Technical Notes and Correspondence

Lyapunov Design of Stabilizing Controllers for Cascaded Systems

Laurent Praly, B. d'Andréa-Novel, and Jean-Michel Coron

Abstract—**We are interested in designing a state feedback law for an affine nonlinear system to render a (as small as possible) compact neighborhood of the equilibrium of interest globally attractive. Following Artstein's theorem [1], the problem can be solved by designing a so called control Lyapunov function. The object of this note is to show how such a function can be explicitly constructed for some cascaded nonlinear systems.**

I. Introduction

We consider the following affine nonlinear system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where x lives in $\mathbb{R}^n$, $n \geq 2$, u is a scalar input, f and g are at least C^1 vector-fields, and $f(0) = 0$. The state being measured, our objective is to design a state feedback rendering a (as small as possible) neighborhood of the origin globally attractive.

To solve this problem, inspired by Lyapunov's second method, we design the control law for the time derivative of a scalar function $h(x)$ to be strictly negative, h being positive, C^1 and radially unbounded, i.e., $h(x) \to \infty$ iff $\|x\| \to +\infty$.

This note is organized as follows. In Section II we use Artstein's theorem [1] to show that the solution to our problem can be reduced to designing a so called control Lyapunov function (clf) (see Definition 1). In Section III, we propose such a design for a cascaded system and give illustrative examples. We give our conclusion in Section IV. The proofs of all Lemmas are given in the Appendix.

In the following, for the sake of simplicity, our assumptions will be global in x. However, if they are satisfied only in the open set $\{x \mid h(x) < M \neq 0\}$, the corresponding conclusion applies to solutions whose initial conditions are in this set.

II. The Control Lyapunov Function Approach

Let h be a C^1 function, its time derivative at x along the solutions of (1) is: ($L_f h$ denoting the Lie derivative of h along f)

$$\dot{h}(x) = L_f h(x) + u L_g h(x). \tag{2}$$

Our stabilization problem is solved if we can assign some strictly negative value to $\dot{h}(x)$. Clearly, this is possible at all points x where $L_g h(x)$ is not zero. The difficulty is to deal with the points where $L_g h(x)$ is actually zero. This justifies the following definition.

Definition 1 (clf) [1], [11]: A control Lyapunov function (clf) for system (1) is a positive function h which is zero only at zero and satisfies

1) h is C^1.
2) $h(x) \to \infty$ iff $\|x\| \to +\infty$.
3) $L_g h(x) = 0 \Rightarrow (L_f h(x) < 0$ or $x = 0)$

From this definition, the origin is the only stationary point of a clf. Moreover, such a function is proper, i.e., the preimage of a compact set is also compact.

To study continuity of the feedback law, we will need the notion of "small control property" introduced by Artstein [1].

Definition 2 (scp) [1], [11]: A clf h is said to satisfy the small control property (scp) if for all strictly positive ϵ, we can find a strictly positive δ such that, for all x, $\|x\| < \delta$, $x \neq 0$, there exists u, $|u| < \epsilon$, satisfying

$$L_f h(x) + u L_g h(x) < 0. \tag{3}$$

The following result due to Sontag [11] (see also Artstein [1]) states that our stabilization problem can be solved if we can find a cl f:

Theorem 1 [11]: If there exists a clf h for system (1), then the following control law:

$$u(x) = \begin{cases} 0 & \text{if } L_f h(x) < 0 \text{ and } L_g h(x) = 0 \\ -\dfrac{L_f h(x) + \sqrt{L_f h(x)^2 + L_g h(x)^4}}{L_g h(x)} & \\ & \text{if not} \end{cases} \tag{4}$$

is defined on $\mathbb{R}^n - \{0\}$, has the same regularity as $L_g h$ and $L_f h$ and makes $\dot{h}(x)$ strictly negative for all nonzero x. Therefore, any prespecified compact set containing the origin as an interior point can be made globally continuously attractive.

