad (201)$$
$$+ \left| \frac{\frac{\partial Q}{\partial x_1}(x_1) \left[h_2(x_1, x_2, y, u) - h_2(x_1, x_2, y, 0)\right]}{u} \right|$$
$$+ \left| \frac{\frac{\partial S}{\partial x_2}(x_2) \left[e_2(x_1, x_2, y, u) - e_2(x_1, x_2, y, 0)\right]}{u} \right|$$

We have:

Proposition 36. *Assume the system (180) satisfies Assumptions C1, C2 and C3. Under these conditions, if*[18]

$$\liminf_{y \longrightarrow 0} \frac{W(y)}{\left|\frac{\partial V}{\partial y}(y)\right|^2} > 0 \, , \qquad (202)$$

[18] This condition is satisfied if $y = 0$ is a locally exponentially stable equilibrium point of $\dot{y} = f_0(y)$.

then, for any $\overline{u}$ *in* $(0,+\infty)$*, there exists a positive real number* $\zeta^\star$ *in* $(0,\overline{u}]$ *so that the origin of the system (180) can be made a globally asymptotically stable solution by a state feedback bounded by* $\overline{u}$ *and of the form*

$$u(x_1,x_2,y) \;=\; -\,\frac{\zeta\,\Psi_R(|y|^2)\,\mathcal{B}(x_1,x_2,y)}{\psi_{R,\overline{u}}(x_1,x_2)\,(1+|\mathcal{B}(x_1,x_2,y)|)\,(1+|f_2(x_1,x_2,y,0)|^2)} \tag{203}$$

where ζ *is any real number in* $(0,\zeta^\star]$ *and* $\mathcal{B}$ *is defined in (198).*

4 Results based on $\mathcal{L}_\infty$ input-output stability.

4.1 Introduction.

In this section of the paper, we focus on global asymptotic stability and stabilizability (GAS) for nonlinear systems using input-output methods. In particular, we will work with $\mathcal{L}_\infty$-type stability properties. This setting has recently proved to be a very useful domain in which to analyze nonlinear systems and it yields results very similar to Lyapunov stability. The section is intended to highlight a growing, and significant, group of research results on stabilization based on $\mathcal{L}_\infty$-type stability properties while providing pointers to more detailed information in the literature. It is not intended to be a comprehensive overview of the use of input-output methods for nonlinear stability. Consequently, passivity and other issues pertaining to $\mathcal{L}_2$ stability, for example, are not addressed. Finally, local versions of the results presented here are fairly straightforward and, thus, are not pursued.

The definition of global asymptotic stability for the equilibrium of an autonomous ordinary differential equation consists of three parts: 1) local ($\epsilon - \delta$) stability, 2) global boundedness, and 3) global convergence. Frequently, as in the previous section, this property is established by producing a positive definite, radially unbounded function with a negative definite derivative along the trajectories of the differential equation. In this part of the paper, instead, we will consider nonlinear systems decomposed into subsystems and we will use appropriate characterizations of the subsystems input-output behavior to establish the pieces which constitute the GAS property. (It turns out that there are many connections between the input-output properties we will use and corresponding Lyapunov function properties. The references [88, 85], [37] and [87] are useful supplements in this direction.) Our key tool for analyzing the interconnection of subsystems will be a version of the nonlinear small gain theorem, based on a condition made precise in [54] and [36]. In general, the small gain theorem is a very efficient tool for discussing robustness of the GAS property.

4.2 Clarifications for this section.

Throughout, we will assume that all vector fields are smooth enough so that solutions exist locally and are unique. We will use the so-called ∞-norm for vectors $x \in \mathbb{R}^n$, i.e. $|x| := \max_i |x_i|$. We will use the word 'signal' to refer to a time function that is measurable and locally essentially bounded. We will use the phrase 'bounded signal' for a signal that is essentially bounded. For a bounded signal d we will use

$$||d||_\infty := \sup_{t\in[0,\infty)} |d(t)| \quad , \qquad ||d||_a := \limsup_{t\to\infty} |d(t)| \tag{204}$$

where each supremum is understood to be an essential supremum.

4.3 Cascades as a starting point.

The interest in the GAS property for nonlinear systems decomposed into subsystems accelerated with the development of global, partially linear normal forms via the tools of geometric nonlinear control. See [33, 6] and the references therein. When applying these tools, a common structure that emerges after feedback is that of a cascade of subsystems :

$$\dot{x}_1 = f_1(x_1, x_2) \quad , \qquad \dot{x}_2 = f_2(x_2). \tag{205}$$

For such a system to have the GAS property, the origins of the systems

$$\dot{x}_1 = f_1(x_1, 0) \quad , \qquad \dot{x}_2 = f_2(x_2) \tag{206}$$

must be GAS. Moreover, for the system

$$\dot{x}_1 = f_1(x_1, u), \tag{207}$$

for each initial condition and each input u generated by the solutions of the autonomous x_2 subsystem in (206) according to $u(t) = x_2(t)$, the solution must converge to the origin. For nonlinear systems, this property is not guaranteed by the property that the x_1 subsystem in (206) is GAS. For example, as pointed out in [80], the scalar system

$$\dot{x} = -x + (x^2 + 1)u \tag{208}$$

is GAS when $u \equiv 0$, but with

$$u(t) = \frac{1}{\sqrt{2t+2}} \quad , \qquad x(0) = \sqrt{2} \tag{209}$$

the resulting solution $x(t) = \sqrt{2t+2}$ is unbounded.

