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Feedback Linearization of the Transverse Dynamics for a Class of One Degree Underactuated Systems

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***Abstract*— This paper investigates feedback linearization of one degree underactuated systems the mass matrix of which does not depend on the unactuated variable. We study the dynamics of such systems in a new time scale τ. In this new time scale, we exhibit a set of coordinates for which the dynamics transverse to a particular one-dimensional manifold is dynamic feedback linearizable. To this end, we design a relative degree three output with respect to one input and use dynamic extension. We then introduce a new motion planning algorithm for this class of systems.**

I. INTRODUCTION

Feedback linearization is a very useful property for a system. Not per se, that it can be rendered linear in some specific coordinates, but because design tasks like path planning or stabilizing feedback design are made easier. Especially, due to their strong link with flat systems, the solutions of feedback linearizable systems can be parameterized in terms of m time functions and their derivatives where m is the number of inputs [4]. Also because in some coordinates the dynamics is linear, in these coordinates they admit quadratic functions as Control Lyapunov functions (example 3.45 [3]).

One degree underactuated systems with n degrees of freedom have $n-1$ actuators. Due to underactuation, in the canonical coordinates $(q, \dot{q}) \in \mathcal{TQ}$, where $\mathcal{Q}$ is the n-dimensional configuration space, such systems are partially feedback linearizable. More precisely, when the outputs are the actuated coordinates, the zero dynamics is of dimension 2 (Spong normal form) [5]. We consider here one degree underactuated systems such that the mass matrix does not depend on the unactuated coordinate. Some examples are the Acrobot [9], the compass model [1] or the 5-link robot RABBIT [1] depicted Fig. 1.

[6] investigated the control of such systems. They found a set of outputs which are input-output feedback linearizable and such that when these outputs are zero, the corresponding zero dynamics is of dimension one. In the context of the stabilization of an equilibrium point, the zero dynamics is exponentially stable. But for any other trajectory, this property may be lost. Besides, [6] showed from [7] that there generally exists no linearizing output depending only on the configuration space. To the best of our knowledge, no set of outputs giving an empty zero dynamics has been found for the models we are considering here.

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In this paper, we study input-output feedback linearization for one degree underactuated systems for which the mass matrix does not depend on the unactuated variable. But, complementary to [6], we exploit the fact established in [2] that, when change of time scale is also allowed in the equations of dynamics (*Time Scaling*), conditions for feedback linearization are different. Indeed, by changing the time, we change the dynamics. So, we propose to use as new time a coordinate τ of a one dimensional manifold, corresponding to a curvilinear abscissa of the orbits of the system. This is in the same spirit as in the virtual constraints approach where trajectories are parameterized with respect to a function of the generalized positions named θ [1]. We show that, ignoring the dynamics of τ and of the ordinary time t, the system is feedback linearizable by dynamic feedback in the new time scale τ. Or, in other words, the dynamics transverse to the curve parameterized by τ is dynamic feedback linearizable. This exact linearizability differs from the linearization used as an approximation and exploited in [11], [12] to design a stabilizer of the curve. As for the dynamics of τ, it is uncontrollable but trivial. We then design for the class of systems we consider a new motion planning algorithm from this feedback linearized form.

II. MODEL

We consider here a mechanism comprised of $(n-1)$ planar rigid bodies with non zero mass connected in a tree structure (no closed kinematic chains) via revolute joints. Each joint is assumed to be independently actuated. The mechanism is connected to the ground via a pivot, i.e. an unactuated frictionless revolute joint. It has n degrees of freedom and $(n-1)$ independent actuators. So, its degree of underactuation is equal to one. Let $\mathcal{Q}$ be the n-dimensional configuration space, $q \in \mathcal{Q}$ be the generalized positions of the system, and $\mathcal{TQ}$ be the tangent bundle of $\mathcal{Q}$.

The method of Lagrange leads to the equations of motion :

$$\dot{\overbrace{D(q)\dot{q}}} - \frac{1}{2}\Big(\frac{\partial}{\partial q}\big(\dot{q}^T D(q)\dot{q}\big)\Big)^T + G(q) = Bu \ , \qquad (1)$$

where $D(q)$ is the mass matrix, $G(q)$ is the gradient of the gravity potential and $u \in \mathbb{R}^{n-1}$ is a vector of external torques.

