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Global stabilization by output feedback: examples and counterexamples

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Abstract: We show by means of examples that global complete observability and global stabilizability by state feedback are not sufficient to guarantee global stabilizability by dynamic output feedback. We show that a main obstruction is related to 'unboundedness unobservability'. This is that some unmeasured state components may escape in finite time whereas the measurements remain bounded.

Keywords: Global asymptotic stabilization; dynamic nonlinear feedback; output nonlinear feedback.

1. Introduction and problem statement

These last years have seen very important progress made on the solution of the problem of global asymptotic stabilization by output feedback. Results imposing explicit restrictions on the rate of growth of nonlinearities were known for a long time (see, for example, [8] or [14]). But, with a seminal paper [9], Marino and Tomei have exhibited and characterized a whole class of systems where no growth condition is explicitly needed. This class has been extended to encompass systems for which there exists a globally defined set of coordinates in which, in the single input single output case, the system can be written in the following form:

$$\begin{aligned} \dot{z} &= h(z, x_1), \\ &\vdots \\ \dot{x}_i &= x_{i+1} + f_i(z, x_1), \\ &\vdots \\ \dot{x}_r &= u + f_r(z, x_1), \\ y &= x_1, \end{aligned} \tag{1}$$

where u in $\mathbb{R}$ is the input, y in $\mathbb{R}$ is the output, the x_i's are some of the state components, the remainder being in the vector z in $\mathbb{R}^n$, and the functions f_i's and h are smooth and satisfy

$$f_i(0,0) = 0, \quad h(0,0) = 0. \tag{2}$$

Since [9], many authors have contributed to the study of the output feedback problem for such systems. Kanellakopoulos et al. [6] have given another solution to the problem solved in [9]. Also, extensions to the

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unknown parameter case [11, 4, 5, 7] and to the robust case have been provided [10, 12, 13]. For example, one can now state the following theorem.

Theorem 1 (Praly and Jiang [12]). *If, for the z-subsystem of* (1).

(1) *there exist two functions, β of class $\mathscr{K}\mathscr{L}$ and γ of class $\mathscr{K}$,*[1] *such that, for any initial condition z_0 and any measurable essentially bounded function $x_1(\cdot)$ defined on $[0, +\infty)$, there exists a solution $z(\cdot)$ defined on $[0, +\infty)$ and satisfying, for all t in $[0, +\infty)$,*

$$|z(t)| \leq \beta(|z_0|, t) + \gamma\left(\sup_{\tau\in[0,t]} |x_1(\tau)|\right), \tag{3}$$

(2) *the zero solution of this z-subsystem is locally exponentially stable when x_1 is identically zero, then the zero solution of* (1) *can be globally asymptotically stabilized by a dynamic output feedback.*

But one may wonder if the results, available for the form (1), could be extended to the much less restrictive normal form shown below, characterized by Byrnes and Isidori [1]:

$$\begin{aligned}
\dot{z} &= \mathscr{H}(z, y_1, \ldots, y_r),\\
&\vdots\\
\dot{y}_i &= y_{i+1},\\
&\vdots\\
\dot{y}_r &= \mathscr{F}(z, y_1, \ldots, y_r) + \mathscr{G}(z, y_1, \ldots, y_r)u,\\
y &= y_1.
\end{aligned} \tag{4}$$

The objective of this paper is to show that, in fact, no extension can be done without introducing extra assumptions. Indeed, we shall prove the following results in the next section.

For any integer n strictly larger than 2:

- *System* 1: the zero solution of

$$\dot{z} = z^n + x_1, \qquad \dot{x}_1 = z + u, \qquad y = x_1 \tag{5}$$

cannot be globally asymptotically stabilized by dynamic output feedback. In fact, this system fails to satisfy assumption (1) of Theorem 1.

- *System* 2: the zero solution of

$$\dot{z} = -z + z^n x_1^2, \qquad \dot{x}_1 = z + u, \qquad y = x_1 \tag{6}$$

cannot be globally asymptotically stabilized by a dynamic output feedback. Again, assumption (1) of Theorem 1 is not satisfied although the zero solution of the z-subsystem is globally asymptotically stable when x_1 is held at 0.