4.4 Addressing cascades via the ISS property.

In general, efficient methods for checking whether converging inputs produce converging states are not known. Instead, and also motivated by robustness issues to be discussed later, we will impose a stronger property which can be verified using Lyapunov arguments. In what follows, we will define and employ what is essentially the input-to-state stability (ISS) property introduced by Sontag in [78]. In our definition, we will call a function from the nonnegative real numbers to the nonnegative real numbers which is continuous, zero at zero and nondecreasing a **gain function**.

Consider the system

$$\dot{x} = f(x, d_1, d_2) \quad , \qquad x(0) = x_\circ \tag{210}$$

where $x \in \mathbb{R}^n$, $d_1 \in \mathbb{R}^{m_1}$, and $d_2 \in \mathbb{R}^{m_2}$, along with an 'output' function $h(x, d_1, d_2)$. Let γ_0, γ_1 and γ_2 be gain functions. We will say that h satisfies an

a-$\mathcal{L}_\infty$ stability bound[19] **with gain** $(\gamma_\circ, \gamma_1, \gamma_2)$ if, for each $x_\circ \in \mathbb{R}^n$ and each pair of bounded signals $(d_1(\cdot), d_2(\cdot))$, the solution to (210) exists for all $t \geq 0$ and satisfies

$$\begin{aligned} ||h(x,d_1,d_2)||_\infty &\leq \max\left\{ \gamma_0\left(|x_\circ|\right) , \quad \gamma_1\left(||d_1||_\infty\right) , \quad \gamma_2\left(||d_2||_\infty\right) \right\} \\ ||h(x,d_1,d_2)||_a &\leq \max\left\{ \gamma_1\left(||d_1||_a\right) , \quad \gamma_2\left(||d_2||_a\right) \right\} . \end{aligned} \tag{211}$$

It will be convenient to think of the subscripts on the gain functions as channel numbers. We will always use channel 0 for the initial conditions. When we are working with the interconnection of two subsystem, each with a channel 1 gain for example, we will use γ_{11} to refer to the channel 1 gain for system 1 and γ_{12} to refer to the channel 1 gain for system 2.

There is a connection between this property when $h(x,d) := x$ and the existence of a Lyapunov-type function with a particular property for its derivative.

Fact 37 [78]. *If there exist a function V, globally invertible gain functions $\underline{\alpha}$ and $\overline{\alpha}$ and a gain function γ such that*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{212}$$

and

$$|x| > \gamma(|u|) \quad \Longrightarrow \quad \frac{\partial V}{\partial x}(x) f(x,u) < 0 \tag{213}$$

then the state x of the system $\dot{x} = f(x,u)$ satisfies an a-$\mathcal{L}_\infty$ stability bound with gain

$$(\gamma_0, \gamma_1) := \left(\underline{\alpha}^{-1} \circ \overline{\alpha} \, , \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \gamma\right) . \tag{214}$$

Remark 38. A converse of this result, for Sontag's ISS property, is reported on in [88] (see also [52] and [85]).

Now, returning to the cascade system in (205), if the state of the system

$$\dot{x}_1 = f_1(x_1, d) \tag{215}$$

satisfies an a-$\mathcal{L}_\infty$ stability bound then converging inputs produce converging states and, thus, the cascade is GAS. Thus, with this condition on the x_1 subsystem, the GAS control problem for the system

$$\dot{x}_1 = f_1(x_1, x_2) \quad , \qquad \dot{x}_2 = f_2(x_2, u) \tag{216}$$

reduces to the GAS control problem for the x_2 subsystem.

---

[19] You may wish to read the 'a' in 'a-$\mathcal{L}_\infty$' as 'augmented' or 'asymptotic' or simply as the 'a' subscript in (204).

Adding integrators. When it is not the case that the state of the x_1 subsystem in (216) satisfies an a-$\mathcal{L}_\infty$ stability bound, certain control systems in the form (216) benefit from the following lemma:

Lemma 39 [81]. *If the origin of the system $\dot{x} = f(x, 0)$ is GAS then there exists a smooth, globally invertible matrix-valued function[20] $\beta(x)$ such that the state of the system $\dot{x} = f(x, \beta(x)d)$ satisfies an a-$\mathcal{L}_\infty$ stability bound.*

Consider the GAS control problem for the system

$$\dot{x}_1 = f_1(x_1, x_2) \quad , \qquad \dot{x}_2 = u \tag{217}$$

where $x_1 \in \mathbb{R}^n$ and u and x_2 belong to $\mathbb{R}^m$. (The idea presented here easily generalizes to the case of adding chains of integrators.) Suppose that there exists a smooth function $k(x_1)$ so that the origin of the system

$$\dot{x}_1 = f_1(x_1, k(x_1)) \tag{218}$$

is GAS. Using the above lemma, there exists $\beta(x_1)$, smooth and globally invertible, so that the state of the system

$$\dot{x}_1 = f_1(x_1, k(x_1) + \beta(x_1)d) \tag{219}$$

satisfies an a-$\mathcal{L}_\infty$ stability bound. Define $\tilde{x}_2 := \beta^{-1}(x_1)\,[x_2 - k(x_1)]$, so that the original system (217) becomes

$$\begin{cases} \dot{x}_1 = f_1\left(x_1, k(x_1) + \beta(x_1)\tilde{x}_2\right) \\ \dot{\tilde{x}}_2 = \beta^{-1}(x_1)u + g(x_1, \tilde{x}_2) \end{cases} \tag{220}$$

where the definition of g follows from differentiating $\tilde{x}_2$. Then choosing $u = \beta(x_1)[\alpha(\tilde{x}_2) - g(x_1, \tilde{x}_2)]$, where the origin of $\dot{x} = \alpha(x)$ is GAS, achieves the GAS property for the system (217).