Without loss of generality, we assume that the generalized positions are chosen such that the $(n-1)$ first components of $q = (q_1, q_2, ..., q_{n-1})$ are actuated, and the last component q_n, which is the angle of the pivot between the system and the ground, is not. So, we have :

$$B = \begin{pmatrix} I_{(n-1)\times(n-1)} \\ 0_{1\times(n-1)} \end{pmatrix} .$$

In the following, we assume the mass matrix D does not depend on the unactuated generalized position q_n. It is appropriate in the following to manipulate more compact notations. For this we rewrite (1) as :

$$\dot{x} = f(x) + g(x)u \ , \tag{2}$$

with $x = (q, \dot{q}) \in \mathcal{TQ}$, f and g C^1 vector fields on $\mathcal{TQ}$, and $u \in \mathbb{R}^{n-1}$ the vector control of inputs.

III. Feedback linearization of the transverse dynamics

In this section, we show that the dynamics transverse to the curvilinear abscissa of the orbits, called *transverse dynamics* [8], is dynamic feedback linearizable. For that, we work in the $2n+1$ dimensional space-time set $\mathbb{R} \times \mathcal{TQ}$. We introduce a new time scale and exhibit $n-1$ outputs controlling $2n-1$ state components, that correspond to the transverse dynamics. The two uncontrolled state components are the ordinary time t and the new time scale τ.

A. Equivalence of systems written in different time scales

Given an input $t \to u(t)$, defined in open or closed loop, let be $t \to \Phi(t)$ a solution of (2). The data of the function $t \in \mathbb{R} \to \Phi(t) \in \mathcal{TQ}$ is equivalent to the data of a subset of the space-time set $\mathbb{R} \times \mathcal{TQ}$ which is the graph of the function. With this interpretation, $x = \Phi(t)$ is the Cartesian representation of this subset which is a curve in a $2n+1$ dimensional set. Instead, one can adopt a parametric representation by introducing a one dimensional parameter $\tau \in \mathbb{R}$ which is to play the role of a curvilinear abscissa. The parametric representation of the curve is of the form :

$$\begin{cases} x = X(\tau) \\ t = T(\tau) \end{cases} \tag{3}$$

To select the functions X and T above, we observe that, once the derivative $\tau \to T'(\tau)$ is fixed and has no zero values, we do obtain a parametric representation of the curve associated to a solution if and only if the function $\tau \to (X(\tau), T(\tau))$ is obtained as a solution of :

$$\begin{cases} \dfrac{dX}{d\tau}(\tau) = T'(\tau)\Big(f\big(X(\tau)\big) + g\big(X(\tau)\big)u\Big) \\ \dfrac{dT}{d\tau}(\tau) = T'(\tau) \end{cases} \tag{4}$$

Specifically, we can come back to the Cartesian representation in $\mathbb{R} \times \mathcal{TQ}$ by eliminating τ which, according to the implicit function theorem, is possible when $T'(\tau)$ is not zero. Actually, to preserve the sense of motion, one needs :

$$T'(\tau) > 0 \ .$$

Besides this constraint, the function T' being free, we are motivated for considering :

$$\begin{cases} \dfrac{dx}{d\tau} = u_T(f(x) + g(x)u) \\ \dfrac{dt}{d\tau} = u_T \end{cases} \tag{5}$$

where u_T is a new control (=T'). With this expression we see that the curvilinear abscissa τ also plays the role of a new time. As we shall see below, the interest of this process is in the new possibilities offered for feedback-linearizing the dynamics of the systems we consider here. Indeed, [2] showed "that there exist nonlinear systems which cannot be linearized in ordinary time scale t but can be linearized in a new time scale τ ". In particular, thanks to the equivalence we have just highlighted above, linearizabilty of our model can be studied in the form given in (5).

B. Choice of the control u_T

Although other choices are possible (and could be fruitful), inspired by [1], we select the control u_T in such a way that the curvilinear abscissa τ is directly related to the generalized positions q. For this, we split them into two parts : a one dimensional component θ and a $n-1$ dimensional component r.