Moreover, u is at least continuous at the origin if the clf h satisfies scp. In this case, the origin can be made globally continuously asymptotically stable.

Alternative explicit expressions of suitable control laws $u(x)$ are proposed by Tsinias [14] or Praly, d'Andréa and Coron [10].

III. Design of a clf

Let us apply the aforementioned theorem to the following cascaded system:

$$\begin{cases} \dot{z} = k(y, z) \\ \dot{y} = u \end{cases} \tag{5}$$

where z is in $\mathbb{R}^{n-1}$ and y in $\mathbb{R}$ and such that a positive C^1 proper function $h_0(z)$ and a C^0 control law $u_0(z)$ are known and satisfy

$$u_0(0) = 0,\ h_0(z) = 0 \Rightarrow z = 0,$$

$$\text{and } L_{k(u_0(z), z)} h_0(z) < 0 \qquad \forall z \in \mathbb{R}^{n-1} - \{0\}. \tag{6}$$

A typical example for (5) is the case of a system which has been maximally linearized by feedback and diffeomorphism [8].

From (6), a C^1 proper function h satisfying the following two implications is necessarily a clf for system (5)

$$\frac{\partial h}{\partial y}(y, z) = 0 \Rightarrow y = u_0(z) \tag{7}$$

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$$y = u_0(z) \Rightarrow \frac{\partial h}{\partial z}(y, z)k(y, z) = L_{k(u_0(z), z)}h_0(z). \quad (8)$$

Indeed, in this case, if $L_g h \left(= \frac{\partial h}{\partial y}\right)$ is zero, $L_f h \left(= \frac{\partial h}{\partial z} k\right)$ is negative. A simple solution to (7) and (8) is

$$h(y, z) = \int_{u_0(z)}^{y} (s - u_0(z))\, ds + h_0(z)$$

$$= \frac{1}{2}(y - u_0(z))^2 + h_0(z). \quad (9)$$

If u_0 is smooth enough, this function h is an appropriate *clf*. For example, it can be used to reestablish the Property [7, Corollary 3.2] and [15, Theorem 3c]: if $\dot{z} = k(v, z)$ is smoothly stabilizable, the cascaded system (5) is smoothly stabilizable as well.

However, in general, the given control law u_0 is not smooth enough for h in (9) to be a C^1 proper function. To overcome this difficulty, one may replace u_0 by a smoother global stabilizer. This is always possible from Artstein's Theorem [12, Sect. 7, Corollary]. Unfortunately, for engineering applications, this regularization is usually not practical. Another solution is to replace, in (9), $y - u_0(z)$ by a so called "desingularizing" function $\varphi(y, z)$. We have the following lemma.

Lemma 1: If there exist a positive C^1 proper function h_0 and a C^0 function u_0 such that:

1) after possibly a C^1 change of the z-coordinates in $\mathbb{R}^{n-1}$, for all i in $\{1, \cdots, n-1\}$ and all $(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{n-1})$ in $\mathbb{R}^{n-2}$, the real numbers z_i where $\frac{\partial u_0}{\partial z_i}(z_1, \cdots, z_i, \cdots, z_{n-1})$ is not defined are isolated in $\mathbb{R}$;

2) there exists a scalar C^0 function $\varphi(y, z)$ such that

$$\varphi(y, z) = 0 \Leftrightarrow y = u_0(z) \quad (10)$$

$\Phi(y, z)$ is C^1 in $\mathbb{R}^n$ and, for all z in $\mathbb{R}^{n-1}$, $\Phi(y, z) \to +\infty$ if $|y| \to +\infty$, where Φ denotes the antiderivative

$$\Phi(y, z) = \int_0^y \varphi(s, z)\, ds; \quad (11)$$

3) for all nonzero z, we have

$$L_{k(u_0(z), z)}h_0(z) < 0 \quad (12)$$

then

$$h(y, z) = \Phi(y, z) - \Phi(u_0(z), z) + \beta h_0(z)^\alpha \quad (13)$$

is a *clf* for system (5) for all real α such that h_0^α is a C^1 function and for all strictly positive real β.