4.5 GAS for feedback interconnections.

Imposing an a-$\mathcal{L}_\infty$ stability bound on the state of the x_1 subsystem in (205) is much stronger than is needed. However, it leads to natural robustness conditions for the GAS property made clear via a nonlinear small gain theorem. In particular, consider a nominally cascaded system which is perturbed in a way so that the x_1 subsystem feeds into the x_2 subsystem:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2, x_1). \end{cases} \tag{221}$$

[20] We will have use later for the fact that this matrix-valued function can be chosen to be globally bounded and equal to the identity matrix on a neighborhood of the origin.

Suppose that for each subsystem its state satisfies an a-$\mathcal{L}_\infty$ stability bound. Let γ_{11} denote the channel 1 gain function for the x_1 subsystem and let γ_{12} denote the channel 1 gain function for the x_2 subsystem. We will say that the composition of these two functions (the channel 1 gains) is a **simple contraction** if $\gamma_{11}(\gamma_{12}(s)) < s$ (equivalently $\gamma_{12}(\gamma_{11}(s)) < s$) for all $s > 0$.

Theorem 40. *If the composition of the channel 1 gains is a simple contraction then the origin of (221) is GAS.*

Proof. Given a particular initial condition, let $[0, T)$ be the maximal interval of definition for the system (221). We will use x_τ to represent the truncation of the signal x at time τ. By causality, we have, for each $\tau \in [0, T)$,

$$\begin{aligned} \|x_{1_\tau}\|_\infty &\leq \max\left\{ \gamma_{01}(|x_{1_\circ}|)\ ,\quad \gamma_{11}\left(\|x_{2_\tau}\|_\infty\right) \right\} \\ \|x_{2_\tau}\|_\infty &\leq \max\left\{ \gamma_{02}(|x_{2_\circ}|)\ ,\quad \gamma_{12}\left(\|x_{1_\tau}\|_\infty\right) \right\}. \end{aligned} \tag{222}$$

Combining, we get

$$\begin{aligned} \|x_{1_\tau}\|_\infty &\leq \max\left\{ \gamma_{01}(|x_{1_\circ}|),\quad \gamma_{11}\circ\gamma_{02}(|x_{2_\circ}|),\quad \gamma_{11}\circ\gamma_{12}\left(\|x_{1_\tau}\|_\infty\right) \right\} \\ \|x_{2_\tau}\|_\infty &\leq \max\left\{ \gamma_{02}(|x_{2_\circ}|),\quad \gamma_{12}\circ\gamma_{01}(|x_{1_\circ}|),\quad \gamma_{12}\circ\gamma_{11}\left(\|x_{2_\tau}\|_\infty\right) \right\}. \end{aligned} \tag{223}$$

But, since the composition of γ_{11} and γ_{12} is a simple contraction, it follows that

$$\begin{aligned} \|x_{1_\tau}\|_\infty &\leq \max\left\{ \gamma_{01}(|x_{1_\circ}|)\ ,\quad \gamma_{11}\circ\gamma_{02}(|x_{2_\circ}|) \right\} \\ \|x_{2_\tau}\|_\infty &\leq \max\left\{ \gamma_{02}(|x_{2_\circ}|)\ ,\quad \gamma_{12}\circ\gamma_{01}(|x_{1_\circ}|) \right\}. \end{aligned} \tag{224}$$

Now, since the right hand sides are independent of τ, we have that x_1 and x_2 are uniformly bounded on $[0, T)$ which tells us that $T = \infty$ and that $\|x_1\|_a$ and $\|x_2\|_a$ are well-defined. So, we can also use

$$\begin{aligned} \|x_1\|_a &\leq \gamma_{11}(\|x_2\|_a) \\ \|x_2\|_a &\leq \gamma_{12}(\|x_1\|_a). \end{aligned} \tag{225}$$

Combining these inequalities and again using that the composition of γ_{11} and γ_{12} is a simple contraction, it follows that $\|x_1\|_a = \|x_2\|_a = 0$. □

Remark 41. A similar result can be stated when the interconnection is made via generic output functions. Such a generalization is important for the result in the last part of section 4.7. This idea is also discussed in more detail in [36].

Remark 42. From the proof, it follows that if the composition of the gains is a contraction only for sufficiently small values of s then local asymptotic stability can be established. On the other hand, if the composition is a contraction only for sufficiently large values of s then global boundedness can be established. Indeed, suppose that $\gamma_1(\gamma_2(s)) < s$ for $s > s^*$ and we have $x \le \max\{a, b, \gamma_1(\gamma_2(x))\}$. Then, we claim that $x \le \max\{a, b, s^*\}$. If this is not the case then necessarily $x > s^*$ and $x \le \gamma_1(\gamma_2(x))$ which contradicts the assumption.

Such situations are addressed in [96] and [36].