Let M_r be a $(n-1)\times n$ matrix and M_θ be a n dimensional row vector. Both are chosen state independent and such that there exist N_r a $n \times (n-1)$ matrix and N_θ a n dimensional column vector satisfying :

$$N_r M_r + N_\theta M_\theta = I \ .$$

This allows us to define new coordinates as :

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} M_r \\ M_\theta \end{pmatrix} q \ .$$

They satisfy :

$$q = N_r r + N_\theta \theta \ .$$

With the notations :

$$\omega_\theta = \dot{\theta} \ , \quad \omega_r = \dot{r} \ ,$$

$(r, \theta, \omega_r, \omega_\theta)$ is a new set of coordinates for $(q, \dot{q})$. As explained above, due to the constraint on u_T, we restrict our attention to the subset Ω of $\mathbb{R} \times \mathcal{TQ}$ where :

$$\omega_\theta > 0 \ . \tag{$\mathbf{C_1}$}$$

In this set we choose the extra control u_T as the state feedback :

$$u_T = \frac{1}{\omega_\theta} \ .$$

This choice leads to :

$$\frac{d\tau}{dt} = \omega_\theta \ ,$$

or equivalently :

$$\tau = \int \omega_\theta \, \mathrm{d}t \ .$$

As a consequence, τ behaves as θ with respect to the standard time t, except that τ is a real number, i.e. lives in $\mathbb{R}$, whereas θ is an angle, i.e. lives in the circle $\mathbb{S}^1$.

C. Dynamics with respect to τ

Now that the extra control u_T is defined, we come back to the model given as in (5) expressed with the coordinates $(t, r, \theta, \omega_r, \omega_\theta)$ defined above.

We denote the derivative of x relative to τ as :

$$\mathring{x} = \frac{dx}{d\tau} = \frac{\dot{x}}{\omega_\theta} \ .$$

Also it is useful to replace the coordinate ω_r by $\varpi_r = \frac{\omega_r}{\omega_\theta}$. The motivation is that it gives simply :

$$\frac{dr}{d\tau} \ = \ \varpi_r \ .$$

Also we have :

$$\begin{aligned}\mathring{\varpi}_r &= \frac{1}{\omega_\theta}\frac{d(\frac{\omega_r}{\omega_\theta})}{dt} = \frac{1}{\omega_\theta^2}\Big(\dot{\omega}_r - \varpi_r\dot{\omega}_\theta\Big) = \frac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big)\ddot{q} \\ &= \frac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big) \\ &\qquad \times D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G + Bu\Big),\end{aligned}$$

where :

$$C(q,\dot{q}) = \dot{D}(q)\dot{q} - \frac{1}{2}\Big(\frac{\partial}{\partial q}(\dot{q}^T D(q)\dot{q})\Big).$$

Hence, the dynamics in the coordinates $(t, r, \theta, \varpi_r, \omega_\theta)$ is :

$$\begin{cases} \mathring{t} = \frac{1}{\omega_\theta} \\ \mathring{r} = \varpi_r \\ \mathring{\theta} = 1 \\ \mathring{\varpi}_r = \frac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big) \\ \qquad \times D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G(q) + Bu\Big) \\ \mathring{\omega}_\theta = \frac{1}{\omega_\theta} M_\theta \\ \qquad \times D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G(q) + Bu\Big) \end{cases}$$

Here the argument q stands for (r, θ) and $\dot{q}$ for $(\varpi_r, \omega_\theta)$.

Let us consider now the generalized (or conjugate) momentum associated to the unactuated position q_n [10] :

$$\sigma = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \ ,$$

with $\mathcal{L} = \mathcal{T} - \mathcal{V}$, where $\mathcal{L}$ is the Lagrangian of the system, $\mathcal{T} = \frac{1}{2}\dot{q}^T D(q)\dot{q}$ the kinetic energy, and $\mathcal{V} = V(q)$ the potential energy. We obtain :

$$\sigma = B^\perp D(q)\dot{q} \ ,$$

where :

$$B^\perp = \begin{pmatrix} 0_{1\times(n-1)} & 1 \end{pmatrix} \ .$$

It corresponds to the angular momentum of the mechanism expressed at the point of contact with the ground. This quantity has a strong physical meaning, it quantifies the rotating motion of the system about this point of contact, and has been shown to be useful to build relative degree three outputs [13], [6], [1]. Its expression with our coordinates $(\omega_r, \omega_\theta)$ and $(\varpi_r, \omega_\theta)$ is :

$$\sigma = B^\perp D(q)(N_r\omega_r + N_\theta\omega_\theta) = B^\perp D(q)(N_r\varpi_r + N_\theta)\omega_\theta \ . \tag{6}$$

This shows that when :

$$B^\perp D(q)(N_r\varpi_r + N_\theta) \neq 0 \ , \tag{$\mathbf{C_2}$}$$

σ can be used as a coordinate replacing ω_θ. In this case the condition $(\mathbf{C_1})$ is equivalent to

$$\varepsilon\sigma > 0 \ , \tag{$\mathbf{C_1'}$}$$

with $\epsilon = 1$ or -1 depending on the orientation convention.