Hence, we have reduced the design of a *clf* to that of searching for a desingularizing function φ. If we can find a C^1 function s and a strictly positive integer p such that $(u_0(z) - s(z))^{2p-1}$ is C^1, a simple choice is given by

$$\varphi(y, z) = (y - s(z))^{2p-1} - (u_0(z) - s(z))^{2p-1}. \quad (14)$$

Having a *clf* h, (4) is an appropriate control law for stabilizing any compact neighborhood of the origin. However, the particular expression (13) for h allows us to propose the following more practical expression for such a feedback

$$u(y, z) = \left\{\left(\frac{\partial \Phi}{\partial z}(u_0(z), z) - \frac{\partial \Phi}{\partial z}(y, z)\right)k(y, z) + \beta\alpha h_0(z)^{\alpha-1}\frac{\partial h_0}{\partial z}(z)\big(k(u_0(z), z) - k(y, z)\big)\right\} \times \left(\frac{\partial \Phi}{\partial y}(y, z)\right)^{-1} - \Theta(y, z) \quad (15)$$

where Θ is a C^0 function with the same sign as $\frac{\partial \Phi}{\partial y}$. It makes sense when a continuous extension at the zeros of $\frac{\partial \Phi}{\partial y}$ exists.

Example 1: Consider the following planar system studied by Kawski [5]:

$$\begin{cases} \dot{z} = z - y^3 \\ \dot{y} = u. \end{cases} \quad (16)$$

The system linearized at the origin is not stabilizable and therefore there is no C^1 control law asymptotically stabilizing this point. However, Kawski has proposed a general method for small-time locally controllable systems in the plane which, in this example, gives a locally Hölder control law guaranteeing asymptotic stabilization.

To apply our method, we check that points 1 and 2 of Lemma 1 are satisfied when, according to (14), we choose

$$u_0(z) = (cz)^{\frac{1}{3}},\ s(z) = 0,\ \varphi(y, z) = y^{2p-1} - (cz)^{\frac{2p-1}{3}}, \quad p \geq 2 \quad (17)$$

with c strictly larger than 1. Point 3 holds also with

$$h_0(z) = z^2. \quad (18)$$

This leads to the following *clf*:

$$h(y, z) = \frac{y^{2p}}{2p} - y(cz)^{\frac{2p-1}{3}} + \frac{2p-1}{2p}(cz)^{\frac{2p}{3}} + \beta(z^2)^\alpha, \quad \alpha \geq \frac{1}{2}. \quad (19)$$

About scp, we observe that (16) exhibits the following homogeneity property:

$$(\lambda^3 z) - (\lambda y)^3 = \lambda^3(z - y^3). \quad (20)$$

With our choice, u_0 satisfies the same homogeneity

$$u_0(\lambda^3 z) = \lambda u_0(z). \quad (21)$$

Then choosing $\alpha = p/3$, h in (19) is also homogeneous and consequently

$$L_f h(\lambda y, \lambda^3 z) = \lambda^{2p} L_f h(y, z), \quad L_g h(\lambda y, \lambda^3 z) = \lambda^{2p-1} L_g h(y, z). \quad (22)$$

This implies that scp is satisfied. It follows from Theorem 1 that the origin can be made asymptotically stable with a control law continuous on $\mathbb{R}^2$. Moreover, the larger p, the smoother on $\mathbb{R}^2 - \{(0, 0)\}$ this stabilizing control is.

