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Small-Gain Theorem for ISS Systems and Applications*

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Abstract. We introduce a concept of input-to-output practical stability (IOpS) which is a natural generalization of input-to-state stability proposed by Sontag. It allows us to establish two important results. The first one states that the general interconnection of two IOpS systems is again an IOpS system if an appropriate composition of the gain functions is smaller than the identity function. The second one shows an example of gain function assignment by feedback. As an illustration of the interest of these results, we address the problem of global asymptotic stabilization via partial-state feedback for linear systems with nonlinear, stable dynamic perturbations and for systems which have a particular disturbed recurrent structure.

Key words. Input-to-state stability, Nonlinear systems, Partial-state feedback, Global stability.

1. Introduction

Studying uncertain dynamical systems is not only practical but a means of addressing the control problem for a large class of nonlinear systems based on a simplified model (see [BCL], [CL], [JP], and the references therein). In this paper we introduce some new design tools which, when combined together, allow us to address the problem of stabilizing systems with intricate structure. In particular we prove that the following uncertain dynamical system can be robustly stabilized by means of partial-state feedback:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(x_1, \dots, x_i, Z_i), & 1 \le i \le n-1, \\ \dot{x}_n = u + f_n(x_1, \dots, x_n, Z_n), \\ \dot{Z}_i = q_i(x_1, \dots, x_i, Z_i), & 1 \le i \le n, \end{cases} \tag{1}$$

where $u \in \mathbb{R}$ is the input, $(x_1, \dots, x_n)^\top \in \mathbb{R}^n$ are measured components of the state vector, $(Z_1, \dots, Z_n)$ are unmeasured components, and the q_i's satisfy the following assumption:

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(H) For each i in $\{1, \dots, n\}$ the system

$$\dot{Z}_i = q_i(x_1, \dots, x_i, Z_i) \tag{2}$$

is *input-to-state stable* with $(x_1, \dots, x_i)$ as input.

The basic concept which is used throughout this paper is the input-to-state stability (ISS) property introduced by Sontag [S2], [S4]. This notion allows us to address dynamic uncertainties in addition to static uncertainties [KKM1], [KKM2], [MT2], [FK], and it allows us to deal with systems which, due to their overcomplicated structure, we prefer to consider as uncertain. In particular we show that partial-state feedback can be designed this way for systems whose dynamics are related to a feedback form. Along the way, we generalize the standard "adding one integrator technique" (see, for instance, [T2] and [S4]). Some different approaches on global stabilization for interconnected systems using the notion of ISS may be found in [T3] and [T4].

In Section 2 we begin with a notion of *input-to-output practical stability* (*IOpS*) which is a natural generalization of ISS. It is worth remarking that the IOpS notion refines the classical input–output L^∞ operator approach by making explicit the role of initial conditions for stability analysis. With this concept, we establish a *generalized small-gain theorem.* This theorem completes a recent and important result of Mareels and Hill [MH] about monotone stability under a nonlinear-type small-gain condition. It is also related to the "topological separation" concept introduced by Safonov in [S1]. Section 2 also contains a result on gain assignment by feedback. A consequence of these main results is that, for some special nonlinear systems, a partial-state feedback exists to render the system ISS with respect to input additive disturbances. This is in the spirit of a theorem for more general systems proved by Sontag [S2] stating that smooth stabilizability implies smooth input-to-state stabilizability by full-state feedback. In Section 4.3 we show how, by being able to propagate the ISS property through integrators, we have at our disposal a tool to design a stabilizing partial-state feedback for system (1). The result generalizes the result of [PJ] on output feedback stabilization. Section 5 is devoted to the proofs of the main theorems.

Facts and Notations

- Throughout this paper positive, negative, increasing, decreasing, smaller, etc., refers to the strict corresponding property.
- $|\cdot|$ stands for the Euclidean norm, and Id denotes the identity function.
- In what follows we are concerned with measurable input functions. There, measurable has to be taken with respect to the Lebesgue measure. Also, as a consequence of dealing with this very general class of input functions, the results have to be considered only for almost every time.
- For any measurable function $u\colon \mathbb{R}_+ \to \mathbb{R}^m$, $\|u\|$ denotes ess. sup. $\{|u(t)|, t \geq 0\}$ and, for any pair of times $0 \leq t_1 \leq t_2$, the truncation $u_{[t_1,t_2]}$ is defined as follows:

$$u_{[t_1,t_2]} = \begin{cases} u(t) & \text{if} \quad t \in [t_1, t_2], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In particular, $u_{[0,T]}$ is the usual truncated function and to simplify the notation we let

$$u_T = u_{[0,T]}. \tag{4}$$

- A function $V: \mathbb{R}^n \to \mathbb{R}_+$ is said to be positive definite if $V(x)$ is positive for all nonzero x and is zero at zero.
- A function $V: \mathbb{R}^n \to \mathbb{R}_+$ is said to be proper if $V(x)$ tends to $+\infty$ as $|x|$ tends to $+\infty$. A proper function is often called radially unbounded in the automatic control literature.
- A function $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class K if it is continuous, increasing, and is zero at zero. It is of class K_∞ if, in addition, it is proper.
- A function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class KL if, for each fixed t, the function $\beta(\cdot, t)$ is of class K and, for each fixed s, the function $\beta(s, \cdot)$ is nonincreasing and tends to zero at infinity.
- For any function γ of class K_∞, its inverse function γ^{-1} is well defined and is again of class K_∞.
- **Completing the squares.** For any a and b in $\mathbb{R}^n$, and for any positive real number ε, we have

$$a^\top b \le \frac{1}{4\varepsilon} a^\top a + \varepsilon b^\top b. \tag{5}$$

- **Weak triangular inequality.** For any function γ of class K, any function ρ of class K_∞ such that $\rho - \mathrm{Id}$ is of class K_∞, and any nonnegative real numbers a and b we have

$$\gamma(a+b) \le \gamma(\rho(a)) + \gamma(\rho \circ (\rho - \mathrm{Id})^{-1}(b)). \tag{6}$$

  This inequality generalizes (12) of [S2] and is established by remarking that, for any function σ of class K_∞, we have

$$\gamma(a+b) \le \max_{0 \le s \le \sigma(a)} \{\gamma(a+s)\} + \max_{0 \le s \le \sigma^{-1}(b)} \{\gamma(s+b)\}. \tag{7}$$

- GAS stands for globally asymptotically stable and LES stands for locally exponentially stable.
- UO (resp. SUO) stands for (resp. strong) unboundedness observability (see Definitions 2.1 and 3.1 below).
- ISS stands for input-to-state stable and IOpS stands for input-to-output practically stable (see Definition 2.2 below).

2. Definitions and Main Results

2.1. *Input-to-Output Practical Stability*

Consider the following control system having x as state, u as input, and y as output:

$$\begin{cases} \dot{x} = f(x,u), & x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\ y = h(x,u), & y \in \mathbb{R}^p, \end{cases} \tag{8}$$

where f and h are smooth functions.

Definition 2.1. System (8) is said to have the *unboundedness observability* (*UO*) property if a function α^0 of class K and a nonnegative constant D^0 exist such that, for each measurable essentially bounded control $u(t)$ on $[0, T)$ with $0 < T \leq +\infty$, the solution $x(t)$ of (8) right maximally defined on $[0, T')$ $(0 < T' \leq T)$ satisfies

$$|x(t)| \leq \alpha^0(|x(0)| + \|(u_t^\top, y_t^\top)^\top\|) + D^0, \qquad \forall t \in [0, T'). \tag{9}$$

Definition 2.2. System (8) is *input-to-output practically stable* (*IOpS*) if a function β of class KL, a function γ of class K, called a (*nonlinear*) *gain from input to output*, and a nonnegative constant d exist such that, for each initial condition $x(0)$, each measurable essentially bounded control $u(\cdot)$ on $[0, \infty)$ and each t in the right maximal interval of definition of the corresponding solution of (8), we have

$$|y(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|) + d. \tag{10}$$

When (10) is satisfied with $d = 0$, system (8) is said to be *input-to-output stable* (*IOS*).

Remark 1.

1. For a multi-input system, it is sometimes very useful to specify one gain function for each different input (see (17) for instance).
2. The notions of UO and IOS introduced here differ slightly from the *strong observability* and IOS properties introduced by Sontag in (38) of [S2] and, respectively, (10) of [S2] in that dependence on the initial condition of the particular state space representation (8) is made explicit. In addition, the offset D^0 has been introduced in the UO property. When $y = x$ in (8), IOpS is called *input-to-state practical stability* (*ISpS*). In this case, if $d = 0$ in (10), then IOpS becomes *input-to-state stability* (*ISS*) as proposed by Sontag in [S2] and [S4].
3. If a system has the UO property and is IOpS, then the system has the "bounded input bounded state (BIBS)" property. If a system has the UO property and is IOS, then, in addition, the system has the "converging input converging output (CICO)" property. If a system has the UO property with $D^0 = 0$ and is IOS then, in addition, it is stable in the sense of Lyapunov when $u = 0$ (see (12)).