Remark 43. When there are multiple inputs, it is very common in the literature to work with an input-output bound which is expressed in terms of a summation rather than a maximum. In this case, the small gain condition needs to be stronger. The implication of the composition of γ_1 and γ_2 being a simple contraction is that, for each pair of positive real numbers b_1 and b_2, there exists a positive real number s^* such that the curve $\left(s\,, \max\{b_1, \gamma_1(s)\}\right)$ is below the curve $\left(\max\{b_2, \gamma_2(r)\}\ , r\right)$ for all $s > s^*$. Analogously for the summation case, we would want that, for each pair of positive real numbers b_1 and b_2, there exists a positive real number s^* such that the curve $(s, b_1 + \gamma_1(s))$ is below the curve $(b_2 + \gamma_2(r), r)$ for all $s > s^*$. But since b_1 and b_2 can be arbitrarily large, the curve $(s, b_1 + \gamma_1(s))$ can be shifted an arbitrarily large vertical distance from the curve $(s, \gamma_1(s))$ while the curve $(b_2 + \gamma_2(r), r)$ can be shifted an arbitrarily large horizontal distance from the curve $(\gamma_2(r), r)$. Thus, it is not enough for the curve $(s, \gamma_1(s))$ to simply be below the curve $(\gamma_2(r), r)$, i.e. for the composition to be simple contraction. Indeed what is required is that the distance between these two curves grows without bound. One way to characterize this is to require that there exists a globally invertible gain function ρ such that the composition of the functions $\gamma_1 + \rho$ and $\gamma_2 + \rho$ is a simple contraction. This is essentially the condition used in [54] and [36]. It is easy to see that this is sufficient by first using the fact that, for any globally invertible gain function ρ,

$$b + \gamma(a) \le \max\left\{ (\mathrm{Id} + \gamma \circ \rho^{-1})(b)\,, (\gamma + \rho)(a) \right\} \tag{226}$$

and then using theorem 40. The inequality (226) follows from considering the two case: $b \le \rho(a)$ and $a \le \rho^{-1}(b)$.

These ideas are strongly connected to the ideas found in [73]. For the case where the gain functions are linear, there is no difference between a simple contraction and the stronger notion discussed above. Instead, both properties become that the product of the coefficients of the linear gains be less than one, i.e. the condition of the classical small gain theorem [102].

Example 1. As a simple example, consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + \gamma_1(|x_2|) \\ \dot{x}_2 = -x_2 + \gamma_2(|x_1|) \end{cases} \tag{227}$$

where the functions γ_1 and γ_2 are gain functions. Using fact 37 and the function $V_i = x_i^2$, it follows that x_1 satisfies an a-$\mathcal{L}_\infty$ stability bound with channel 1 gain

γ_1 and x_2 satisfies an a-$\mathcal{L}_\infty$ stability bound with channel 1 gain γ_2. Therefore, if the composition of γ_1 and γ_2 is a simple contraction then the system is GAS.

It can be shown that if the composition is not a simple contraction then the origin is not GAS. For example, if there exists an $s^* > 0$ such that $\gamma_1(\gamma_2(s^*)) = s^*$ then $(s^*, \gamma_2(s^*))$ is a nonzero equilibrium. Otherwise, if $\gamma_1(\gamma_2(s)) > s$ for all $s > 0$ then the set

$$\left\{ (x_1, x_2) : \ 0 < x_2 \leq \gamma_2(|x_1|) \ , 0 < x_1 \leq \gamma_1(|x_2|) \right\}$$

is nonempty, positively invariant, and the function $x_1^2 + x_2^2$ is strictly increasing inside this set.

4.6 $\mathcal{L}_\infty$ stability for feedback interconnections.

For purposes of iteration, we may be interested in establishing an a-$\mathcal{L}_\infty$ stability bound for the composite state of the interconnection with respect to an external input rather than simply the GAS property. Consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, d_1) \\ \dot{x}_2 = f_2(x_2, x_1, d_2) \end{cases} \tag{228}$$

and suppose that for each subsystem its state satisfies an a-$\mathcal{L}_\infty$ stability bound.

Theorem 44. *If the composition of the channel 1 gains is a simple contraction then the composite state* $\binom{x_1}{x_2}$ *for the system (228) satisfies an a-*$\mathcal{L}_\infty$ *stability bound.*

Remark 45. The proof of this result uses the same calculations as in the proof of theorem 40. In working through the proof, one can easily construct the gain functions for the closed loop system. They are simple combinations of the gain functions for the subsystems. In particular, with respect to $\binom{x_{1_\circ}}{x_{2_\circ}}$, d_1 and d_2, respectively, the gains are

$$\begin{aligned} \gamma_0(s) &= \max\left\{ \gamma_{01}(s) \ , \quad \gamma_{11} \circ \gamma_{02}(s) \ , \quad \gamma_{02}(s) \ , \quad \gamma_{12} \circ \gamma_{01}(s) \right\} \\ \gamma_1(s) &= \max\left\{ \gamma_{21}(s) \ , \quad \gamma_{12} \circ \gamma_{21}(s) \right\} \\ \gamma_2(s) &= \max\left\{ \gamma_{11} \circ \gamma_{22}(s) \ , \quad \gamma_{22}(s) \right\} . \end{aligned} \tag{229}$$

Remark 46. Similar to the comments in remark 42, if the composition of the gains is a contraction only for s sufficiently small then the a-$\mathcal{L}_\infty$ stability bound holds for sufficiently small initial conditions and inputs d_1 and d_2 with a sufficiently small $\mathcal{L}_\infty$-norm. Also, if the composition of the gains is a contraction only for s sufficiently large then the state of the closed loop system satisfies a modified a-$\mathcal{L}_\infty$ stability bound where a positive offset is included in the maximums on the right hand side of (211).