On the other hand, with a suitable choice of Θ, expression (15) satisfies the following homogeneity (expected from (22)):

$$u(\lambda y, \lambda^3 z) = \lambda u(y, z). \tag{23}$$

For example, with $p = 2$, we get

$$u(y, z) = -\frac{c(y^3 - z)}{y^2 + (cz)^{\frac{1}{3}}y + (cz)^{\frac{2}{3}}} + \frac{4}{3}\beta z^{\frac{1}{3}} - (y - (cz)^{\frac{1}{3}}). \tag{24}$$

Finally, we remark, as already done by Kawski [5], that the simpler control law

$$u(y, z) = 3c^2 z^{\frac{1}{3}} - 3c(y - z^{\frac{1}{3}}) \tag{25}$$

is sufficient for this stabilization. This may be checked by looking at the time derivative of (see Appendix A.3)

$$h(y, z) = \frac{y^4}{4} - y(cz) + \frac{3}{4}(1 + 3c^{\frac{2}{3}})(cz)^{\frac{4}{3}} \tag{26}$$

obtained from (19) by choosing $p = 2$, $\alpha = \frac{2}{3}$, and $\beta = \frac{9}{4}c^2$. Note that on the contrary of [5], this Lyapunov function is strictly decreasing along solutions of (16)–(25).

For the case, where we have a cascaded system more involved than a simple integrator, we proceed by induction. At each step, the difficulty is to find a desingularizing function to avoid the lack of smoothness of the control given by the previous step.

Example 2: Consider the following three-dimensional system proposed by Kawski in [6]:

$$\begin{cases} \dot z = z - y^3 \\ \dot y = y - x^3 \\ \dot x = u. \end{cases} \tag{27}$$

From (25) in Example 1, we have to find a desingularizing function φ which is zero iff

$$y - x^3 = 3c^2 z^{\frac{1}{3}} - 3c(y - z^{\frac{1}{3}}). \tag{28}$$

A solution is

$$\varphi(x, y, z) = \left[27c^3 z(c+1)^3 + \left[x^3 - (1 + 3c)y\right]^3\right]^{1+m} \tag{29}$$

with m an even integer. With φ a C^∞ function and h_0 given by (26), (13) provides an appropriate *clf* h. Moreover, we have

$$\varphi(\lambda x, \lambda^3 y, \lambda^9 z) = \lambda^{9(1+m)}\varphi(x, y, z),$$
$$h_0(\lambda^3 y, \lambda^9 z) = \lambda^{12} h_0(y, z). \tag{30}$$

It follows by choosing

$$m = 2, \qquad \alpha = \frac{28}{12} \tag{31}$$

that h in (13) is a homogeneous *clf* function with degree 28. Hence, scp is satisfied. This allows us to conclude that the origin of (27) is continuously globally asymptotically stabilizable.

The above arguments show that, to solve the problem of asymptotically stabilizing the origin of system (5), it is essential to find an appropriate C^1 proper function h_0 and a C^0 function u_0 satisfying (6) for all nonzero z, i.e., Condition 3 of Lemma 1. Such a remark has been written many times in the literature (see [9, Theorem 5] or [7, Corollary 3.2] for example). For system (5) in $\mathbb{R}^2$, existence of h_0 and u_0 satisfying (6) is necessary.

Lemma 2 (see also [13, Lemma 3.1]): A necessary condition for the existence of a continuous control law u making all the solutions of

$$\begin{cases} \dot z = k(y, z) \\ \dot y = u \end{cases} \tag{32}$$

enter a connected compact set $K \subset \mathbb{R}^2$, containing the origin, within finite time is: for every C^1 proper function h_0 with no stationary point outside the set $\{z \mid \exists y, (z, y) \in K\}$, and for every z outside this set, there exists y such that $\frac{dh_0}{dz}(z)k(y, z)$ is strictly negative.

What may not be necessary in Lemma 1 are the smoothness assumptions in conditions 1 and 2. In particular, Dayawansa and Martin [4] have established that, if k is a real analytic function, then condition 3 only is necessary and sufficient for the existence of a locally asymptotically stabilizing C^0 control law.