Associated with a detectability property, the UO and IOpS properties imply global asymptotic stability (GAS). To state such a result, we recall the following definition:

Definition 2.3. Let $\Phi(t, x, u)$ be the flow of system (8) at time t starting from the point x under the input u. System (8) is said to be *zero-state detectable* if, for all $x \in \mathbb{R}^n$,

$$\{u \equiv 0,\ y(t) = 0,\ \forall t \geq 0\} \quad \Rightarrow \quad \left\{\lim_{t\to\infty} \Phi(t, x, 0) = 0\right\}. \tag{11}$$

Proposition 2.1. *Assume system* (8) *has the UO property with* $D^0 = 0$ *and is IOS. Under this condition, the origin of* (8) *is GAS when* $u = 0$ *if and only if* (8) *is zero-state detectable.*

Proof. Clearly, GAS when $u = 0$ implies zero-state detectability. For the sufficiency, stability follows from combining (9), with $D^0 = 0$ and $u = 0$, and (10), with $d = 0$ and $u = 0$, so that the norm of the solution, right maximally defined on $[0, T')$, is bounded by a class K function of the initial state:

$$|x(t)| \leq \alpha^0(|x(0)| + \beta(|x(0)|, 0)), \qquad \forall t \in [0, T'). \tag{12}$$

By contradiction, $T' = \infty$. For convergence, the IOS property implies that y converges to zero when $u = 0$. Then, since the solution is bounded, it converges toward its ω-limit set which is nonempty and compact (see Theorem I.8.1 of [H]). By continuity this set is contained in the set $\{x: h(x, 0) = 0\}$. Thus, by zero-state detectability the solution converges to zero. ■

Other properties of IOpS systems are given in Section 3.

2.2. Main Results

Consider now the following general interconnected system:

$$\dot{x}_1 = f_1(x_1, y_2, u_1), \qquad y_1 = h_1(x_1, y_2, u_1), \tag{13}$$

$$\dot{x}_2 = f_2(x_2, y_1, u_2), \qquad y_2 = h_2(x_2, y_1, u_2), \tag{14}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^{p_i}$. The functions f_1, f_2, h_1, and h_2 are smooth and a smooth function h exists such that

$$(y_1, y_2) = h(x_1, x_2, u_1, u_2) \tag{15}$$

is the unique solution of

$$\begin{cases} y_1 = h_1(x_1, h_2(x_2, y_1, u_2), u_1), \\ y_2 = h_2(x_2, h_1(x_1, y_2, u_1), u_2). \end{cases} \tag{16}$$

We have:

Theorem 2.1 (Generalized Small-Gain Theorem). *Suppose* (13) *and* (14) *are IOpS with* (y_2, u_1) *(resp.* (y_1, u_2)*) as input,* y_1 *(resp.* y_2*) as output, and* $(\beta_1, (\gamma_1^y, \gamma_1^u), d_1)$ *(resp.* $(\beta_2, (\gamma_2^y, \gamma_2^u), d_2)$*) as triple satisfying* (10), *namely,*

$$\begin{aligned} |y_1(t)| &\leq \beta_1(|x_1(0)|, t) + \gamma_1^y(\|y_{2t}\|) + \gamma_1^u(\|u_1\|) + d_1, \\ |y_2(t)| &\leq \beta_2(|x_2(0)|, t) + \gamma_2^y(\|y_{1t}\|) + \gamma_2^u(\|u_2\|) + d_2. \end{aligned} \tag{17}$$

Also, suppose that (13) *and* (14) *have the UO property with couple* (α_1^0, D_1^0) *(resp.* (α_2^0, D_2^0)*). If two functions* ρ_1 *and* ρ_2 *of class* K_∞ *and a nonnegative real number* s_l *satisfying*

$$\left.\begin{aligned} (\mathrm{Id} + \rho_2) \circ \gamma_2^y \circ (\mathrm{Id} + \rho_1) \circ \gamma_1^y(s) \leq s, \\ (\mathrm{Id} + \rho_1) \circ \gamma_1^y \circ (\mathrm{Id} + \rho_2) \circ \gamma_2^y(s) \leq s, \end{aligned}\right\} \quad \forall s \geq s_l, \tag{18}$$

exist, then system (13)–(14) *with* $u = (u_1, u_2)$ *as input,* $y = (y_1, y_2)$ *as output, and* $x = (x_1, x_2)$ *as state is IOpS and has the UO property (is IOS and has the UO property with* $D^0 = 0$ *when* $s_l = d_i = D_i^0 = 0$ $(i = 1, 2)$*).*

More specifically, for each pair of class K_∞ functions (r_3, ρ_3), a function β of class KL and a nonnegative constant d (equal to zero when $s_l = d_i = D_i^0 = 0\,(i = 1, 2)$) exist such that system (13)–(14) is IOpS with the triple $(\beta, r_1 + r_2 + r_3, d)$ where

$$\begin{cases} r_1(s) = (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)^2 \circ [\gamma_1^u + \gamma_1^y \circ (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)^2 \circ \gamma_2^u](s), \\ r_2(s) = (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)^2 \circ [\gamma_2^u + \gamma_2^y \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)^2 \circ \gamma_1^u](s). \end{cases} \tag{19}$$

Remark 2.

1. The two inequalities (18) are equivalent. Both are written here for ease of future notation.
2. Condition (18) has been introduced by Mareels and Hill in [MH] to state an input–output stability result in the operator setting without making the role of initial conditions explicit. This condition is a nonlinear version of the classical small-gain condition (see, for intance, [DV]). Sufficient conditions to check condition (18) are given in [MH]. Our task here was to complete the result of [MH] in order to take into account the effects of the initial conditions and to express the gain function γ of the closed-loop system in terms of the gains of the two subsystems. Our result can also be used to conclude asymptotic stability for the internal variables under the conditions of Corollaries 2.1 and 2.2.
3. Theorem 2.1 deals with global practical stability. Its complement, local asymptotic stability, holds (as in [TP]) when $d_i = D_i^0 = 0$ and $\forall s \geq s_l$ is replaced by $\forall s \leq s_l$ in (18).
4. The IOpS properties (17) and the small gain condition (18) with $s_l = 0$ imply that the topological separation condition of Theorem 2.1 of [S1] holds. Indeed, to each t in $\mathbb{R}_+$ and each output pair (y_1, y_2), we can associate the real number

$$d_t(y_1, y_2) = \|y_{2t}\| - \gamma_2^y(\|y_{1t}\|). \tag{20}$$

Then (17) implies readily that (2.3.2) of [S1] holds with the symbol v representing $x_2(0)$, d_2, and u_2. Also (17) ansd (18) imply that (2.3.1) of [S1] holds for some function ϕ_1 of class K_∞ and with the symbol u representing $x_1(0)$, d_1, and u_1.

Corollary 2.1. *Under the conditions of Theorem* 2.1, *if* $s_l = d_i = D_i^0 = 0$ $(i = 1, 2)$ *and systems* (13) *and* (14) *are zero-state detectable, then system* (13)–(14) *is GAS when* $u = 0$.

Proof. The result follows readily from Theorem 2.1 and Proposition 2.1 after it is recognized that, if (13) and (14) are zero-state detectable, the interconnection (13)–(14) is zero-state detectable. ■

Remark 3. When establishing GAS results using Corollary 2.1 we, in certain instances, assume that each subsystem is ISS (see Remark 1.2) since this is sufficient to guarantee that each subsystem has the UO property and is zero-state detectable. See Proposition 3.1 and Corollary 3.1 for another motivation of the ISS assumption.

As stated in Remark 1.2, IOpS (resp. IOS) is ISpS (resp. ISS) when the state is seen as an output. In this case the UO property with $D^0 = 0$ is obviously satisfied. The following corollary is a particular case of Theorem 2.1.

Corollary 2.2. *Consider system* (13)–(14) *with* $y_1 = x_1$ *and* $y_2 = x_2$, *i.e.*,

$$\dot{x}_1 = f_1(x_1, y_2, u_1), \qquad y_1 = x_1, \tag{21}$$

$$\dot{x}_2 = f_2(x_2, y_1, u_2), \qquad y_2 = x_2. \tag{22}$$

Assume that both the x_1 *and* x_2 *subsystems are ISpS (resp. ISS) with* (y_2, u_1) *and* (y_1, u_2) *considered as inputs, i.e.,* (17) *holds. If, in addition, the small-gain condition* (18) *is satisfied, then the complete system* (21)–(22) *is ISpS (resp. ISS when* s_l *in* (18) *is equal to zero) with* (u_1, u_2) *as input.*

Another interesting result relying upon the notion of IOpS is the following gain assignment theorem:

Theorem 2.2 (Gain Assignment). *Consider the control system*

$$\begin{cases} \dot{\zeta} = A\zeta + B(H\xi + \omega_0), \\ \dot{\xi} = F\xi + Gu + \omega, \end{cases} \tag{23}$$

with $u \in \mathbb{R}$ *as input,* $\zeta \in \mathbb{R}^l$, $\xi \in \mathbb{R}^n$ *as components of the state,* $(\omega_0, \omega) \in \mathbb{R} \times \mathbb{R}^n$ *as perturbations, and* ζ *as output. Assume* (A, B) *is stabilizable,* (F, G) *is controllable,* (F, H) *is observable, and* (H, F, G) *has maximal relative degree. Under these conditions, for any function* γ *of class* K_∞, *a smooth function* $u_n(\zeta, \xi)$, *with* $u_n(0, 0) = 0$, *exists such that system* (23) *in closed loop with* $u = u_n(\zeta, \xi) + v$ *is:*

1. *ISS with* (ω_0, ω, v) *as input.*
2. *IOpS with* (ω_0, ω, v) *as input,* ζ *as output, and the function* γ *as gain.*

Moreover, if the inverse function γ^{-1} *of* γ *is linearly bounded on a neighborhood of* 0, *the closed-loop system* (23) *can be rendered IOS with* (ω_0, ω, v) *as input,* ζ *as output, and the function* γ *as gain.*

Remark 4.