- *System* 3: the zero solution of

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_2^n + u, \qquad y = y_1 \tag{7}$$

cannot be globally asymptotically stabilized by a dynamic output feedback. This time, assumption (1) of Theorem 1 is trivially satisfied since there is no z-subsystem. But, for n strictly larger than 2, this system, in the form (4), cannot be written in the form (1).

Although these three systems fail to satisfy the assumptions of Theorem 1 for different reasons, we shall see below that, in fact, they all share the same 'unboundedness unobservability' problem. It is important to

[1] See [3] for a definition.

remark, however, that the zero solution of each of these three systems can be globally stabilized by static state feedback. Also, these systems are written exactly in the form of [2, Theorem 2]. It follows that they are completely observable, i.e., for any C^∞ input $u(t)$, the state initial condition can be reconstructed from the output function $y(\cdot)$.

Definition 1 (*The unboundedness observability property*). A system

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{8}$$

with state x in $\mathbb{R}^n$, input u in $\mathbb{R}^p$ and output y in $\mathbb{R}^m$ is said to have the unboundedness observability property if, for any solution $x(\cdot)$ right maximally defined on $[0, T)$, with T finite, corresponding to a bounded input function in $u(\cdot)$ in $L^\infty([0, T); \mathbb{R})$, we have

$$\limsup_{t \to T} |h(x(t))| = +\infty. \tag{9}$$

In words, this property means that the system cannot have one of its solutions escaping to infinity in finite time without having its input or its output escaping also at infinity at the same time, i.e. the finite escape time phenomenon can be observed. As mentioned above, our three systems do not satisfy this property. In particular, for system 3, we readily see that if u is identically zero and the initial condition $y_2(0)$ is strictly positive, we have the following results.

- For $n \geq 2$ and all t in $[0, 1/y_2(0)^{n-1}(n-1))$,

$$y(t) = y_1(t) = y_1(0) + y_2(0)^{2-n}\frac{1}{n-2}[1 - (1 - y_2(0)^{n-1}(n-1)t)^{(n-2)/(n-1)}], \tag{10}$$

$$y_2(t) = \frac{y_2(0)}{(1 - y_2(0)^{n-1}(n-1)t)^{1/(n-1)}}. \tag{11}$$

Therefore, we have

$$\lim_{\left\{t \to \frac{1}{y_2(0)^{n-1}(n-1)}\right\}} \begin{pmatrix} y(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_1(0) + y_2(0)^{2-n}\dfrac{1}{n-2} \\ +\infty \end{pmatrix}. \tag{12}$$

So the unmeasured state component y_2 escapes to infinity whereas the output remains finite.

- For $n = 2$ and all t in $[0, 1/y_2(0))$,

$$y(t) = y_1(t) = y_1(0) - \log(1 - y_2(0)t), \tag{13}$$

$$y_2(t) = \frac{y_2(0)}{(1 - y_2(0)t)}. \tag{14}$$

Therefore, we have

$$\lim_{\left\{t \to \frac{1}{y_2(0)}\right\}} \begin{pmatrix} y(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} +\infty \\ +\infty \end{pmatrix}. \tag{15}$$

So the unmeasured state component y_2 escapes to infinity with the output.

It follows that system 3 does not have the unboundedness observability property if n is strictly greater than 2. But it does if $n = 2$, although, in this case, the finite escape time phenomenon does exist for y_2. But the key point, as mentioned above, is that it is now seen from the output y. In fact, in this latter case, a globally asymptotically stabilizing output feedback does exist.