IV. Conclusion

For an affine nonlinear system, we have studied the problem of rendering globally attractive a (as small as possible) compact neighborhood of the equilibrium. A solution consists of assigning the dynamical behavior of a Lyapunov function. The resulting control law has singularities. But, according to Artstein's theorem, a smooth control law exists if and only if we can find a Lyapunov function such that the open-loop dynamic makes this function decrease at the singular points.

For systems which (possibly after feedback and diffeomorphism) are in a cascade form, we design a Lyapunov function meeting Artstein's conditions, assuming the knowledge of a control law stabilizing the equilibrium of the head nonlinear subsystem. In particular, for planar systems, this gives sufficient conditions and necessary conditions for a compact neighborhood of the equilibrium to be stabilized.

Appendix

A.1. Proof of Lemma 1

First Step: h is a C^1 proper function.

1) h is C^1: To show that h in (13) is C^1, it is sufficient to prove that $\Phi(u_0(z), z)$ is C^1. Let us denote

$$\Psi(z) = \Phi(u_0(z), z). \tag{33}$$

For all $z = (z_1, \cdots, z_{n-1})$ were $\frac{\partial u_0}{\partial z_i}(z)$ exists, we have

$$\frac{\partial \Psi}{\partial z_i}(z) = \frac{\partial \Phi}{\partial y}(u_0(z), z)\frac{\partial u_0}{\partial z_i}(z) + \frac{\partial \Phi}{\partial z_i}(u_0(z), z). \tag{34}$$

Since the definition of Φ implies

$$\frac{\partial \Phi}{\partial y}(u_0(z), z) = \varphi(u_0(z), z) = 0 \tag{35}$$

we obtain, for all $z = (z_1, \cdots, z_{n-1})$ where $\frac{\partial u_0}{\partial z_i}(z)$ exists

$$\frac{\partial \Psi}{\partial z_i}(z) = \frac{\partial \Phi}{\partial z_i}(u_o(z), z). \tag{36}$$

Now, for $(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{n-1})$ fixed at any arbitrary value in $\mathbb{R}^{n-2}$ and for any i, $\Psi(z)$ and $\frac{\partial \Phi}{\partial z_i}(u_0(z), z)$ are C^0 functions of z_i and (36) is satisfied maybe except at isolated points of $\mathbb{R}$. It follows from [3, Proposition I.2.6] that $\frac{\partial \Psi}{\partial z_i}(z)$ is defined and continuous on whole $\mathbb{R}^{n-1}$. Since this holds for all i, Ψ is C^1 (see [2, Statement II.1.3]).

2) h is Positive and Proper: It is sufficient to show that for every y and z we have

$$\Phi(y, z) - \Phi(u_0(z), z) \geq 0. \tag{37}$$

Indeed if this is the case, when $h(y, z)$ is bounded, $h_0(z)$ is bounded. Moreover, h_0 being proper, the same holds for z and consequently for $\Phi(u_o(z), z)$, $\Phi(y, z)$, and y.

To prove (37), we study $\Phi(y, z) - \Phi(u_0(z), z)$ as a function of y with z fixed. It is positive at infinity, zero if $y = u_0(z)$, and continuously differentiable. Its derivative satisfies

$$\frac{\partial \Phi}{\partial y}(y, z) = \varphi(y, z) \tag{38}$$

and therefore vanishes only at $y = u_0(z)$. This allows us to conclude that (37) holds.

3) h is Zero Only at Zero: With (6), (13), and (37), if $h(y, z) = 0$, then

a) $h_0(z) = 0$ and therefore $z = 0$ and $u_0(z) = 0$,

b) $\Phi(y, z) - \Phi(u_0(z), z) = 0$ and therefore $y = u_0(z)$.

Hence $h(y, z) = 0$ implies $y = 0$ and $z = 0$.

Second Step: h is a clf: By construction, we have:

$$L_g h(y, z) = \varphi(y, z) = 0 \quad \text{iff } y = u_0(z). \tag{39}$$

Hence, when $L_g h(y, z)$ is zero, we obtained with (36)

$$L_f h(y, z) = \alpha\beta h_0^{\alpha-1}(z) L_{k(u_0(z), z)} h_0(z) \tag{40}$$

which is strictly negative for all nonzero z.