1. There is no contradiction between the ISS and IOpS properties. The "practical" in the latter means only that, in general, γ is actually assigned only outside a neighborhood of 0.
2. The first point of Theorem 2.2 guarantees that the closed-loop system (23) with (ω_0, ω, v) as input and ζ as output has the UO property with $D^0 = 0$ and is zero-state detectable. In fact, it has the stronger SUO property (with $d^0 = 0$) of Definition 3.1 below.
3. The motivation for assigning a gain function with an inverse that is linearly bounded on a neighborhood of 0 comes from Theorem 2.1 together with Lemma A.2.

The proofs of Theorems 2.1 and 2.2 are given in Section 5. To illustrate the interest of these two theorems, let us consider the following single-input system:

$$\begin{cases} \dot{x} = f(x,z) + u, \\ \dot{z} = q(x,z), \end{cases} \tag{24}$$

where (u, x, z) is in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^p$, and f and q are smooth functions. When $\dot{z} = q(x, z)$ is ISS with x as input the whole system (24) is made ISS with v as input by a feedback law such as

$$u := -x - f(x, z) + v. \tag{25}$$

This follows, for example, from the first part of Proposition 3.2 which is a special case of Theorem 2.1. However, making system (24) ISS with a feedback law as

$$u = \vartheta(x) + v \tag{26}$$

is still an open issue. Nevertheless, the next corollary shows that system (24) can be made ISpS with a partial-state feedback control $\vartheta(x)$.

Corollary 2.3.

1. *If in* (24) *the z-subsystem is ISpS with x as input, then we can find a smooth partial-state feedback* $\vartheta(x)$ *which is zero at zero and such that, with*

$$u = \vartheta(x) + v, \tag{27}$$

system (24) *is ISpS with v as input.*
2. *If the z-subsystem, with x as input, is ISS and, with* $f(x, z)$ *as output, is IOS with a gain function linearly bounded on a neighborhood of* 0, *then, with* (27), *system* (24) *becomes ISS with v as input.*
3. *If the z-subsystem, with x as input, is ISS,* $f(0, 0) = 0$, *and the matrix* $(\partial q/\partial z)(0, 0)$ *is asymptotically stable, then* $\vartheta(x)$ *in closed loop with* (24) *gives GAS and LES.*

Remark 5. The conditions of point 3 of this corollary are sufficient, but not necessary, to give the conditions of point 2. See Lemma A.2.

The proof of this corollary is given later. This result extends to the partial-state feedback case or dynamic uncertain case the "adding one integrator technique" (compare with Theorem 4 of [T2]). In Section 4.3 we see that Corollary 2.3 can be used as one of the tools to design a stabilizing partial-state feedback for system (1).

3. Further Facts About the IOpS Property

The main purpose of this section is to establish other properties of IOpS systems.

We first point out that the notions of IOpS (resp. IOS) and ISpS (resp. ISS) are strongly related. Indeed, consider again system (8) where f and h are smooth functions.

Definition 3.1. System (8) is said to have the *strong unboundedness observability* (*SUO*) property if a function β^0 of class KL, a function γ^0 of class K, and a nonnegative constant d^0 exist such that, for each measurable control $u(t)$ defined on $[0, T)$ with $0 < T \leq \infty$, the solution $x(t)$ of (8) right maximally defined on $[0, T')$ $(0 < T' \leq T)$ satisfies

$$|x(t)| \leq \beta^0(|x(0)|, t) + \gamma^0(\|(u_t^\top, y_t^\top)^\top\|) + d^0, \qquad \forall t \in [0, T'). \tag{28}$$

Remark 6. The SUO property implies the UO property and, when $d^0 = 0$, the zero-state detectability property.

We have, similar to Propositions 3.2 and 7.1 of [S2], and in the spirit of Proposition 2.1:

Proposition 3.1. *If the x-system is ISpS (resp. ISS), then system* (8) *with y as output has the SUO property (resp. the SUO property with* $d^0 = 0$*) and is IOpS (resp. IOS if, in addition,* $h(0, 0) = 0$*). Conversely, if system* (8) *is IOpS (resp. IOS) and has the SUO property, then the x-system is ISpS (resp. ISS if, in addition,* $d^0 = 0$*).*

Proof. With the help of (6), the first assertion is directly proved just by remarking that two functions α_x and α_u of class K exist such that, for all (x, u),

$$|h(x, u)| \leq |h(0, 0)| + \alpha_x(|x|) + \alpha_u(|u|). \tag{29}$$

For instance, α_x and α_u in (29) may be taken as

$$\alpha_x(s) = s + \max_{|u| \leq |x| \leq s} |h(x, u) - h(0, 0)|, \tag{30}$$

$$\alpha_u(s) = s + \max_{|x| \leq |u| \leq s} |h(x, u) - h(0, 0)|. \tag{31}$$

Conversely, if system (8) is IOpS and has the SUO property, then, by a contradiction argument from inequalities like (32) and (34) below, we can show that, for every measurable essentially bounded input u on $[0, \infty)$, y and x exist and are bounded on $[0, \infty)$. Moreover, two functions β and β^0 of class KL, two functions γ and γ^0 of class K, and two nonnegative constants d and d^0 exist such that, using time invariance and causality, for all $t \geq t_0 \geq 0$,

$$|y(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|) + d, \tag{32}$$

$$|x(t)| \leq \beta^0(|x(t_0)|, t - t_0) + \gamma^0(\|(u^\top, y_{[t_0,t]}^\top)^\top\|) + d^0. \tag{33}$$

By substituting (32) with $t_0 = 0$ into (33) with $t_0 = t/2$, we obtain

$$|x(t)| \leq \beta^0\left(\left|x\left(\frac{t}{2}\right)\right|, \frac{t}{2}\right) + \gamma^0\left(\|u\| + \beta\left(|x(0)|, \frac{t}{2}\right) + \gamma(\|u\|) + d\right) + d^0. \tag{34}$$

Moreover, from (32) and (33), we have, for any function ρ of class K_∞ and all $t \geq 0$,

$$\begin{aligned} \left|x\left(\frac{t}{2}\right)\right| &\leq [\beta^0(|x(0)|, 0) + \gamma^0 \circ (\mathrm{Id} + \rho^{-1})^2(\beta(|x(0)|, 0))] \\ &\quad + \gamma^0 \circ (\mathrm{Id} + \rho) \circ (\mathrm{Id} + \gamma)(\|u\|) + \gamma^0 \circ (\mathrm{Id} + \rho^{-1}) \circ (\mathrm{Id} + \rho)(d) + d^0 \quad (35) \\ &:= s_\infty. \quad (36) \end{aligned}$$

We can conclude by replacing $|x(t/2)|$ in (34) by the bound s_∞ given in (35). ■

Remark 7. We remark that, by following the same lines as in the proof of Theorem 2.1 (see (98)–(109)), from (35) and (34), we can obtain the following more precise statement:

For any pair of class K_∞-functions (r, ρ), a function β_x of class KL and a nonnegative constant d_x ($d_x = 0$ when $d = d^0 = 0$) exist such that

$$|x(t)| \leq \beta_x(|x(0)|, t) + (r + \gamma^0 \circ (\mathrm{Id} + \rho) \circ (\mathrm{Id} + \gamma))(\|u\|) + d_x. \tag{37}$$

Corollary 3.1. *Under the conditions of Theorem* 2.1, *if systems* (13) *and* (14) *have the SUO property, system* (13)–(14) *is ISpS* (*resp. ISS if* $s_i = d_i^0 = d_i = 0$ $(i = 1, 2)$).

Proof. The result follows readily from Theorem 2.1 and Proposition 3.1 after it is recognized that if (13) and (14) have the SUO property (resp. the SUO property with $d_i^0 = 0$), then the interconnection (13)–(14) has the SUO property (resp. the SUO property with $d^0 = 0$). ■

In Theorem 2.1 we gave a small-gain condition under which the interconnected system made of two IOpS systems is again IOpS. In some cases this condition is trivially checked. Precisely, when system (13)–(14) takes the following form,

$$\begin{cases} \dot{x} = f(x, z, u), \\ \dot{z} = g(z, u), \end{cases} \tag{38}$$

as a straightforward consequence of our previous results, we have:

Proposition 3.2. *If the x-subsystem of* (38) *is ISpS* (*resp. ISS*) *with* (z, u) *as input and the z-subsystem of* (38) *is ISpS* (*resp. ISS*) *with u as input, then system* (38) *is ISpS* (*resp. ISS*) *with u as input.*

This proposition shows that the ISpS property is closed under composition. This fact has already been noticed by Sontag [S2, Proposition 7.2] for input–output stability.