4.7 Robust control via gain assignment.

In this section we will state some important results on assigning closed loop gains for nonlinear control systems. These results will be used to solve certain robust stabilization problems.

Gain assignment for a scalar system. Consider the scalar system

$$\dot{x} = u + \phi(x, d_1) + d_2 =: f(u, x, d_1, d_2) \tag{230}$$

and suppose gain functions ρ_0 and ρ_1 are known so that

$$|\phi(x, d_1)| \leq \max\{\, \rho_0(|x|),\, \rho_1(|d_1|) \,\}\,. \tag{231}$$

Lemma 47 [36]. *Let γ_1 be a globally invertible gain function and suppose that $\rho_1 \circ \gamma_1^{-1}$ and ρ_0 are locally Lipschitz at the origin. Then there exists a smooth function $k(x)$ so that the state x of the closed loop system (230) with $u = k(x)$ satisfies an a-$\mathcal{L}_\infty$ stability bound with gain* $(\mathrm{Id}, \gamma_1, \mathrm{Id})$.

Remark 48. If $\rho_1 \circ \gamma_1^{-1}$ is not locally Lipschitz at the origin, for each strictly positive real number δ, one can always find a globally invertible gain function $\bar{\gamma}_1$ so that $\rho_1 \circ \bar{\gamma}_1^{-1}$ is locally Lipschitz at the origin and $\bar{\gamma}_1(s) \leq \max\{\gamma_1(s), \delta\}$. So, even without the extra condition on $\rho_1 \circ \gamma_1^{-1}$, one can still achieve the channel 1 gain γ_1 with a smooth control if one is willing to tolerate a small positive offset in the a-$\mathcal{L}_\infty$ stability bound. Also, if one is willing to settle for a control which is continuous and smooth everywhere except at the origin, the extra Lipschitz conditions are not needed.

Proof. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be smooth, zero at zero, strictly increasing, odd and, with the definition $\tilde{\alpha}(s) := \alpha(s)$ for $s \geq 0$, satisfies

$$\max\left\{ \rho_0(s) \ , \rho_1 \circ \gamma_1^{-1}(s) \right\} \leq \tilde{\alpha}(s). \tag{232}$$

Such a smooth function exists from the local assumptions made for the functions on the left hand side. Then pick

$$u = -x - \alpha(x) =: k(x) \tag{233}$$

and consider the derivative of the function $V(x) = x^2$ along the solutions of the closed loop system. We have

$$\begin{aligned} &\frac{\partial V}{\partial x}(x) f(k(x), x, d_1, d_2) \\ &= 2x \Big[- x - \alpha(x) + \phi(x, d_1) + d_2 \Big] \\ &\leq 2|x| \Big[- |x| - \tilde{\alpha}(|x|) + \max\{\, \rho_0(|x|),\, \rho_1(|d_1|) \,\} + |d_2| \Big]\,. \end{aligned} \tag{234}$$

Suppose

$$|x| > \max\{\gamma_1(|d_1|), |d_2|\}. \tag{235}$$

Then, using (232), this implies

$$|x| > \max\left\{\tilde{\alpha}^{-1} \circ \rho_1(|d_1|), |d_2|\right\} \tag{236}$$

which in turn implies

$$\tilde{\alpha}(|x|) > \rho_1(|d_1|) \quad , \qquad |x| > |d_2|. \tag{237}$$

Since we have from (232) that $\tilde{\alpha}(|x|) \geq \rho_0(|x|)$, it follows that (235) implies $\dot{V} < 0$. The lemma then follows from fact 37. □

Adding perturbed integrators. (For more details see [36] and the related problem in [67].) Consider the system

$$\begin{cases} \dot{z} = f_1(z, x_1) \\ \dot{x}_1 = x_2 + \phi_1(z, x_1) \\ \dot{x}_2 = u + \phi_2(z, x_1, x_2) \end{cases} \tag{238}$$

where the state of the z subsystem satisfies an a-$\mathcal{L}_\infty$ stability bound with channel 1 gain γ_{11}, and where ϕ_1 and ϕ_2 are locally Lipschitz and vanish at the origin. In particular, consider the GAS control problem using feedback of x_1 and x_2 only. To solve this problem, we will use lemma 47 twice. We will assume the data of the problem is such that the local Lipschitz conditions in the lemma hold for each application. Otherwise, according to remarks 48 and 46, we can achieve a type of "practical GAS". Let γ_{12} be a gain function so that the composition of γ_{11} and γ_{12} is a simple contraction and let k be the solution to the gain assignment problem for γ_{12}. Now define $\zeta_2 = x_2 - k(x_1)$ so that we have

$$\begin{cases} \dot{z} = f_1(z, x_1) \\ \dot{x}_1 = k(x_1) + \phi_1(z, x_1) + \zeta_2 \\ \dot{\zeta}_2 = u + \tilde{\phi}_2(z, x_1, \zeta_2) \end{cases} \tag{239}$$

where the definition of $\tilde{\phi}$ follows from differentiating ζ_2. From the solution to the gain assignment problem, and according to remark 45, the state of the (z, x_1) subsystem satisfies an a-$\mathcal{L}_\infty$ stability bound with respect to ζ_2 with channel 1 gain

$$\tilde{\gamma}_{11}(s) = \max\{\gamma_{11}(s), s\}. \tag{240}$$

We then apply lemma 47 a second time, this time for the ζ_2 subsystem, for a gain function whose composition with $\tilde{\gamma}_{11}$ is a simple contraction. From theorem 44, this will give us that the state of the closed loop system satisfies an a-$\mathcal{L}_\infty$ stability bound with respect to initial conditions and an additive disturbance at the input. When this additive disturbance is zero, we have the GAS property.