A.2. Proof of Lemma 2

Since u is continuous, the solutions $(z(t), y(t))$ of (32) exist and are C^1 for any initial conditions. Hence, by assumption, for any initial condition, there exists a C^1 time function $y(t)$ such that the corresponding solution of

$$\dot{z} = k(y(t), z) \tag{41}$$

enters the compact set $\{z \mid \exists y:(z, y) \in K\}$, within finite time. Following a trivial extension of [13, Lemma 3.1], this implies: for every z outside the set $\{z \mid \exists y:(z, y) \in K\}$, there exists y such that $zk(y, z)$ is strictly negative.

To conclude, we note that, h_0 being a C^1 proper function with no stationary point outside the connected set $\{z \mid \exists y:(z, y) \in K\}$, which contains 0, z and $\frac{dh_0}{dz}(z)$ have same sign outside this set. □

A.3. (25) Stabilizes (16)

Noticing that $L_f h$ can be written

$$L_f h = -c(y - (cz)^{\frac{1}{3}})(z - y^3) + 3c^2 z^{\frac{1}{3}}(cz - y^3) - 3c^2 z^{\frac{1}{3}}(cz - z) \tag{42}$$

$\dot{h}$ satisfies

$$\dot{h} = -c(y - (cz)^{\frac{1}{3}})(y - z^{\frac{1}{3}})(2y^2 + (3c^{\frac{1}{3}} - 1) yz^{\frac{1}{3}} + (3c^{\frac{2}{3}} - 1) z^{\frac{2}{3}}) - 3c^2(c - 1) z^{\frac{4}{3}} \tag{43}$$

where, c being strictly larger than 1, the quadratic form

$$(2y^2 + (3c^{\frac{1}{3}} - 1) yz^{\frac{1}{3}} + (3c^{\frac{2}{3}} - 1) z^{\frac{2}{3}})$$

in y and $(cz)^{\frac{1}{3}}$ is always positive. Therefore, the only difficulty is when $(y - (cz)^{\frac{1}{3}})(y - z^{\frac{1}{3}})$ is negative. In this case, y is between $z^{\frac{1}{3}}$ and $(cz)^{\frac{1}{3}}$ and we get the following inequalities:

$$-(y - (cz)^{\frac{1}{3}})(y - z^{\frac{1}{3}}) \leq \frac{(c^{\frac{1}{3}} - 1)^2}{4} z^{\frac{2}{3}} \tag{44}$$

$$(2y^2 + (3c^{\frac{1}{3}} - 1) yz^{\frac{1}{3}} + (3c^{\frac{2}{3}} - 1) z^{\frac{2}{3}}) \leq 8c^{\frac{2}{3}} z^{\frac{2}{3}}. \tag{45}$$

This allows us to state that in any case

$$\dot{h} \leq -\min\{c^2(c^{\frac{1}{3}} - 1), 3c^2(c - 1)\} z^{\frac{4}{3}}. \tag{46}$$

To conclude that $\dot{h}$ is strictly negative for all nonzero (z, y), we note that for $z = 0$, we get

$$\dot{h} = -2cy^4. \tag{47}$$

References

[1] Z. Artstein, "Stabilization with relaxed controls," *Nonlinear Analysis*, vol. TMA-7, pp. 1163–1173, 1983.

[2] W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry.* New York: Academic, 1975.

[3] N. Bourbaki, "Fonctions d'une variable réelle: Théorie élémentaire," *Éléments de mathématique.* Fascicule IX, Livre IV, Hermann, 1958.

[4] W. Dayawansa and C. Martin, "Asymptotic stabilization of two dimensional real analytic systems," *Syst. Contr. Lett.*, vol. 12, pp. 205–211, 1989.