We finally note the following useful fact:

Fact 1. *If the system*

$$\dot{x} = f(x, v), \qquad v \in \mathbb{R}^m, \tag{39}$$

is ISpS with v as input, and if $\varphi\colon \mathbb{R}^m \to \mathbb{R}^m$ *is a continuous function, then* (39) *is also ISpS with u as input when*

$$v = \varphi(u). \tag{40}$$

Proof. With $y = x$, let (β, γ, d) be the triple given by the ISpS property (see Remark 1.3) of (10). Let

$$d_0 = |\varphi(0)|. \tag{41}$$

The system

$$\dot{x} = f(x, \varphi(u)) \tag{42}$$

is ISpS with u as input and with $(\beta, \gamma \circ 2\gamma_0, d + \gamma(2d_0))$ as triple satisfying (10) where γ_0 is the function of class K defined as

$$\gamma_0(s) = \max_{|u| \leq s} \{|\varphi(u) - \varphi(0)|\} + s, \qquad \forall s \geq 0. \quad \blacksquare \tag{43}$$

4. Applications

4.1. A Detour from the Center Manifold Reduction Theorem

Consider the following system:

$$\begin{cases} \dot{z} = q(z, \zeta), \\ \dot{\zeta} = f(\zeta) + \omega(z, \zeta), \end{cases} \tag{44}$$

with $(z, \zeta) \in \mathbb{R}^p \times \mathbb{R}^n$ as state and $\omega \in \mathbb{R}^n$ as coupling nonlinearity. Assume:

1. The vector field f is homogeneous with degree r and $\zeta = 0$ is an asymptotically stable equilibrium point of $\dot{\zeta} = f(\zeta)$.
2. The z-subsystem with ζ as input and $\omega(z, \zeta)$ as output has the SUO property with $d_z^0 = 0$ in (28) and is IOS with gain $\gamma_z(s) \leq \mu |s|^r$ for some nonnegative real number μ.

Proposition 4.1. *Under these conditions and if μ is sufficiently small, the zero solution of* (44) *is GAS.*

This result generalizes the lemma on p. 442 of [I] or Lemma 4.3 of [BI] where the local counterpart of this result is proved by applying the center manifold reduction theorem which imposes $f(\zeta) = F\zeta$ with F an asymptotically stable matrix. System (44) has been treated in a different way in Section 4 of [JP].

Proof. From [R] for example, for any $k > 1$, a homogeneous C^1 function V and four positive real numbers c_1 to c_4 exist so that, for all ζ,

$$c_1 |\zeta|^k \leq V(\zeta) \leq c_2 |\zeta|^k, \qquad \left| \frac{\partial V}{\partial \zeta}(\zeta) \right| \leq c_3 |\zeta|^{k-1}, \tag{45}$$

$$\frac{\partial V}{\partial \zeta}(\zeta) f(\zeta) \leq -c_4 |\zeta|^{k+r-1}. \tag{46}$$

Now, for all measurable essentially bounded $\omega(t)$ defined on $[0, +\infty)$ and any initial condition $\zeta(0)$, let $\zeta(t)$ be the solution of the ζ-subsystem right-maximally defined on $[0, T)$. Along this solution, we get

$$\dot{V} \leq -c_4 |\zeta|^{k+r-1} + c_3 |\zeta|^{k-1} |\omega| \tag{47}$$

$$\leq -\tfrac{1}{2} c_4 |\zeta|^{k+r-1} - |\zeta|^{k-1} (\tfrac{1}{2} c_4 |\zeta|^r - c_3 |\omega|). \tag{48}$$

It follows from [S2] that the ζ-subsystem with ω as input is ISS with gain

$$\gamma_\zeta(s) = \left(\frac{c_2}{c_1}\right)^{1/k} \frac{2c_3}{c_4} |s|^{1/r}. \tag{49}$$

Our conclusion follows readily from Corollary 3.1. ■

4.2. *Linear Systems with Nonlinear, Stable Dynamic Perturbations*

Consider the following system:

$$\begin{cases} \dot{z} = q(z, \zeta), \\ \dot{\zeta} = A\zeta + B(H\xi + \omega_0(z, \zeta)), \\ \dot{\xi} = F\xi + Gu + \omega(z, \zeta), \end{cases} \tag{50}$$

with $u \in \mathbb{R}$ as input, $(z, \zeta, \xi) \in \mathbb{R}^p \times \mathbb{R}^l \times \mathbb{R}^n$ as state, and $(\omega_0, \omega) \in \mathbb{R} \times \mathbb{R}^n$ as coupling nonlinearities. Assume:

1. (A, B) is stabilizable, (F, G) is controllable, (F, H) is observable, and (H, F, G) has maximal relative degree.
2. The z-subsystem with ζ as input and (ω_0, ω) as output has the SUO property with a $d_z^0 = 0$ in (28) and is IOS with a gain function γ_z linearly bounded on a neighborhood of 0.

Proposition 4.2. *Under these conditions, we can design a smooth partial-state global asymptotic stabilizer $u(\zeta, \xi)$ for system* (50) *such that system* (50) *with $u = u(\zeta, \xi) + v$ is ISS with respect to v.*

This proposition belongs to the class of results known for these so-called partially linear composite systems studied, for example, in [SK], [SKS], [S5], [T1], and [LS]. As proved in [S5], when $l > 1$, extra assumptions must be imposed on the z-subsystem to guarantee controllability to the origin even when (A, B) is controllable and the coupling terms (ω_0, ω) are not present (see also Theorem 3 of [SKS]). These extra assumptions are in place to guarantee that the state z remains bounded while the state ζ converges to zero, as in [S3]. For example, growth conditions on q may be imposed [SKS, Proposition 5], [SK, Theorems 6.2 and 6.4]. Here, to address the coupling terms, we impose the SUO and IOS properties on the z-subsystem with (ω_0, ω) as outputs. According to Corollary 2.1, this could be relaxed to UO + IOS + zero-state detectable if only GAS is desired.

Proof. From Corollary 3.1, the result holds if we can find a control law $u(\zeta, \xi)$ which makes the (ζ, ξ)-subsystem, with (ω_0, ω) as input and ζ as output, IOS, with gain function $(2\gamma_z(2s))^{-1}$ and to have the SUO property with a $d_{(\zeta,\xi)}^0 = 0$ in (28). However, such a fact is proved under the assumptions of Proposition 4.2 in Theorem 2.2. ■

4.3. *Pure Feedback Systems with Dynamic Uncertainties*

Let us now consider the single-input system mentioned in the introduction:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(X_i, Z_i), & 1 \le i \le n-1, \\ \dot{x}_n = u + f_n(X_n, Z_n), \\ \dot{Z}_i = q_i(X_i, Z_i), & 1 \le i \le n, \end{cases} \tag{51}$$

where, for each i in $\{1, \dots, n\}$, the vector X_i in $\mathbb{R}^i$ is defined as

$$X_i = (x_1, \dots, x_i) \tag{52}$$

and is part of the measured system state components, Z_i in $\mathbb{R}^{m_i}$ is part of the remaining unmeasured state components, u in $\mathbb{R}$ is the input, and the f_i's, q_i's are smooth functions.

Our objective is to design a control law u, involving the components $(x_1, \dots, x_n)$ only and rendering any trajectory of the closed-loop system (51) globally bounded and if possible to guarantee global asymptotic stability.

We make the following assumption about the unmeasured dynamics of system (51):

(H1) For each i in $\{1, \dots, n\}$, the system

$$\dot{Z}_i = q_i(X_i, Z_i) \tag{53}$$

is ISpS with X_i as input.

For proving not only boundedness but also asymptotic stability of an equilibrium point, we need the following extra assumption:

(H2) For each i in $\{1, \dots, n\}$, $(\partial q_i/\partial Z_i)(0, 0)$ is an asymptotically stable matrix, we have

$$f_i(0, 0) = 0, \tag{54}$$

and the system

$$\dot{Z}_i = q_i(X_i, Z_i) \tag{55}$$

is ISS with X_i as input.

This type of system has been extensively studied by many researchers with different viewpoints including state feedback stabilization, or (dynamic) output feedback stabilization (see [KKM2], [MT1], and the references therein). In the absence of the dynamic uncertainties characterized here by z, results on the global stabilization of (51) are available in [KKM1], [KKM2], [MT2], and [FK].

To solve our problem the idea is to use, recursively, Corollary 2.3, Proposition 3.2, and Fact 1 established in the previous sections as three basic tools.

Step 1. Consider first the subsystem of (51):

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1, Z_1), \\ \dot{Z}_1 = q_1(x_1, Z_1). \end{cases} \tag{56}$$

By applying Corollary 2.3 to system (56), we get a smooth function $\vartheta_1(x_1)$ which is zero at zero and such that, with

$$x_2 = \vartheta_1(x_1) + x_2^*, \tag{57}$$

system (56) is ISpS with x_2^* as input. Moreover, if (H2) holds, this system is also ISS with x_2^* as input and LES when x_2^* is zero. We denote

$$x_1^* := x_1, \qquad Z_1^* := Z_1, \qquad \tilde{q}_1 := q_1. \tag{58}$$

Step i $(2 \leq i < n)$. Assume that we have designed a smooth function ϑ_{i-1} so that the system

$$\begin{cases} \dot{x}_{i-1}^* = x_i + \tilde{f}_{i-1}(x_{i-1}^*, Z_{i-1}^*), \\ \dot{Z}_{i-1}^* = \tilde{q}_{i-1}(x_{i-1}^*, Z_{i-1}^*) \end{cases} \tag{59}$$

is ISpS with x_i^* as input if

$$x_i = x_i^* + \vartheta_{i-1}(x_{i-1}^*). \tag{60}$$

Here, by induction, $(x_1^*, x_2^*, \ldots, x_{i-2}^*)$ are part of the components of Z_{i-1}^*.