Gain assignment for a general system. Consider the system

$$\dot{x} = f(x) + g(x)[u + d]. \tag{241}$$

Lemma 49 [68]. *Suppose the system (241) with $d = 0$ can be made GAS with smooth static state feedback. Let γ_x and γ_u be globally invertible gain functions. If the functions γ_x^{-1} and γ_u^{-1} are suitably smooth at the origin then there exists a smooth function $k(x)$ such that, for the closed loop system (241) with $u = k(x)$, the state x and the function $k(x)$ satisfy a-$\mathcal{L}_\infty$ stability bounds, the former with channel 1 gain γ_x and the latter with channel 1 gain $\mathrm{Id} + \gamma_u$.*

Remark 50. Remark 48 applies when the functions γ_x^{-1} and γ_u^{-1} aren't suitably smooth.

We use this result to discuss the GAS control problem for nonlinear systems affine in u where stable, unmodeled dynamics enter additively at the input. (For more details see [68]. Compare also with [44].) In particular we consider the GAS control problem for the system

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)[u + \phi(x_2, x_1, u)] \\ \dot{x}_2 = f_2(x_2, x_1, u) \end{cases} \tag{242}$$

using only feedback of x_1. We suppose that 1) the x_1 subsystem with $\phi \equiv 0$ can be made GAS with smooth static state feedback; 2) for the system

$$\dot{x}_2 = f_2(x_2, d_1, d_2), \tag{243}$$

the state x_2 and the function $\phi(x_2, d_1, d_2)$ satisfy a-$\mathcal{L}_\infty$ stability bounds, the function ϕ with channel 1 gain γ_{12} and channel 2 gain γ_{22}; and 3) there exist globally invertible gain functions γ_x and γ_u (with a suitably smooth inverses at the origin as required in lemma 49) so that the composition of γ_{12} and γ_x, as well as the composition of γ_{22} and $\mathrm{Id} + \gamma_u$, are simple contractions.

Apply lemma 49 to find a function $k(x_1)$ so that the closed loop system (242) with $u = k(x_1)$ satisfies: for each initial condition there is a maximal interval of definition $[0, T)$ and for each $\tau \in [0, T)$,

$$\begin{aligned} ||\phi_\tau||_\infty &\leq \max\left\{ \gamma_{0\phi}(|x_{2_\circ}|) \,,\; \gamma_{12}(||x_{1_\tau}||_\infty) \,,\; \gamma_{22}(||k_\tau||_\infty) \right\} \\ ||k_\tau||_\infty &\leq \max\left\{ \gamma_{0k}(|x_{1_\circ}|) \,,\; (\mathrm{Id} + \gamma_u)(||\phi_\tau||_\infty) \right\} \\ ||x_{1_\tau}||_\infty &\leq \max\left\{ \gamma_{0x_1}(|x_{1_\circ}|) \,,\; \gamma_x(||\phi_\tau||_\infty) \right\} \end{aligned} \tag{244}$$

and, if $T = \infty$ and all signals are bounded,

$$
\begin{aligned}
||\phi||_a &\leq \max\left\{ \quad \gamma_{12}(||x_1||_a) \ , \quad \gamma_{22}(||k||_a) \quad \right\} \\
||k||_a &\leq (\mathrm{Id} + \gamma_u)(||\phi||_a) \\
||x_1||_a &\leq \gamma_x(||\phi||_a) \ .
\end{aligned}
\tag{245}
$$

Using the same type of calculations as in the proof of theorem 40, and then combining with the a-$\mathcal{L}_\infty$ stability bound for the state x_2, it follows that all signals are defined on $[0, \infty)$, are bounded by a gain function of the initial conditions and converge to zero. Thus, the GAS property is established.

4.8 'Saturated' interconnections.

Having summarized several results for feedback interconnections, we draw attention back to cascades for a moment. We have worked with cascades where the state of the driven subsystem satisfied an a-$\mathcal{L}_\infty$ stability bound. However, we pointed out that this was stronger than was really needed. In fact, although for cascades it is somewhat awkward to think of it this way, we only need the second part of the a-$\mathcal{L}_\infty$ stability bound in (211) and, even then, we only need it for inputs that converge to zero. The reason that this is the case is that we know a priori, regardless of what the driven subsystem does, that the state of the autonomous system will converge to zero. This observation suggests a final class of interconnections that we will consider. This class will be such that the state of one subsystem is guaranteed to converge to a ball of a certain radius. This will be guaranteed using the second inequality of an a-$\mathcal{L}_\infty$ stability bound with a globally bounded gain function. (The state of the autonomous subsystem of a cascade can be thought of as satisfying such a bound with gain function identically zero.) The state of the second subsystem will also be assumed to satisfy the second inequality of an a-$\mathcal{L}_\infty$ stability bound but only for inputs that converge to a sufficiently small ball. For global boundedness, which is the second piece of the GAS property (cf. section 4.1), we will simply need that these two balls match. Global convergence will be guaranteed if the composition of the gain functions in the mentioned inequalities is a simple contraction. This is the third piece of the GAS property. Typically the remaining piece of the GAS property, namely the LAS property, can be check via the Jacobian linearization or a local version of theorem 40.