Lemma 1. *The zero solution of*

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_2^2 + u, \qquad y = y_1 \tag{16}$$

is globally asymptotically stabilized by the following dynamic output feedback:

$$\begin{aligned} \dot{\chi} &= -(3\chi + 4y)\exp(y), \\ u &= -(5\chi + 7y)\exp(2y). \end{aligned} \tag{17}$$

Proof. Consider the positive-definite and proper function

$$V(y_1, y_2, \chi) = \tfrac{1}{2}(y_1^2 + [\chi + y_1]^2 + [\chi + y_1 - y_2\exp(-y_1)]^2). \tag{18}$$

Its Lie derivative, $\dot{V}$, along the vector field given by (16)–(17) satisfies

$$\dot{V} = -y_2^2\exp(-y_1) - (\chi + y_1)^2\exp(y_1). \tag{19}$$

The conclusion follows readily from LaSalle's theorem. □

2. Main result

The fact that the zero solutions of systems 1–3 cannot be stabilized by a finite-dimensional dynamic output feedback will be proved with the help of two technical lemmas.

The first lemma gives estimates on the solution of the following differential inequality:

$$ax^n \geq \dot{x} \geq bx^n, \tag{20}$$

where a, b and n are strictly positive real numbers.

Lemma 2. *If n is strictly greater than 2, then any solution of* (20), *with strictly positive initial condition x_0, has a finite escape time T satisfying*

$$T \leq \frac{1}{b(n-1)x_0^{n-1}}. \tag{21}$$

Moreover, for all t in $[0, T)$, we have

$$x(t) \leq \left(\frac{1}{a(n-1)(T-t)}\right)^{1/(n-1)}. \tag{22}$$

Finally, for any real number m in $(0, n-1)$ and all t in $[0, T)$, we have

$$\int_0^t [x(s)]^m \,\mathrm{d}s < \left(\frac{1}{a(n-1)}\right)^{m/(n-1)} \frac{n-1}{n-m-1} T^{(n-m-1)/(n-1)}. \tag{23}$$

Proof. We remark that, for all $n > 2$, the differential equation

$$\dot{w} = cw^n, \quad w(0) = w_0, \tag{24}$$

where w_0 and c are strictly positive real numbers, has a unique C^0 solution given by

$$w(t) = \frac{w_0}{(1 - cw_0^{n-1}(n-1)t)^{1/(n-1)}}. \tag{25}$$

Therefore, from standard comparison theorems we find

(i) the inequality on the escape time T, and

(ii) that any solution of the differential inequality (20) can be estimated as

$$x(t) \leq \frac{x(\tau)}{(1 - ax(\tau)^{n-1}(n-1)(t-\tau))^{1/(n-1)}}, \tag{26}$$

where t and τ are in $(0, T)$. The two inequalities (22) and (23) follow. □

The second lemma states that, under some appropriate properties, the following system admits solutions whose component x_2 only escapes within finite time,

$$\dot{x}_1 = f_1(x_1, x_2, z), \qquad \dot{x}_2 = f_2(x_1, x_2, z), \qquad \dot{z} = \psi(x_1, z), \tag{27}$$

where (x_1, x_2) is in $\mathbb{R}^2$, z is in $\mathbb{R}^k$ for some k and f_1, f_2 and ψ are continuous functions.

Lemma 3. *If the functions f_1 and f_2 are such that there exist nonempty open sets $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^k$, and positive real numbers a, b, c, d, n, m such that:*

$$2 < n, \qquad m < n - 1, \qquad b < a, \tag{28}$$

for all x_1 in U, all x_2 strictly greater than d and all z in V, we have

$$|f_1(x_1, x_2, z)| < cx_2^m, \qquad bx_2^n < f_2(x_1, x_2, z) < ax_2^n, \tag{29}$$

then system (27) *admits a solution whose component x_2 only escapes within finite time.*

Proof. Fix initial conditions $x_1(0) = x_{10}$ and $z(0) = z_0$ in U and V, respectively. We will show that for large enough initial condition $x_2(0) = x_{20}$ the system has a finite escape time.