Consider the new variable

$$x_i^* = x_i - \vartheta(x_{i-1}^*) \tag{61}$$

and let

$$Z_i^* = (Z_i, (x_{i-1}^*, Z_{i-1}^{*\top})^\top)^\top. \tag{62}$$

We can rewrite the system

$$\begin{cases} \dot{Z}_i = q_i(x_1, \ldots, x_i, Z_i), \\ \dot{x}_{i-1}^* = x_i + \tilde{f}_{i-1}(x_{i-1}^*, Z_{i-1}^*), \\ \dot{Z}_{i-1}^* = \tilde{q}_{i-1}(x_{i-1}^*, Z_{i-1}^*) \end{cases} \tag{63}$$

as

$$\dot{Z}_i^* = \tilde{q}_i(x_i^*, Z_i^*) := \begin{bmatrix} q_i(x_1^*, \vartheta_1(x_1^*) + x_2^*, \ldots, \vartheta_{i-1}(x_{i-1}^*) + x_i^*, Z_i) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \begin{pmatrix} x_i^* + \vartheta_{i-1}(x_{i-1}^*) + \tilde{f}_{i-1}(x_{i-1}^*, Z_{i-1}^*) \\ \tilde{q}_{i-1}(x_{i-1}^*, Z_{i-1}^*) \end{pmatrix} \end{bmatrix}. \tag{64}$$

Since the z_i-subsystem of (51) is ISpS with $(x_1, \ldots, x_i)$ as input, and the map which transforms $(x_1, \ldots, x_i)$ into $(x_1^*, \ldots, x_i^*)$ is a global diffeomorphism preserving the origin, a direct application of Fact 1 shows that the system

$$\dot{Z}_i = q_i(x_1^*, \vartheta_1(x_1^*) + x_2^*, \ldots, \vartheta_{i-1}(x_{i-1}^*) + x_i^*, Z_i) \tag{65}$$

is ISpS with $(x_1^*, \ldots, x_i^*)$ as input. So, by applying Proposition 3.2, we see that system (64) is ISpS with x_i^* as input. The ISS and LES properties also hold if (H2) is satisfied.

Also, our change of variable gives

$$\dot{x}_i^* = x_{i+1} + f_i(X_i, Z_i) - \nabla\vartheta_{i-1}(x_{i-1}^*)(x_i^* + \tilde{f}_{i-1}(x_{i-1}^*, Z_{i-1}^*)), \tag{66}$$

where $\nabla\vartheta_{i-1}$ stands for the gradient of ϑ_{i-1}, or, in a form compatible with Corollary 2.3,

$$\dot{x}_i^* = x_{i+1} + \tilde{f}_i(x_i^*, Z_i^*), \tag{67}$$

where $\tilde{f}_i$ is given as

$$\begin{aligned}\tilde{f}_i(x_i^*, Z_i^*) = f_i(x_1^*, \vartheta_1(x_1^*) + x_2^*, \ldots, \vartheta_{i-1}(x_{i-1}^*) + x_i^*, Z_i)\\ - \nabla\vartheta_{i-1}(x_{i-1}^*)(x_i^* + \tilde{f}_{i-1}(x_{i-1}^*, Z_{i-1}^*)).\end{aligned} \tag{68}$$

Now we apply Corollary 2.3 to system (67) and we get a function $\vartheta_i(x_i^*)$ which is zero at zero and such that, with

$$x_{i+1} = \vartheta_i(x_i^*) + x_{i+1}^*, \tag{69}$$

system (67)–(64) is ISpS with x_{i+1}^* as input. It is also ISS and LES if (H2) holds.

Step n. As above, we get a control law $\vartheta_n(x_n^*)$ such that

$$u = \vartheta_n(x_n^*) + v \tag{70}$$

makes the system, derived from the previous $n - 1$ steps,

$$\begin{cases}\dot{x}_n^* = u + \tilde{f}_n(x_n^*, Z_n^*),\\ \dot{Z}_n^* = \tilde{q}_n(x_n^*, Z_n^*)\end{cases} \tag{71}$$

ISpS with v as input. Therefore, the solutions $(x_n^*(t), Z_n^*(t))$ of (71) with

$$u = \vartheta_n(x_n^*) \tag{72}$$

are uniformly ultimately bounded. Since the map which transforms (x_n^*, Z_n^*) into $(x_n, Z_n, \ldots, x_1, Z_1)$ is a global diffeomorphism and preserves the origin, this implies that, for any initial condition, the solutions (x, z) of the closed-loop system (51) are bounded. The ultimate bound for the transformed coordinates (x_n^*, Z_n^*) depends mainly on the d_i's associated with the Z_i's subsystem. However, for the original coordinates $(x_n, Z_n, \ldots, x_1, Z_1)$, their ultimate bound depends also, and in a very intricate manner, on the controller. We even have the possibility that, by trying to push the ultimate bound for (x_n^*, Z_n^*) to zero, the ultimate bound for $(x_n, Z_n, \ldots, x_1, Z_1)$ will go to infinity. This is known as the peaking phenomenon [SK]. However, if assumption (H2) holds, system (71) with

$$u = \vartheta_n(x_n^*) + v \tag{73}$$

is ISS with v as input and LES when v is zero.

We summarize with the following result:

Proposition 4.3. *Under assumption* (H1), *we can design a smooth partial-state feedback $u(x_1, \ldots, x_n)$ such that, for any initial conditions, all the trajectories of system*

(51) *in closed loop with*

$$u = u(x_1, \dots, x_n) \tag{74}$$

are bounded. Moreover, if assumption (H2) *holds, we can design a global asymptotic partial-state stabilizer* $u(x_1, \dots, x_n)$ *for system* (51).

5. Proofs

5.1. *Proof of Theorem 2.1*

A first fact to be noticed is that (18) implies the existence of a nonnegative real number d_3 such that

$$\left.\begin{aligned} \gamma_2^y \circ (\mathrm{Id} + \rho_1) \circ \gamma_1^y(s) \le (\mathrm{Id} + \rho_2)^{-1}(s) + d_3, \\ \gamma_1^y \circ (\mathrm{Id} + \rho_2) \circ \gamma_2^y(s) \le (\mathrm{Id} + \rho_1)^{-1}(s) + d_3, \end{aligned}\right\} \quad \forall s \ge 0, \tag{75}$$

with $d_3 = 0$ when $s_l = 0$.

Step 1: Existence and Boundedness of Solutions on $[0, \infty)$. For any pair of measurable essentially bounded controls $(u_1(t), u_2(t))$ defined on $[0, +\infty)$, for any initial condition $x(0)$, by hypothesis of smoothness, a unique solution $x(t)$ of (13)–(14) right maximally defined on $[0, T)$ with $T > 0$ possibly infinite exists. Also, since (13) and (14) are IOpS, for any τ in $[0, T)$ and any

$$0 \le t_{10} \le t_{20} \le t_{11} \le t_{21} < T - \tau, \tag{76}$$

we have, using time invariance and causality,

$$|y_1(t_{11} + \tau)| \le \beta_1(|x_1(t_{10} + \tau)|, t_{11} - t_{10}) + \gamma_1^y(\|y_{2[t_{10}+\tau, t_{11}+\tau]}\|) + \gamma_1^u(\|u_1\|) + d_1, \tag{77}$$

$$|y_2(t_{21} + \tau)| \le \beta_2(|x_2(t_{20} + \tau)|, t_{21} - t_{20}) + \gamma_2^y(\|y_{1[t_{20}+\tau, t_{21}+\tau]}\|) + \gamma_2^u(\|u_2\|) + d_2. \tag{78}$$

For ease of notation, set $\gamma_i = \gamma_i^y$ and $v_i = \gamma_i^u(\|u_i\|)$. Then pick an arbitrary T_0 in $[0, T)$ and let

$$\tau = t_{10} = t_{20} = 0, \qquad t_{21} = T_0, \qquad t_{11} \in [0, T_0]. \tag{79}$$

By applying (6) and using (75), we get successively

$$\|y_{2T_0}\| \le \beta_2(|x_2(0)|, 0) + \gamma_2(\beta_1(|x_1(0)|, 0) + \gamma_1(\|y_{2T_0}\|) + v_1 + d_1) + v_2 + d_2 \tag{80}$$

$$\le \beta_2(|x_2(0)|, 0) + \gamma_2 \circ (\mathrm{Id} + \rho_1) \circ \gamma_1(\|y_{2T_0}\|) + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1})(\beta_1(|x_1(0)|, 0) + v_1 + d_1) + v_2 + d_2 \tag{81}$$

$$\le \beta_2(|x_2(0)|, 0) + (\mathrm{Id} + \rho_2)^{-1}(\|y_{2T_0}\|) + d_3 + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1})(\beta_1(|x_1(0)|, 0) + v_1 + d_1) + v_2 + d_2, \tag{82}$$

$$\le (\mathrm{Id} + \rho_2^{-1})(\beta_2(|x_2(0)|, 0) + d_3 + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1})(\beta_1(|x_1(0)|, 0) + v_1 + d_1) + v_2 + d_2). \tag{83}$$

Since T_0 is arbitrary in $[0, T)$ and the right-hand side of (83) is independent of T_0, $y_2(t)$ is bounded on $[0, T)$. By symmetry, the same argument shows that $y_1(t)$ is bounded on $[0, T)$. Since the x_1-subsystem and x_2-subsystem satisfy the UO property, we conclude that $x_1(t)$ and $x_2(t)$ are bounded on $[0, T)$. It follows by contradiction that $T = +\infty$.