Then, given that D is independent of q_n :

$$\frac{\partial D(q)}{\partial q_n}(q) = 0, \text{ and } B^\perp B = 0_{1\times n-1} \ ,$$

we get from (1) :

$$\begin{aligned} \dot{\sigma} &= -B^\perp G(q) \\ \mathring{\sigma} &= -\frac{B^\perp G(q)}{\omega_\theta} = -\frac{B^\perp G(q)}{\sigma}[B^\perp D(q)(N_r\varpi_r + N_\theta)] \ . \end{aligned}$$

Hence, the dynamics in the coordinates $(t, r, \theta, \varpi_r, \sigma)$ is :

$$\begin{cases} \mathring{t} = \dfrac{B^\perp D(q)(N_r\varpi_r + N_\theta)}{\sigma} \\ \mathring{r} = \varpi_r \\ \mathring{\theta} = 1 \\ \mathring{\varpi}_r = \dfrac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big) \\ \qquad \times D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G(q) + Bu\Big) \\ \mathring{\sigma} = -\dfrac{B^\perp G(q)}{\sigma}[B^\perp D(q)(N_r\varpi_r + N_\theta)] \end{cases}$$

Here the argument q stands for (r, θ), $\dot{q}$ for (ϖ_r, σ) and ω_θ is obtained from (r, ϖ_r, σ) with (6).

This can be further simplified when the $(n-1)\times(n-1)$ matrix

$$\tfrac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big)D(q)^{-1}B \ \text{ is invertible } \ . \tag{$\mathbf{C_3}$}$$

Indeed in this case, the control u is equivalent to[1] :

$$\begin{aligned} \mathrm{v} &= \frac{1}{\omega_\theta^2}\Big(M_r - \varpi_r M_\theta\Big) \\ &\qquad \times D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G(q) + Bu\Big) \end{aligned}$$

and we have more simply :

$$\begin{cases} \mathring{t} = \dfrac{B^\perp D(q)(N_r\varpi_r + N_\theta)}{\sigma} \\ \mathring{r} = \varpi_r \\ \mathring{\theta} = 1 \\ \mathring{\varpi}_r = \mathrm{v} \\ \mathring{\sigma} = -\dfrac{B^\perp G(q)}{\sigma}[B^\perp D(q)(N_r\varpi_r + N_\theta)] \end{cases} \tag{7}$$

Here the dynamics of θ is trivial, and the dynamics of (r, ϖ_r) with v as input is linear. But, in the τ time scale, the conjugate momentum σ has relative degree two whereas in the t time scale, its relative degree is three as exploited in [13], [6], [1] . This is due in particular to the presence of ϖ_r in $\mathring{\sigma}$.

[1]Note that to obtain the Spong normal form [5], i.e. the input-output linearization with respect to the standard time t with r as output, we should pick v as :
$\mathrm{v} = M_r D(q)^{-1}\Big(- C(q,\dot{q})[N_r\varpi_r + N_\theta]\omega_\theta - G(q) + Bu\Big).$

D. Designing a relative degree three output

Another interesting property of (7) is that the $n-1$ dimensional control v acts on ϖ_r in a series structure. But then ϖ_r acts in parallel on both the $n-1$ dimensional vector r and the scalar σ. This parallel structure is not suited for feedback linearization. For this, we need a series structure. To this end, we design from σ a relative degree three output with respect to one component of r. To extract this specific component of r, we choose [2] M_{x_1} a $(n-2)\times(n-1)$ matrix and M_s a $n-1$ dimensional row vector, independent of the state and such that there exists N_{x_1} a $(n-1)\times(n-2)$ matrix and N_s a $n-1$ dimensional column vector satisfying :

$$N_{x_1}M_{x_1} + N_s M_s = I.$$

With this, we decompose r as :

$$\begin{pmatrix} x_1 \\ s \end{pmatrix} = \begin{pmatrix} M_{x_1} \\ M_s \end{pmatrix} r\ ,$$

where the component s is the one mentioned above. Since we have :

$$r = \begin{pmatrix} N_{x_1} & N_s \end{pmatrix} \begin{pmatrix} x_1 \\ s \end{pmatrix}\ ,$$