[5] M. Kawski, "Stabilization of nonlinear systems in the plane," *Syst. Contr. Lett.*, vol. 12, pp. 169–175, 1989.

[6] —, "Continuous feedback laws in dimension three," presented at SIAM Conf. Contr. in the 1990's, San Francisco, CA, May 1989.

[7] P. Kokotovic and H. Sussmann, "A positive real condition for global stabilization of nonlinear systems," *Syst. Contr. Lett.*, vol. 13, pp. 125–133, 1989.

[8] R. Marino, "On the largest feedback linearizable subsystem," *Syst. Contr. Lett.*, vol. 6, pp. 345–351, 1986.

[9] —, "Feedback stabilization of single-input nonlinear systems," *Syst. Contr. Lett.*, vol. 10, pp. 201–206, 1988.

[10] L. Praly, B. d'Andréa-Novel, and J. M. Coron, "Lyapunov design of stabilizing controllers," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1989.

[11] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Syst. Contr. Lett.*, vol. 13, pp. 117–123, 1989.

[12] —, "Feedback stabilization of nonlinear systems," Rep. SYCON-89-09; also in *Proc. MTNS-89*, to be published.

[13] E. Sontag and H. Sussmann, "Remarks on continuous feedback," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1980.

[14] J. Tsinias, "Stabilization of affine in control nonlinear systems," *Nonlinear Analysis*, vol. 12, no. 11, pp. 1283-1296, 1988.
[15] —, "Sufficient Lyapunov-like conditions for stabilization," *Math. Contr. Signals Syst.*, vol. 2, no. 4, 1989.

Piecewise Monotone Filtering in Discrete-Time with Small Observation Noise

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Abstract—**A discrete-time model for filtering with small observation noise is considered in this note. A piecewise linear observation function is considered with two intervals of monotonicity. A sequential quadratic variation test is found to detect intervals of linearity of the observation function. Diffusion approximations to certain discrete processes are made to estimate the mean times for reaching a decision and the error probabilities.**

I. Introduction

There is substantial literature on the problem of optimal nonlinear filtering. In continuous time, an unobserved state X_t and observation Y_t are modeled according to

$$\begin{cases} dX_t = f(X_t)\,dt + g(X_t)\,dU_t, & 0 \le t \le T \\ dY_t = h(X_t)\,dt + \epsilon\,dV_t, & Y_0 = 0 \end{cases} \quad (1.1)$$

where U_t, V_t are independent Brownian motions and T is a finite number. To find the mean square optimal estimate $\hat{X}_t$ for X_t given Y_s for $0 \le s \le t$ requires knowing the conditional distribution of X_t. Since the dynamics of the conditional distribution are governed by the nonlinear functional partial differential equation of nonlinear filtering, the problem is inherently infinite dimensional [8].

If X_t and Y_t are of the same dimension and h is one-to-one, then X_t would be known exactly if $\epsilon = 0$. For small $\epsilon > 0$ good finite-dimensional approximate filters have been described in [7] and [9]-[11]. An extended Kalman filter, or even a simpler approximation of Picard [9], [10], can be used to obtain an approximation $\tilde{X}_t$ to the conditional mean $\hat{X}_t$, such that $\tilde{X}_t = \hat{X}_t + O(\epsilon^p)$ for some $p \ge 2$ and $E(X_t - \tilde{X}_t)^2 = O(\epsilon)$. However, if $h(x)$ is many-to-one, then for $\epsilon = 0$, $h^{-1}(Y_t)$ is a set rather than a point. For small $\epsilon > 0$ such approximations as the extended Kalman filter will not give accurate approximations to the optimal filter. References [3]-[5] are concerned with the case when h is piecewise one-to-one. Under a certain "detectability" condition, see (1.3) for scalar-valued processes, one can perform a hypothesis test based on observations on Y_t to decide that X_t belongs to a region on which h is one-to-one. Once this is done, an approximate filter of the type in [7], [9]-[11] is used to estimate $\hat{X}_t$.