Step 2: The IOpS Property. Continuing from (83), we can establish bounds on the outputs in the following manner. From (6), for any function ρ_3 of class K_∞, we have

$$\begin{aligned}|y_2(t)| &\le (\mathrm{Id} + \rho_2^{-1})(\beta_2(|x_2(0)|, 0) + d_3 + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3^{-1})(\beta_1(|x_1(0)|, 0)) \\ &\qquad + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)(v_1 + d_1) + v_2 + d_2) \qquad (84)\\ &\le (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3^{-1})(\beta_2(|x_2(0)|, 0) + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3^{-1})(\beta_1(|x_1(0)|, 0))) + (\mathrm{Id} + \rho_2^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3)(d_3 + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3)(v_1 + d_1) + v_2 + d_2). \qquad (85)\end{aligned}$$

So, by symmetry, we have established

$$|y_1(t)| \le \delta_1(|x(0)|) + \Delta_1, \qquad |y_2(t)| \le \delta_2(|x(0)|) + \Delta_2, \tag{86}$$

with

$$\begin{aligned}\delta_1(s) &= (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3^{-1})(\beta_1(s, 0) + \gamma_1 \circ (\mathrm{Id} + \rho_2^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3^{-1})(\beta_2(s, 0))), \\ \delta_2(s) &= (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3^{-1})(\beta_2(s, 0) + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \\ &\qquad \circ (\mathrm{Id} + \rho_3^{-1})(\beta_1(s, 0))), \\ \Delta_1 &= (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)(d_3 + \gamma_1 \circ (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(v_2 + d_2) \\ &\qquad + v_1 + d_1), \\ \Delta_2 &= (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)(d_3 + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)(v_1 + d_1) \\ &\qquad + v_2 + d_2).\end{aligned} \tag{87}$$

With these bounds on the outputs we can use the UO property to establish bounds on the states x_i. In particular, let (α_i^0, D_i^0), $i = 1, 2$, be two couples satisfying (9) respectively for the subsystems (13) and (14). In this case any solution $x(t)$ of (13)–(14) satisfies, for all $t \ge 0$,

$$|x_1(t)| \le \alpha_1^0(|x_1(0)| + \|(u_{1t}^\top, y_{2t}^\top, y_{1t}^\top)^\top\|) + D_1^0, \tag{88}$$

$$|x_2(t)| \le \alpha_2^0(|x_2(0)| + \|(u_{2t}^\top, y_{1t}^\top, y_{2t}^\top)^\top\|) + D_2^0. \tag{89}$$

From (86), (88), (89), and (6), we have

$$\begin{aligned}\|x\| &\le (\alpha_1^0 + \alpha_2^0) \circ (2\mathrm{Id} + 2\delta_1 + 2\delta_2)(|x(0)|) \\ &\quad + [(\alpha_1^0 + \alpha_2^0)(2\|u\| + 2\Delta_1 + 2\Delta_2) + D_1^0 + D_2^0] \\ &:= \delta_3(|x(0)|) + \Delta_3 \\ &:= s_\infty \end{aligned} \tag{90}$$

with δ_i and Δ_i $(i = 1, 2)$ defined in (87).

With this bound on the state, inequalities (86) can be completed as follows: Let

$$t_{10} = \frac{t}{4}, \qquad t_{20} = \frac{t}{2}, \qquad t_{21} = t, \qquad t_{11} \in \left[\frac{t}{2}, t\right], \tag{91}$$

and substitute (77) in (78), so that we have, for any $t \ge 0$ and $\tau \ge 0$,

$$\begin{aligned}|y_2(t+\tau)| &\ge \beta_2\left(s_\infty, \frac{t}{2}\right) + v_2 + d_2 \\ &\quad + \gamma_2\left(\gamma_1(\|y_{2[t/4+\tau,\infty)}\|) + \beta_1\left(s_\infty, \frac{t}{4}\right) + v_1 + d_1\right).\end{aligned} \tag{92}$$

Thus, by applying (6) and using (75), we obtain, for all $t \ge 0$ and $\tau \ge 0$,

$$\begin{aligned}|y_2(t+\tau)| &\le \left[\beta_2\left(s_\infty, \frac{t}{2}\right) + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3^{-1})\left(\beta_1\left(s_\infty, \frac{t}{4}\right)\right)\right] \\ &\quad + (\mathrm{Id} + \rho_2)^{-1}(\|y_{2[t/4+\tau,\infty)}\|) + \gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)(v_1 + d_1) \\ &\quad + v_2 + d_2 + d_3.\end{aligned} \tag{93}$$

Note that the term between brackets in (93) is a function of class KL with respect to (s_∞, t). Further,

$$\begin{aligned}&\gamma_2 \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3)(v_1 + d_1) + v_2 + d_2 + d_3 \\ &\quad = [(\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3)]^{-1}(\Delta_2).\end{aligned} \tag{94}$$

So we apply Lemma A.1 to (93) with τ fixed, $z(t) = |y_2(t+\tau)|$, $\mu = \frac{1}{4}$, $\lambda = (\mathrm{Id} + \rho_3)$, and $\rho = (\mathrm{Id} + \rho_2)^{-1}$. It follows, using symmetry, that two functions $\hat{\beta}_1$ and $\hat{\beta}_2$ of class KL exist such that, for all $t \ge 0$ and $\tau \ge 0$,

$$\begin{aligned}|y_1(t+\tau)| &\le \hat{\beta}_1(s_\infty, t) + \Delta_1, \\ |y_2(t+\tau)| &\le \hat{\beta}_2(s_\infty, t) + \Delta_2.\end{aligned} \tag{95}$$

Then from (86) on one hand and (95) on the other hand, we have, for all $t \ge 0$, the two inequalities

$$|y(t)| \le \hat{\beta}_1(s_\infty, t) + \hat{\beta}_2(s_\infty, t) + \Delta_1 + \Delta_2, \tag{96}$$

$$|y(t)| \le (\delta_1 + \delta_2)(|x(0)|) + \Delta_1 + \Delta_2. \tag{97}$$

This is not yet the IOpS property since s_∞ (in (90)) depends not only on $x(0)$ but also on u and the d_i's. To split this dependence, we define the following function on $\mathbb{R}^3_+$:

$$\sigma(s, \Delta, t) := \min\{(\hat{\beta}_1 + \hat{\beta}_2)(\delta_3(s) + \Delta, t), (\delta_1 + \delta_2)(s)\}. \tag{98}$$

Then, for any function α of class K_∞ and for each (s, Δ, t), we have

$$\sigma(s, \Delta, t) \leq \sigma(s, \alpha^{-1}(s), t) + \sigma(\alpha(\Delta), \Delta, t) \tag{99}$$

$$\leq (\hat{\beta}_1 + \hat{\beta}_2)(\delta_3(s) + \alpha^{-1}(s), t) + (\delta_1 + \delta_2) \circ \alpha(\Delta). \tag{100}$$

The first inequality follows from considering the two cases, $\Delta \leq \alpha^{-1}(s)$ and $s \leq \alpha(\Delta)$, and using the fact that, for each t, the function $\sigma(s, \Delta, t)$ is increasing as s and Δ increase.

In view of (90), (98), and (100), (96) and (97) imply, for all $t \geq 0$,

$$|y(t)| \leq \sigma(|x(0)|, \Delta_3, t) + \Delta_1 + \Delta_2 \tag{101}$$

$$\leq (\hat{\beta}_1 + \hat{\beta}_2)(\delta_3(|x(0)|) + \alpha^{-1}(|x(0)|), t) + (\delta_1 + \delta_2) \circ \alpha(\Delta_3) + \Delta_1 + \Delta_2. \tag{102}$$

The first term on the right-hand side of (102) is a class KL function of $(|x(0)|, t)$. The definitions of Δ_i in (87) and (90), the fact that $u = (u_1^\top, u_2^\top)^\top$, and simple computations based on (6) give, using the notation (19),

$$\Delta_1 \leq r_1(\|u\|) + \tilde{d}_1, \tag{103}$$

$$\Delta_2 \leq r_2(\|u\|) + \tilde{d}_2, \tag{104}$$

$$\Delta_3 \leq (\alpha_1^0 + \alpha_2^0) \circ (4\mathrm{Id} + 4r_1 + 4r_2)(\|u\|) + (\alpha_1^0 + \alpha_2^0)(4\tilde{d}_1 + 4\tilde{d}_2) + D_1^0 + D_2^0. \tag{105}$$

where

$$\tilde{d}_1 = (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3) \circ (\mathrm{Id} + \rho_3^{-1})[d_1 + d_3 + \gamma_1^y \circ (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3) \circ (\mathrm{Id} + \rho_3^{-1})(d_2)], \tag{106}$$

$$\tilde{d}_2 = (\mathrm{Id} + \rho_2^{-1}) \circ (\mathrm{Id} + \rho_3) \circ (\mathrm{Id} + \rho_3^{-1})[d_2 + d_3 + \gamma_2^y \circ (\mathrm{Id} + \rho_1^{-1}) \circ (\mathrm{Id} + \rho_3) \circ (\mathrm{Id} + \rho_3^{-1})(d_1)]. \tag{107}$$

Then, using (6) again, we have

$$(\delta_1 + \delta_2) \circ \alpha(\Delta_3) \leq (\delta_1 + \delta_2) \circ \alpha \circ (2\alpha_1^0 + 2\alpha_2^0) \circ (4\mathrm{Id} + 4r_1 + 4r_2)(\|u\|) + (\delta_1 + \delta_2) \circ \alpha((2\alpha_1^0 + 2\alpha_2^0)(4\tilde{d}_1 + 4\tilde{d}_2) + 2D_1^0 + 2D_2^0). \tag{108}$$