(x_1, s) is another set of coordinates for r. Accordingly, we decompose ϖ_r as :

$$\begin{pmatrix} \varpi_{x_1} \\ \varpi_s \end{pmatrix} = \begin{pmatrix} M_{x_1} \\ M_s \end{pmatrix} \varpi_r.$$

With this, the expression of $\mathring{\sigma}$ becomes :

$$\mathring{\sigma} = -\frac{B^\perp G(q)}{\sigma}[B^\perp D(q)\big(N_r(N_{x_1}\varpi_{x_1} + N_s\varpi_s) + N_\theta\big)],$$

or equivalently, for σ non zero,

$$\mathring{\overbrace{\frac{\sigma^2}{2}}} = -B^\perp G(q)[B^\perp D(q)(N_r(N_{x_1}\varpi_{x_1} + N_s\varpi_s) + N_\theta)].$$

Here the argument q stands for (x_1, s, θ). To remove the term $-(B^\perp G(q)B^\perp D(q)N_r N_s)\varpi_s$ on the right hand side, we replace $\frac{\sigma^2}{2}$ by the equivalent coordinate :

$$y_1 = \frac{\sigma^2}{2} + \int_0^s (B^\perp G(q)B^\perp D(q)N_r N_s)\,\mathrm{d}l\ .$$

In this expression, the argument q appearing in the integrand represents (x_1, s, θ) and the integration is with respect to the second argument s for which we use the dummy variable l. With the condition $(\mathbf{C_1'})$, we have :

$$\sigma = \epsilon\sqrt{2\Big(y_1 - \int_0^s (B^\perp G(q)B^\perp D(q)N_r N_s)\,\mathrm{d}l\Big)} \qquad (8)$$

when :

$$y_1 > \int_0^s (B^\perp G(q)B^\perp D(q)N_r N_s)\,\mathrm{d}l. \qquad (\mathbf{C_1''})$$

Note that (C_1'') and (C_2) give (C_1'), and so (C_1) once an orientation convention is chosen.

[2]The problem of the choice of one specific component of r only appears for $n > 2$. So, it does not concern the Acrobot or the compass model.

Under (C_1''), y_1 can be used as a coordinate instead of σ. It gives :

$$\begin{aligned}
\mathring{y}_1 &= -B^\perp G(q)[B^\perp D(q)(N_r(N_{x_1}\varpi_{x_1} + N_s\varpi_s) + N_\theta)] \\
&\quad + B^\perp G(q)B^\perp D(q)N_r N_s\varpi_s \\
&\quad + \int_0^s \Big[(N_r N_s)^T\Big(\frac{\partial}{\partial x_1}\big(B^\perp G(q)B^\perp D(q)\big)^T\varpi_{x_1} \\
&\qquad\qquad + \frac{\partial}{\partial\theta}\big(B^\perp G(q)B^\perp D(q)\big)^T\Big)\Big]\,\mathrm{d}l \\
&= -(N_r N_{x_1}\varpi_{x_1} + N_\theta)^T(B^\perp G(q)B^\perp D(q))^T \\
&\quad + \int_0^s \Big[(N_r N_s)^T\Big(\frac{\partial}{\partial x_1}\big(B^\perp G(q)B^\perp D(q)\big)^T\varpi_{x_1} \\
&\qquad\qquad + \frac{\partial}{\partial\theta}\big(B^\perp G(q)B^\perp D(q)\big)^T\Big)\Big]\,\mathrm{d}l \\
&= f_2(x_1, s, \varpi_{x_1}, \theta) = y_2.
\end{aligned}$$

Hence we do obtain that $\mathring{y}_1$ depends on s but not on ϖ_s. Moreover, according to the Implicit Function Theorem, we can replace the coordinate s by $y_2\ (= \mathring{y}_1)$ if we have :

$$\frac{\partial f_2}{\partial s}(x_1, s, \varpi_{x_1}, \theta) \neq 0 \qquad (\mathbf{C_4})$$

with :

$$\begin{aligned}
&\frac{\partial f_2}{\partial s}(x_1, s, \varpi_{x_1}, \theta) \\
&\quad = -(N_r N_{x_1}\varpi_{x_1} + N_\theta)^T \tfrac{\partial}{\partial s}(B^\perp G(q)B^\perp D(q))^T \\
&\qquad + (N_r N_s)^T\Big(\frac{\partial}{\partial x_1}\big(B^\perp G(q)B^\perp D(q)\big)^T\varpi_{x_1} \\
&\qquad\qquad + \frac{\partial}{\partial\theta}\big(B^\perp G(q)B^\perp D(q)\big)^T\Big).
\end{aligned}$$