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In this note, we consider the following discrete-time analog of (1.1):

$$\begin{cases} x_{k+1} = x_k + \epsilon f(x_k) + g(x_k)\sqrt{\epsilon}\,u_k, & k = 0, 1, 2, \cdots, [T/\epsilon] \\ y_k = h(x_k) + \sqrt{\epsilon}\,v_k, & y_0 = 0 \end{cases} \quad (1.2)$$

where u_k, v_k, $k = 0, 1, 2, \cdots$, are independent standard normal random variables, and $[T/\epsilon]$ is the largest integer less than T/ϵ.

Actually, (1.2) approximates (1.1) in the following way. One discretizes (1.1) with time-step size ϵ and replaces $X_{k\epsilon}$ by x_k, $U_{(k+1)\epsilon} - U_{k\epsilon}$ by $\sqrt{\epsilon}\,u_k$, $V_{(k+1)\epsilon} - V_{k\epsilon}$ by $\sqrt{\epsilon}\,v_k$, and $\epsilon^{-1}(Y_{(k+1)\epsilon} - Y_{k\epsilon})$ by y_k.

To illustrate the ideas without undue technical complications, we assume that all of the processes x_k, y_k, u_k, v_k in (1.2) are 1-dimensional, and that $h(x)$ has just two intervals of monotonicity. In fact, we suppose that $h(x)$ is strictly decreasing for $x < 0$ and strictly increasing for $x > 0$, with $h(0) = 0$. The detectability condition is the following: if $x^+ > 0$, $x^- < 0$ are such that $h(x^+) = h(x^-) = y$, then

$$(gh')^2(x^+) \neq (gh')^2(x^-). \quad (1.3)$$

Our method is as follows, roughly speaking. First, we test for zero crossings of x_k on a time interval $K \le k \le K + M$. If $|y_k| \ge c > 0$ for $K \le k \le K + M$, then for small ϵ no zero crossings occur with probability very near 1 (Section II). Next, we apply a test based on quadratic variations to decide whether $x_k > 0$ for $K \le k \le K + M$ or $x_k < 0$ for $K \le k \le K + M$. This test is based on observing y_k for $K \le k \le K + N$, where $N \le M$ is either fixed (Section III) or random (Section IV). The third step is to apply an extended Kalman filter (Section V).

To further simplify matters, we suppose as in [4] that $g(x)$ is constant and $h(x)$ piecewise linear. In fact, we now assume in (1.2) that

$$g(x) = 1, h(x) = \begin{cases} \alpha x \text{ if } x \ge 0 \\ \beta x \text{ if } x < 0, & \alpha > 0, \beta < 0, \text{ and } \alpha^2 \neq \beta^2. \end{cases} \quad (1.4)$$

The last condition $\alpha^2 \neq \beta^2$ is just the detectability condition (1.3). We also assume in (1.2) that f is smooth with $f'(x)$ bounded. For the sequential hypothesis test in Section IV, the assumption (1.4) allows us to find explicit estimates for the probabilities of incorrect decisions between positivity and negativity of x_k, and for the mean decision time. This is done by a diffusion approximation technique as $\epsilon \to 0$.

In this note, we focus on description of the method, on the diffusion approximation technique, and on numerical results reported in Section VI. Proofs of underlying mathematical results are omitted. In many instances, they are very similar to proofs given in [3] for the corresponding results for the continuous-time model. Another test for positivity or negativity of x_k on an interval $K \le k \le K + M$, based on likelihood ratios, is described in [5]. While this likelihood ratio test is mathematically appealing, it was found in [5] that the sequential quadratic variation test gave consistently better results in numerical experiments.

In a sequel to this note, extensions of our results without the special assumption (1.4) will be considered. In addition, in that sequel corrections between u_k, u_l and between v_k, v_l will be allowed for $|k - l| \le R$ with fixed $R < \infty$. This is done to