Now, given any function r_3 of class K_∞, we can pick α such that

$$(\delta_1 + \delta_2) \circ \alpha \circ (2\alpha_1^0 + 2\alpha_2^0) \circ (4\mathrm{Id} + 4r_1 + 4r_2)(s) \leq r_3(s), \qquad \forall s \geq 0 \tag{109}$$

(for example, $\alpha = (\mathrm{Id} + \delta_1 + \delta_2)^{-1} \circ r_3 \circ (\mathrm{Id} + (2\alpha_1^0 + 2\alpha_2^0) \circ (4\mathrm{Id} + 4r_1 + 4r_2))^{-1}$). This in conjunction with (102), (103), (104), and (108) implies the IOpS property for system (13)–(14) with the triple $(\beta, r_1 + r_2 + r_3, d)$, where

$$\beta(s, t) = (\hat{\beta}_1 + \hat{\beta}_2)(\delta_3(s) + \alpha^{-1}(s), t), \tag{110}$$

$$d = \tilde{d}_1 + \tilde{d}_2 + (\delta_1 + \delta_2) \circ \alpha((2\alpha_1^0 + 2\alpha_2^0)(4\tilde{d}_1 + 4\tilde{d}_2) + 2D_1^0 + 2D_2^0). \tag{111}$$

When $d_i = D_i^0 = 0$ $(i = 1, 2)$ and $d_3 = 0$ (i.e., $s_i = 0$), we get $d = 0$ implying the IOS property holds. Finally, the UO property for the interconnection follows from the UO property for each subsystem. ■

5.2. *Proof of Theorem 2.2*

Theorem 2.2 is a direct consequence of Lemmas 3.2–3.4 of [PJ].

Introduce the new variables

$$(\bar{\xi}_1, \ldots, \bar{\xi}_n) = \Phi\xi, \qquad (\bar{\omega}_1, \ldots, \bar{\omega}_n) = \Phi\omega, \qquad \bar{\omega}_0 = \omega_0, \tag{112}$$

where Φ is the (invertible) observability matrix of (F, H):

$$\Phi = [H^\top F^\top H^\top \cdots (F^\top)^{n-1} H^\top]^\top. \tag{113}$$

By hypothesis, system (23) is rewritten as

$$\begin{cases} \dot{\zeta} = A\zeta + B(\bar{\xi}_1 + \bar{\omega}_0), \\ \dot{\bar{\xi}}_i = \bar{\xi}_{i+1} + \bar{\omega}_i, \qquad 1 \le i < n, \\ \dot{\bar{\xi}}_n = (HF^{n-1}G)u + HF^n\xi + \bar{\omega}_n. \end{cases} \tag{114}$$

Note that our assumptions imply that $HF^{n-1}G \neq 0$.

Following Lemma A.1 of [PJ], for any function γ_1 of class K_∞ and each positive real number η, a smooth function k_0 of class K_∞ exists such that

$$k_0(s + \eta) \ge \gamma_1(s), \qquad \forall s \ge 0. \tag{115}$$

Then we define

$$k(s) = s \sup_{0 \le t \le 1} \left\{ \frac{dk_0}{ds}(t) \right\} + \int_0^{2s} k_0(t)\, dt. \tag{116}$$

This function is of class K_∞, is convex, and satisfies

$$k(s + \eta) \ge k_0(s + \eta) \ge \gamma_1(s), \qquad s\frac{dk}{ds}(s) \ge k(s), \qquad \forall s \ge 0. \tag{117}$$

In (115) η can be chosen as 0 whenever γ_1 is linearly bounded on a neighborhood of 0. Since (A, B) is stabilizable, a matrix K exists such that $A - BK$ is stable. Let P be the positive definite solution of

$$(A - BK)^\top P + P(A - BK) = -I \tag{118}$$

and consider the function

$$V_0(\zeta) = k(\zeta^\top P\zeta). \tag{119}$$

Along the trajectories of

$$\dot{\zeta} = A\zeta + B(u_0 + \bar{\omega}_0), \tag{120}$$

the time derivative of V_0 satisfies (using (117))

$$\dot{V}_0 = \frac{dk}{ds}(\zeta^\top P\zeta)[-\zeta^\top\zeta + 2\zeta^\top PB(u_0 + K\zeta + \bar{\omega}_0)] \tag{121}$$

$$\le -\lambda_{\max}(P)^{-1} V_0 + \bar{\omega}_0^2 \tag{122}$$

provided that we take

$$u_0 = -K\zeta - \frac{dk}{ds}(\zeta^\top P\zeta)B^\top P\zeta. \tag{123}$$

By applying recursively the generalization of the adding one integrator technique given in Lemmas 3.2 and 3.3 of [PJ], we get a smooth, positive definite and proper function $V_n(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n)$ and a smooth function $u_n(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n)$ such that (see [PJ] and p. 135 of [J])

$$u_n(0) = 0, \qquad V_0(\zeta) \leq V_n(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n), \qquad \forall (\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n) \in \mathbb{R}^{l+n}. \tag{124}$$

Along with the solutions of (114) in closed loop with $u_n + v$, the derivative of this function V_n satisfies

$$\dot{V}_n(t) \leq -\lambda_{\max}(P)^{-1} V_n(\zeta(t), \bar{\xi}_1(t), \ldots, \bar{\xi}_n(t)) + \frac{(n+1)(n+2)}{2} \sup_{i \in \{0,1,\ldots,n\}} \{(|\bar{\omega}_i(t)| + |v(t)|)^2\} \tag{125}$$

for all t in $[0, T)$, the domain of definition of the right maximal solution $(\zeta(t), \bar{\xi}_1(t), \ldots, \bar{\xi}_n(t))$. A direct application of the Gronwall lemma implies, for all t in $[0, T)$,

$$V_n(t) \leq e^{-\lambda_{\max}(P)^{-1}t} V_n(0) + \frac{(n+1)(n+2)}{2} \lambda_{\max}(P) \left(\sup_{i \in \{0,1,\ldots,n\}} \{\|\bar{\omega}_i\| + \|v\|\} \right)^2. \tag{126}$$

Since V_n is positive definite and proper, two functions α_1 and α_2 of class K_∞ exist such that

$$\alpha_1(|(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n)^\top|) \leq V_n(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n) \leq \alpha_2(|(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n)^\top|). \tag{127}$$

From (126) and (127), it results that the solutions $(\zeta(t), \bar{\xi}_1(t), \ldots, \bar{\xi}_n(t))$ are bounded on $[0, T)$ and then $T = +\infty$. In fact, the closed-loop system (114) is ISS with $(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_n, v)$ as input. Finally, by (112), we conclude that the original system (23) is ISS with (ω_0, ω, v) as input.

It remains to prove the IOpS and IOS properties. In view of (117), (119), and (124), we have

$$|\zeta| \leq \frac{(\gamma_1^{-1}(V_n(\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n)) + \eta)^{1/2}}{\lambda_{\min}(P)^{1/2}}, \qquad \forall (\zeta, \bar{\xi}_1, \ldots, \bar{\xi}_n) \in \mathbb{R}^{l+n}. \tag{128}$$

Then, using (6) and (126), we obtain, for all $t \geq 0$,

$$|\zeta(t)| \leq \frac{1}{\lambda_{\min}(P)^{1/2}} \gamma_1^{-1}(2e^{-\lambda_{\max}(P)^{-1}t} V_n(0))^{1/2} + \frac{\eta^{1/2}}{\lambda_{\min}(P)^{1/2}} + \frac{1}{\lambda_{\min}(P)^{1/2}} [\gamma_1^{-1}((n+1)(n+2)\lambda_{\max}(P)(1 + |\Phi|)^2(\|(\omega^\top, \omega_0, v)^\top\|)^2)]^{1/2}, \tag{129}$$

with Φ in (113). By (127) and (112), the first term on the right-hand side of (129) is a function of class KL of $(|(\zeta(0), \xi(0))^\top|, t)$. The second term is equal to zero if γ_1 is linearly bounded on a neighborhood of 0 since, as we saw, $\eta = 0$ in this case. Then, given any function γ of class K_∞, it is possible to choose a function γ_1 of class K_∞ such that

$$\frac{1}{\lambda_{\min}(P)^{1/2}}[\gamma_1^{-1}((n+1)(n+2)\lambda_{\max}(P)(1+|\Phi|)^2 s^2)]^{1/2} \le \gamma(s), \qquad \forall s \ge 0, \quad (130)$$

and such that γ_1 is linearly bounded on a neighborhood of 0 whenever γ^{-1} is. From this, the third term on the right-hand side of (129) is dominated by $\gamma(\|(\omega, v)^\top\|)$. ■

5.3. *Proof of Corollary 2.3*

From Proposition 3.1, the z-subsystem with $f(x, z)$ as output and x as input has the SUO property and is IOpS. Let γ_z be its gain function which, without loss of generality, we can assume to be of class K_∞. By applying Theorem 2.2 to the x-subsystem of (24) with

$$l = 1, \qquad n = 0, \qquad A = 0, \qquad B = 1, \quad (131)$$

and

$$H\xi = u, \qquad \omega_0 = f(x, z), \qquad \gamma = \tfrac{1}{2}(2\gamma_z)^{-1} := \gamma_x, \quad (132)$$

we get a smooth feedback law $u_n(x)$, which is zero at zero, such that the x-subsystem of (24) in closed loop with $u = u_n(x) + v$ is IOpS with $(f(x, z), v)$ as input, x as output, and γ_x as gain function. We remark that by defining

$$\vartheta(x) = u_n(x) - Kx, \quad (133)$$

with K any nonnegative real number, the same result holds with $u = \vartheta(x) + v$. So, in particular, the small-gain condition (18) is satisfied between γ_z and γ_x with $s_l = 0$. With Remark 4, we know that the x-subsystem has the SUO property with a $d_x^0 = 0$ in (28). Hence the first point of Corollary 2.3 follows from Proposition 3.1 and Corollary 3.1.