To continue our chain of derivations, we decompose v as :

$$\begin{pmatrix} \mathrm{w}_{x_1} \\ \mathrm{w}_s \end{pmatrix} = \begin{pmatrix} M_{x_1} \\ M_s \end{pmatrix} \mathrm{v}.$$

We obtain :

$$\begin{aligned}
\mathring{\mathring{y}}_1 = \mathring{y}_2 &= \frac{\partial f_2}{\partial x_1}\varpi_{x_1} + \frac{\partial f_2}{\partial s}\varpi_s + \frac{\partial f_2}{\partial \varpi_{x_1}}\mathrm{w}_{x_1} + \frac{\partial f_2}{\partial\theta} \\
&= f_3(x_1, s, \varpi_{x_1}, \varpi_s, \mathrm{w}_{x_1}, \theta)\ =\ y_3.
\end{aligned}$$

Again we are interested in replacing the coordinate ϖ_s by $y_3\ (= \mathring{y}_2)$. This is possible if $\frac{\partial f_3}{\partial \varpi_s}$ is non zero. Having $\frac{\partial f_3}{\partial \varpi_s} = \frac{\partial f_2}{\partial s}$, this replacement can be done when condition $(\mathbf{C_4})$ holds. Specifically we have

$$\varpi_s = \frac{1}{\frac{\partial f_2}{\partial s}}\Big(y_3 - \frac{\partial f_2}{\partial x_1}\varpi_{x_1} - \frac{\partial f_2}{\partial \varpi_{x_1}}\mathrm{w}_{x_1} - \frac{\partial f_2}{\partial\theta}\Big)\ . \qquad (9)$$

We get next :

$$\begin{aligned}
\mathring{\mathring{\mathring{y}}}_1 = \mathring{y}_3 &= \frac{\partial f_3}{\partial x_1}\varpi_{x_1} + \frac{\partial f_3}{\partial s}\varpi_s + \frac{\partial f_3}{\partial \varpi_{x_1}}\mathrm{w}_{x_1} + \frac{\partial f_3}{\partial \varpi_s}\mathrm{w}_s \\
&\quad + \frac{\partial f_3}{\partial \mathrm{w}_{x_1}}\mathring{\mathrm{w}}_{x_1} + \frac{\partial f_3}{\partial\theta}.
\end{aligned}$$

This third τ-derivative of y_1 can be fully controlled by the component w_s of the control under the same condition that $\frac{\partial f_3}{\partial \varpi_s}$ is non zero, i.e. under $(\mathbf{C_4})$.

In this case, the control w_s is equivalent to :

$$\mu_s = \frac{\partial f_3}{\partial x_1}\varpi_{x_1} + \frac{\partial f_3}{\partial s}\varpi_s + \frac{\partial f_3}{\partial \varpi_{x_1}}\mathrm{w}_{x_1} + \frac{\partial f_3}{\partial \varpi_s}\mathrm{w}_s + \frac{\partial f_3}{\partial \mathrm{w}_{x_1}}\mathring{\mathrm{w}}_{x_1} + \frac{\partial f_3}{\partial\theta}.$$

E. Linearization of the transverse dynamics

To cope with the fact that $\mathring{\mathrm{w}}_{x_1}$ appears in $\overset{\circ\circ\circ}{y}_1$, we do a dynamic extension with adding w_{x_1} to the state components and we consider its derivative $\mu_{x_1} = \mathring{\mathrm{w}}_{x_1}$ as control. Then, collecting the results of all the previous steps, we get that, if the conditions $(\mathbf{C_1''})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, and $(\mathbf{C_4})$ are satisfied, we can use the following $3(n-2)+5 = 3n-1$ dimensional state to describe the dynamics :

$$(t\,,\,\theta\,,\,x_1\,,\,\varpi_{x_1}\,,\,y_1\,,\,y_2\,,\,y_3\,,\,\mathrm{w}_{x_1}).$$