- First, let U_1 and V_1 be open balls of radii δ_1 and δ_2 around x_{10} and z_0, respectively, and such that their closures $\mathrm{Cl}(U_1)$ and $\mathrm{Cl}(V_1)$ are subsets of U and V, respectively. Let s be a strictly positive real number small enough so that, for any continuous function $v: [0, s] \to \mathrm{Cl}(U_1)$, the differential equation

$$\dot{z} = \psi(v(t), z), \quad z(0) = z_0, \tag{30}$$

  admits a solution $z(t; z_0, v)$ which remains in V_1 for all t in $[0, s]$. Continuity of ψ and boundedness of U_1 imply that this is possible.
- Second, let us consider the following two real functions on $\mathbb{R}_+$:

$$T(\chi) = \frac{1}{(b(n-1)\chi)^{n-1}}, \tag{31}$$

$$I(\chi) = \frac{1}{c}\left(\frac{1}{a(n-1)}\right)^{m/(n-1)} \frac{n-1}{n-m-1} T(\chi)^{(n-m-1)/(n-1)}. \tag{32}$$

- Finally, pick any positive real number $x_{20} \geq d$ large enough so that

$$T(x_{20}) < s, \qquad I(x_{20}) < \delta_1. \tag{33}$$

We claim that any solution $(x_1(t), x_2(t), z(t))$ of the system with initial condition (x_{10}, x_{20}, z_0) will be such that

(i) there exists τ in $(0, T(x_{20}))$ such that $x_2(t)$ tends to $+\infty$ as t tends to τ,

(ii) the components $(x_1(t), z(t))$ remain in $U_1 \times V_1$ for all t in $[0, \tau)$.

Indeed, let $(x_1(t), x_2(t), z(t))$ be any of these solutions and let $[0, T)$ be its right maximal domain of definition. Let

$$\sigma = \min\{s, T, \inf\{t \in [0, T): x_1(t) \notin U_1\}\}. \tag{34}$$

By continuity, σ is strictly positive. The definition of s and (29) imply

$$z(t)\in V_1, x_2(t) \geq x_{20} \geq d \quad \forall t\in[0,\sigma). \tag{35}$$

With (29), (21) and (33), we get by contradiction

$$\sigma < T. \tag{36}$$

But then, (29), (23) and (33) imply

$$\sigma = T, \tag{37}$$

which concludes the proof again with the help of (29), (21) and (33). □

This lemma gives the condition under which the 'unboundedness unobservability' probem occurs. Indeed, we see that, under conditions (28) and (29), there exist initial conditions so that the state component x_2 escapes in finite time whereas the other components x_1 and z remain bounded. So if x_2 is not involved in the output function, its escape is not observed. This is exactly what is occurring with systems 1–3 of the previous section. And, in fact, as mentioned above, they are not globally asymptotically stabilizable by a finite-dimensional dynamic output feedback. To see this, let us consider an arbitrary continuous dynamic output feedback controller:

$$\dot{z} = \phi(y,z), \qquad u = \theta(y,z), \tag{38}$$

where z is in $\mathbb{R}^k$ for some k and ϕ, θ are two continuous functions. It is easy to see that all of systems 1–3, in closed-loop with this controller, satisfy the hypotheses of Lemma 3. Therefore, none of them is stabilizable by continuous dynamic output feedback.

3. Conclusion

We have given three examples of systems which are globally observable, globally state-feedback-stabilizable, and even strongly globally minimum-phase for two of them. But none of these is globally asymptotically stabilizable by any continuous dynamic output feedback control law. We explained these negative results by exhibiting a phenomenon that we have called 'unboundedness unobservability', i.e. some unmeasured state components may escape in finite time whereas the measurements remain bounded.

In fact, by following the same idea as in our construction of these counterexamples, we can easily see that a wide class of systems will have this 'unboundedness unobservability'. For example, for the following system in the normal form (4),

$$\begin{aligned}
&\dot{z} = \mathscr{H}(z, y_1, \ldots, y_r),\\
&\vdots\\
&\dot{y}_1 = y_{i+1},\\
&\vdots\\
&\dot{y}_r = y_r^n + \mathscr{F}(z, y_1, \ldots, y_{r-1}) + \mathscr{G}(z, y_1, \ldots, y_{r-1})u,\\
&y = y_1,
\end{aligned} \tag{39}$$

the 'unboundedness observability' and the global asymptotic stabilizability by continuous dynamic output feedback do not hold if

$$n \geq \frac{r}{r-1}. \tag{40}$$

This shows that, for global asymptotic stabilization by output feedback, we cannot go very far beyond linearity for relative degrees $r > 2$.

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