Point 2 follows in the same way since, in this case, the x-subsystem of (24) in closed loop with $u = \vartheta(x) + v$ is made IOS with gain γ_x.

Next, under the conditions of point 3 and according to Proposition 3.1 the z-subsystem with x as input and $f(x, z)$ as output is IOS. Moreover, from Lemma A.2, the z-subsystem is ISS with a gain which is linearly bounded on a neighborhood of 0. From the smoothness of f, we can obtain a gain from input to output which is also linearly bounded on a neighborhood of 0. The GAS property follows readily.

Finally, to prove the LES property, we observe that, when we regard x as the output of system (24), (24) is hyperbolically minimum-phase with relative degree 1 (see [BI]). From Theorem 24.I of [L1] on the conditions for stability supplied by the first approximation and the root locus technique of [E], by choosing K large enough, the partial-state feedback $\vartheta(x)$ renders the zero solution of (24) LES. ■

6. Conclusions

The notion of input-to-output practical stability (IOpS) introduced in this paper is a natural generalization of Sontag's input-to-state stability property. We have shown that the notion IOpS allows us to establish a generalized small-gain theorem (see Theorem 2.1 and Corollary 2.1) and a gain assignment theorem (see Theorem 2.2). The first one extends the small monotone gain theorem proved by Mareels and Hill in [MH] by including a stability result of Lyapunov type. With these results, we have been able to prove a result in the spirit of the center manifold reduction theorem (see Proposition 4.1), to give conditions under which a linear system with nonlinear, stable dynamic perturbations is globally asymptotically stabilizable (see Proposition 4.2), and finally to show that the ISS property can be propagated through integrators by choosing an appropriate partial-state feedback (see Corollary 2.3). The latter provides an interesting tool for control design. In particular, for a class of nonlinear control systems composed of a chain of dynamically perturbed integrators, we showed how to design a robust partial-state feedback to render all the trajectories of the system bounded. A sufficient condition for the global asymptotic stabilization of the whole system is that the ISS inverse dynamics are locally exponentially stable.

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Appendix

The following technical lemmas have been used in the proofs of Theorem 2.1 and Corollary 2.3.

Lemma A.1. *Let β be a function of class KL, let ρ be a function of class K such that* $\mathrm{Id} - \rho$ *is of class K_∞, and let μ be a real number in* $(0, 1]$. *For any function λ, such that $\lambda - \mathrm{Id}$ is of class K_∞, a function $\hat{\beta}$ of class KL exists such that, for any nonnegative real numbers s and d and any nonnegative real function $z(t)$, defined and essentially bounded on* $[0, +\infty)$ *and satisfying*

$$z(t) \leq \beta(s, t) + \rho(\|z_{[\mu t, \infty)}\|) + d, \qquad \forall t \in [0, +\infty). \tag{134}$$

we have

$$z(t) \leq \hat{\beta}(s, t) + (\mathrm{Id} - \rho)^{-1} \circ \lambda(d), \qquad \forall t \in [0, +\infty). \tag{135}$$

Proof. With the function λ fixed as stated in the lemma, we associate a function $\bar{z}(t)$ to any function $z(t)$ satisfying (134):

$$\bar{z}(t) := z(t)\chi(\|z_{[\mu t, \infty)}\| - (\mathrm{Id} - \rho)^{-1} \circ \lambda(d)), \tag{136}$$

where $\chi(x) = 1$ if $x > 0$ and $\chi(x) = 0$ if $x \leq 0$. Note that, since $0 < \mu \leq 1$,

$$\|z_{[\mu t, \infty)}\| - (\mathrm{Id} - \rho)^{-1} \circ \lambda(d) \leq 0 \quad \Rightarrow \quad z(t) \leq (\mathrm{Id} - \rho)^{-1} \circ \lambda(d), \tag{137}$$

$$\|z_{[\mu t, \infty)}\| - (\mathrm{Id} - \rho)^{-1} \circ \lambda(d) > 0 \quad \Rightarrow \quad d < \lambda^{-1} \circ (\mathrm{Id} - \rho)(\|z_{[\mu t, \infty)}\|). \tag{138}$$

From (134), (136), and (138), the function $\bar{z}$ satisfies

$$\bar{z}(t) \le \beta(s, t) + \bar{\rho}(\|\bar{z}_{[\mu t, \infty)}\|), \qquad \forall t \in [0, +\infty), \tag{139}$$

where $\bar{\rho}$ is defined by

$$\bar{\rho} := \rho + \lambda^{-1} \circ (\mathrm{Id} - \rho). \tag{140}$$

Note that the function $\bar{\rho}$ is of class K_∞ such that $\mathrm{Id} - \bar{\rho}$ is of class K_∞. If we find a function $\hat{\beta}$ of class KL such that

$$\bar{z}(t) \le \hat{\beta}(s, t), \qquad \forall t \ge 0, \tag{141}$$

then, by definition of $\bar{z}$ and (137), (135) holds for z. The proof of existence of such a function $\hat{\beta}$ follows exactly the same lines as in Lemma 2.1.4 and Proposition 2.1.5 of [L2] (see also proofs of Lemma 3.1 and Proposition 2.5 of [LSW]) once the following claim is established:

Claim. *For any r and $\varepsilon > 0$, a nonnegative real number T exists such that if $\bar{z}(t)$ satisfies* (139) *with $s \le r$, then we also have*

$$\bar{z}(t) \le \varepsilon, \qquad \forall t \ge T. \tag{142}$$

Proof. Let $t_0 = 0$ and $t_1 \ge 0$ be the first time instant such that

$$\beta(r, t_1) \le \left(\frac{\mathrm{Id} - \bar{\rho}}{2}\right) \circ (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)). \tag{143}$$

Then, for any integer $n \ge 1$, let $t_{n+1} \ge 0$ be the first time instant such that

$$\beta(r, t_{n+1}) \le \left(\frac{\mathrm{Id} - \bar{\rho}}{2}\right) \circ \left(\frac{\mathrm{Id} + \bar{\rho}}{2}\right)^n \circ (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)). \tag{144}$$

Since β is a function of class KL, such a t_{n+1} exists. Then we define a sequence of nonnegative real numbers $\{\bar{t}_n\}_{n \ge 1}$ as follows:

$$\bar{t}_0 = 0, \qquad \bar{t}_{n+1} = \max\left\{t_{n+1}, \frac{1}{\mu}\bar{t}_n\right\}. \tag{145}$$

Finally, we remark that, for each $x > 0$, $((\mathrm{Id} + \bar{\rho})/2)^n(x)$ is a decreasing sequence and converges to zero as n goes to ∞.

Now we prove by induction that if $\bar{z}(t)$ satisfies (139) with $s \le r$, then we have, for all $n \ge 0$,

$$\bar{z}(t) \le \left(\frac{\mathrm{Id} + \bar{\rho}}{2}\right)^n \circ (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)), \qquad \forall t \ge \bar{t}_n. \tag{146}$$

Indeed, $s \le r$ and (139) imply

$$\bar{z}(t) \le (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)), \qquad \forall t \ge 0. \tag{147}$$

This implies (146) for $n = 0$. Then suppose (146) holds. With (139), we get

$$\bar{z}(t) \le \beta(r, t) + \bar{\rho} \circ \left(\frac{\mathrm{Id} + \bar{\rho}}{2}\right)^n \circ (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)), \qquad \forall t \ge \frac{1}{\mu}\bar{t}_n. \tag{148}$$

This in conjunction with (144) implies

$$\bar{z}(t) \le \left(\frac{\mathrm{Id} + \bar{\rho}}{2}\right)^{n+1} \circ (\mathrm{Id} - \bar{\rho})^{-1}(\beta(r, 0)), \qquad \forall t \ge \bar{t}_{n+1}. \tag{149}$$

Therefore, (146) holds for every nonnegative integer n. This concludes the proof of the claim and, as mentioned above, the proof of the lemma with Lemma 2.1.4 and Proposition 2.1.5 of [L2] or proofs of Lemma 3.1 and Proposition 2.5 of [LSW]. (An idea about these results can be found in Appendix B.3 of [K].) ■

The following lemma follows straightforwardly from Theorem 4.10 of [K] (see also Theorem 1 of [VV] and Lemma 6.1 of [S2]):

Lemma A.2. *Let $\dot{z} = q(z, u)$ be an ISS system with u as input, and assume that $(\partial q/\partial z)(0, 0)$ is an asymptotically stable matrix, then a function β of class KL, a function γ of class K, and two positive real numbers η and k exist such that*

$$\gamma(s) \le ks, \qquad \forall s \in [0, \eta], \tag{150}$$

and, for any measurable essentially bounded control u,

$$|z(t)| \le \beta(|z(0)|, t) + \gamma(\|u\|). \tag{151}$$

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