It satisfies :

$$\begin{cases} \mathring{t} = \dfrac{B^{\perp}D(q)\big(N_r(N_{x_1}\varpi_{x_1} + N_s\varpi_s) + N_\theta\big)}{\sigma} \\ \mathring{\theta} = 1 \\ \mathring{x}_1 = \varpi_{x_1} \\ \mathring{y}_1 = y_2 \\ \mathring{y}_2 = y_3 \\ \mathring{y}_3 = \mu_s \\ \mathring{\varpi}_{x_1} = \mathrm{w}_{x_1} \\ \mathring{\mathrm{w}}_{x_1} = \mu_{x_1} \end{cases} \tag{10}$$

In the expression of $\mathring{t}$ above, q, ϖ_s and σ are functions of $(\theta, x_1, \varpi_{x_1}, y_1, y_2, y_3, \mathrm{w}_{x_1})$. See (8) and (9) in particular.

We have established :

Proposition 1: Under the conditions $(\mathbf{C_1''})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, and $(\mathbf{C_4})$, (2) and (10) have the same solutions.

Ignoring t and θ, (10) is a linear system.

Ignoring t only, we have on top of this a one dimensional invariant set in which the dynamics is trivial $\mathring{\theta} = 1$ and a (3n-3) dimensional set in which the dynamics, called *transverse dynamics*, is linear.

Keeping t and θ allows to come back to the standard time scale t and so to (2). Note that we implicitly assume the dynamics of t in (10) has no finite escape time within the time domain of interest when the other variables still make sense.

In conclusion, ignoring t, we have obtained, in the τ time scale, a system with a one dimensional zero dynamics in which the dynamics is trivial. But, this holds only when the conditions $(\mathbf{C_1''})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, and $(\mathbf{C_4})$ are satisfied. They involve q, ϖ_{x_1}, but also the matrices M_{x_1}, M_s and M_θ that we choose. So, these matrices should be chosen depending on the region in the state space $\mathcal{TQ}$ where we want the robot to evolve.

IV. Application to motion planning

A. Motion planning and flatness

We address now the problem of motion planning, discussing how the representation (10) can be used instead of (2).

First, note that the dynamics of (x_1, y_1) is a chain of integrators. More precisely, (x_1, y_1) is a flat output. It means that the knowledge of the functions $\tau \to x_1(\tau)$ and $\tau \to y_1(\tau)$, encoding the behavior of (x_1, y_1) along the trajectory, is sufficient to deduce the behavior of the other (except t and θ) state components and the controls by successive derivations. No integration is needed. This is the core idea of flatness theory [4].

As for θ, its dynamics is trivial. On the contrary, the dynamics of t is complex but fortunately it can be ignored in a motion planning algorithm, especially if one prefers to get trajectories parameterized by τ instead of the time t as in the virtual constraints approach [1].

Consequently, if the constraints of the motion planning problem can be easily translated into an appropriate choice of $\tau \to x_1(\tau)$ and $\tau \to y_1(\tau)$, its resolution becomes an easy task with (10) while it was not necessarily obvious with (2). For more details, chapter 7 of [4] gives some examples of motion planning using flatness.

Nonetheless, $(\mathbf{C_1''})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, and $(\mathbf{C_4})$ must be satisfied to have the equivalence of the solutions of (2) and (10). Unfortunately, we have found no obvious relationship between the choice of $\tau \to x_1(\tau)$ and $\tau \to y_1(\tau)$ and the verification of the four conditions. So they have to be checked a posteriori.

B. Example : Motion planning for planar biped robots.

We address here the problem of the design of walking cycles for one degree underactuated biped robots, models of which are described chapter 3 of [1]. A walking cycle is a periodic orbit solution of the dynamics of the robot that must respect constraints such that the no take off and no slip conditions of the stance leg, and a strictly positive altitude of the swing leg. Other additional constraints can be added to the motion planning problem such that torque saturation, a desired mean walking speed, ...

[1] propose to parameterize the trajectories of the actuated coordinates by Bézier polynomials and to select the parameters which are the polynomials coefficients by solving an optimization problem under constraints. These polynomials are chosen as functions not of the time t but of θ, the angle between the vertical line and the line passing by the hip and the point of contact between the stance leg and the ground, depicted Fig. 1 for the 5-link robot RABBIT. Because in [1] stabilization is carried out by an input-output linearization with the actuated positions as output, an extra constraint of stability of a 2 dimensional zero dynamics is imposed.