The description of the above type of interconnection may sound rather contrived. But, in fact, it has proved quite useful in the design and analysis of control laws for systems with saturation and/or with a type of feedforward structure similar to that discussed in the previous section. For references in this direction, see [99, 97, 98]. The same idea is used in [94] and [95] but without the same degree of formalism.

We now modify the notion of an a-$\mathcal{L}_\infty$ stability bound so that the first bound, the $\mathcal{L}_\infty$ bound, is removed while the second bound holds, but perhaps for a restricted class of inputs. Consider the system

$$\dot{x} = f(x, d_1, d_2) \quad , \qquad x(0) = x_\circ \tag{246}$$

where $x \in \mathbb{R}^n$, $d_1 \in \mathbb{R}^{m_1}$ and $d_2 \in \mathbb{R}^{m_2}$, along with an 'output' function $h(x, d_1, d_2)$. Let γ_1 and γ_2 be gain functions and $\Delta_1, \Delta_2 \in \mathbb{R}_{\geq 0} \cup \infty$. We will say that h satisfies an **asymptotic bound with gain** $(\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2})$ **and restriction** $(\boldsymbol{\Delta_1}, \boldsymbol{\Delta_2})$ if, for each $x_\circ \in \mathbb{R}^n$ and each pair of signals $(d_1(\cdot), d_2(\cdot))$ satisfying

$$||d_1||_a \leq \Delta_1 \quad , \quad ||d_2||_a \leq \Delta_2 \tag{247}$$

the solution to (246) exists for all $t \geq 0$ and satisfies

$$||h(x, d_1, d_2)||_a \leq \max \left\{ \gamma_1 \left(||d_1||_a\right) \, , \, \gamma_2 \left(||d_2||_a\right) \right\} . \tag{248}$$

The arguments of the gain functions are not necessarily bounded since we are working here with signals which are not necessarily bounded a priori. Thus, we need the definition $\gamma(\infty) := \lim_{s \to \infty} \gamma(s)$. This quantity may also be infinite. We will refer to $\gamma(\infty)$ as the supremum of γ. Also, analogous to the labeling of gains, Δ_i will be referred to as the channel i restriction.

For an example of when this property holds, it was shown in [94] that, for matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, if the pair of matrices (A, B) is stabilizable and the eigenvalues of A have nonpositive real part then for each strictly positive real number b there exists a smooth function $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and strictly positive real numbers (Δ, N) such that

$$|\alpha(x)| \leq b \qquad \forall x \in \mathbb{R}^n, \tag{249}$$

the state of the system

$$\dot{x} = Ax + B\alpha(x) + d \tag{250}$$

satisfies an asymptotic bound with channel 1 gain $N \cdot \mathrm{Id}$ and channel 1 restriction Δ, and when $d \equiv 0$ the origin of the system (250) is GAS.

We will now consider the interconnection of subsystems in the form (228) where the state of each subsystem satisfies an asymptotic bound. Moreover, we will assume that for all initial conditions and signals (d_1, d_2) defined on $[0, \infty)$ there is no finite escape time.

Theorem 51. *Suppose the system (228) has no finite escape times. If*

1. *the channel 1 restriction for x_2 is ∞, i.e. $\Delta_{12} = \infty$,*
2. *the channel 1 restriction for x_1 is finite, i.e. $\Delta_{11} < \infty$,*
3. *the supremum of the channel 1 gain for x_2 is less than or equal to the channel 1 restriction for x_1, i.e. $\gamma_{12}(\infty) \leq \Delta_{11}$ and*
4. *the composition of the channel 1 gains is a simple contraction*

then the composite state $\binom{x_1}{x_2}$ satisfies an asymptotic bound.

Proof. We will show that the composite state satisfies an asymptotic bound with gain (γ_1, γ_2) given in (229) and with restriction $(\Delta_{21}, \tilde{\Delta}_2)$ where $\tilde{\Delta}_2 \in (\mathbb{R}_{\geq 0} \cup \infty) \cap [0, \Delta_{22}]$ satisfies

$$\max \left\{ \quad \gamma_{12}(\infty) \ , \quad \gamma_{22}(\tilde{\Delta}_2) \quad \right\} \leq \Delta_{11} \ . \tag{251}$$

First, such a $\tilde{\Delta}_2$ exists since $\gamma_{12}(\infty) \leq \Delta_{11}$. Next, since $\tilde{\Delta}_2 \leq \Delta_{22}$ and since $\Delta_{12} = \infty$, $||d_2||_a \leq \tilde{\Delta}_2$ implies

$$||x_2||_a \leq \max \left\{ \quad \gamma_{12}(||x_1||_a) \ , \quad \gamma_{22}(||d_2||_a) \quad \right\} \leq \Delta_{11} \ . \tag{252}$$

This, together with $||d_1||_a \leq \Delta_{21}$ implies that

$$||x_1||_a \leq \max \left\{ \quad \gamma_{11}(||x_2||_a) \ , \quad \gamma_{21}(||d_1||_a) \quad \right\} \ . \tag{253}$$

Now, with the definition of γ_1 in (229), if $\gamma_{21}(||d_1||_a)$ is not finite then there is nothing to prove. Otherwise, both $||x_1||_a$ and $||x_2||_a$ are bounded and the inequalities (252) and (253) can be combined to arrive at the desired result. □

Applications. As a first application, consider the GAS control problem for the system

$$\begin{cases} \dot{x}_1 = Ax_1 + Bu \\ \dot{x}_2 = f(x_2, u) \end{cases} \tag{254}$$

where $u \in \mathbb{R}^m$. This may be a subproblem for the control of this system followed by chains of integrators. Suppose that the origin of the system

$$\dot{x}_2 = f(x_2, 0) \tag{255}$$

is GAS, the pair (A, B) is stabilizable and the eigenvalues of A have nonpositive real part. According to lemma 39, and its footnote, there exist a smooth, globally invertible, globally bounded matrix function $\beta(x_2)$, a gain function γ_2 and a strictly positive real number δ such that $|x_2| \leq \delta$ implies $\beta(x_2) = I_{m \times m}$ and such that the state of the system

$$\dot{x}_2 = f(x_2, \beta(x_2)v) \tag{256}$$

satisfies an a-$\mathcal{L}_\infty$ bound with channel 1 gain γ_2. Let $b > 0$ be such that

$$\gamma_2(b) \leq \delta. \tag{257}$$

Then, according to the result in [94] there exists a smooth function α and strictly positive real numbers N and Δ such that

$$|\alpha(x_1)| \leq b \qquad \forall x_1 \in \mathbb{R}^n, \tag{258}$$

the state of the system

$$\dot{x}_1 = Ax_1 + B(\alpha(x_1) + d) \tag{259}$$

satisfies an asymptotic bound with gain $N \cdot \mathrm{Id}$ and restriction Δ and the origin of the system (259) with $d \equiv 0$ is GAS. With these definitions, consider the control

$$u = \beta(x_2)\,[\alpha(x_1) + d_1] \tag{260}$$

yielding the closed loop system

$$\begin{cases} \dot{x}_1 = Ax_1 + B\alpha(x_1) + B\beta(x_2)d_1 + B(\beta(x_2) - 1)\alpha(x_1) \\ \dot{x}_2 = f(x_2, \beta(x_2)(\alpha(x_1) + d_1)). \end{cases} \tag{261}$$

We will show that the origin of the closed loop is GAS when $d_1 \equiv 0$ and satisfies an asymptotic bound with respect to d_1 otherwise. The signal d_1 may represent a decaying signal from an autonomous system if the original control problem was for the system (254) appended with integrators.

First, the closed loop (261) does not have finite escape times for any signal d_1 defined on $[0, \infty)$ since the x_2 subsystem satisfies an a-$\mathcal{L}_\infty$ stability bound with respect to $\alpha + d_1$ and α and β are globally bounded. Next, it can be shown that the state of the x_2 subsystem satisfies an asymptotic bound with gain $(\gamma_{21}, 0 \cdot \mathrm{Id})$ and with restriction $(\infty, 0)$ where $\gamma_{21}(\infty) \leq \delta$. Also, it can be shown that the state of the x_1 subsystem satisfies an asymptotic bound with respect to x_2 and d_1 with gain $(0 \cdot \mathrm{Id}, 0 \cdot \mathrm{Id})$ with restriction $(\delta, 0)$. The restrictions here are conservative but adequate for our needs. Thus, all of the conditions of the theorem are satisfied and the state of the closed loop system satisfies an asymptotic bound with respect to d_1 with restriction 0. This, in particular, gives us global boundedness and convergence when $d_1 \equiv 0$. Also in this case the LAS property holds since, near the origin, the closed loop system behaves like a cascade.

As a second application, consider a particular system in so-called feedforward form. Consider the GAS control problem for the system

$$\begin{cases} \dot{x}_1 = x_2 + x_3^2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u. \end{cases} \tag{262}$$

We will try the control

$$u = -x_2 - 2x_3 - \lambda \mathrm{sat}(\frac{x_1 + 2x_2 + x_3}{\lambda}) \tag{263}$$

where $0 < \lambda < 0.25$ and $\mathrm{sat}(s) = \mathrm{sgn}(s)\min\{|s|, 1\}$. The Jacobian linearization gives the LAS property. Also, there is no finite escape time since x_2 and x_3 are globally bounded. Now, for global boundedness and convergence of the full state using theorem 51, define $z_1 = x_1 + 2x_2 + x_3$. Then, the state of the (x_2, x_3) subsystem satisfies an a-$\mathcal{L}_\infty$ stability bound with respect to z_1 with gain $\gamma_{21}(s) =$

$\min\{2\lambda, 2s\}$ and also an asymptotic stability bound with this gain and restriction ∞. Also we have that

$$\dot{z}_1 = -\lambda \mathrm{sat}(\frac{z_1}{\lambda}) + x_3^2. \tag{264}$$

So, the state of the z_1 subsystem satisfies an asymptotic bound with respect to x_3 with gain $\gamma_{11}(s) = 2\lambda s$ with restriction $\Delta_{11} = 2\lambda$. Since $\gamma_{21}(\infty) = \Delta_{11}$ and $\gamma_{21}(\gamma_{11})(s)) < s$ for all $s > 0$ (since $4\lambda < 1$), the GAS property follows from theorem 51.

Finally, several other applications of theorem 51, including stabilization of a general class of systems in so-called feedforward form, stabilization with rate saturation and time-delays, and stabilization of mechanical systems like the PV-TOL, the ball and beam, and the inverted pendulum on a cart, are discussed in [99, 97, 98].

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