We follow here exactly the same design process, but working with (10). For the system (10), we choose to parameterize (x_1, y_1) as functions of τ which are also Bézier polynomials. But, on top of the constraints considered in [1], we must deal with the constraints $(\mathbf{C_1''})$, $(\mathbf{C_2})$, $(\mathbf{C_3})$, and $(\mathbf{C_4})$. On the other hand, given that we have a trivial zero dynamics, we don't care about its stability.

Also, we cannot benefit here fully from the flatness of the transverse dynamics, i.e. being freed of numerical integration. Indeed, we have to integrate the dynamics of the ordinary time t along τ to know the duration of a step, and so computing the mean walking speed. Nonetheless, we keep the advantage that we can get the trajectories of all the

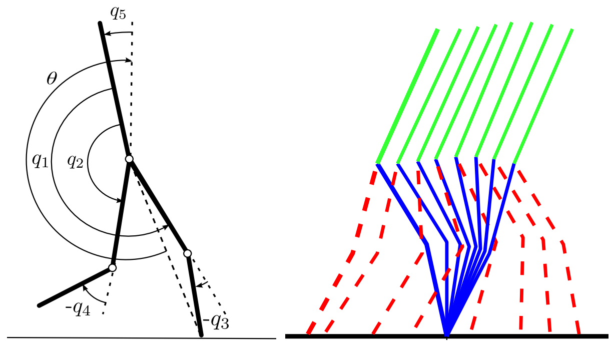

Fig. 1. State description of the planar 5-link robot RABBIT (left). A simulated step of the motion obtained for the RABBIT with our planning algorithm (right).

degrees of freedom without integration, while one integration is needed in [1].

Furthermore, we did not succeed in formulating some constraints, such that the no take off and no slip conditions, in the coordinates $(x_1, \theta, \varpi_{x_1}, y_1, y_2, y_3, \mathrm{w}_{x1})$. So, we have to come back to the original coordinates $(q, \dot{q})$ at each iteration of the optimization process. This has a cost, especially due to the fact that we must invert the function $s \mapsto y_2 = f_2(x_1, s, \varpi_{x_1}, \theta)$ to recover s from the data of $(y_2, x_1, \varpi_{x_1}, \theta)$. This makes the algorithm even more sensitive to the fact that $\frac{\partial f_2}{\partial s}$ must not be zero, i.e. condition $(\mathbf{C_4})$. And, given that the physical meaning of y_1 and its derivatives is abstruse to us, correctly initializing the optimization problem appeared trickier than in the coordinates $(q, \dot{q})$.

We successfully obtained walking cycles for the compass model, described section 3.4.6.1 of [1], in a similar computation time as [1].

We also applied our method to the 5-link robot RABBIT described section 6.6.2.1 of [1] and depicted Fig. 1. In single support phase, the robot has 5 degrees of freedom and 4 actuators. q_1, q_2, q_3 and q_4 are the actuated coordinates.

For that, we choose :

$$\begin{aligned} M_r &= \begin{pmatrix} I_{4\times 4} & 0_{4\times 1} \end{pmatrix}, \\ M_\theta &= \begin{pmatrix} -1 & 0 & -\frac{1}{2} & 0 & -1 \end{pmatrix}, \\ M_{x_1} &= \begin{pmatrix} I_{3\times 3} & 0_{3\times 1} \end{pmatrix}, \\ M_s &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

which leads to the same definition for θ, as in [1]. As explained previously, there may exist a more appropriate choice, especially for M_s, in order to avoid singularities.

We obtained, with some difficulties especially due to linearization singularities, walking trajectories for the RABBIT, but less optimal and with a higher computation cost than [1].

On the other hand, we have not yet exploited (10) in terms of stabilization. But, it is expected that its large dependence on the dynamic parameters of the model could lead to a non robust control law.

V. Conclusions

We discussed feedback linearization of one degree underactuated systems such that the mass matrix is independent of the unactuated coordinate. We showed that by changing the time scale, designing a relative degree three output relative to one input, and using dynamic extension on the other input components, one can exhibit a set of outputs leading to a dynamic feedback linearizable transverse dynamics. The zero dynamics corresponds to the dynamics of the new time scale and of the ordinary time scale t. We then proposed a new motion planning algorithm from this feedback linearized form. The flatness of the transverse dynamics can indeed facilitate the design process. This method was successfully applied for the compass model. But, it appeared that because our linearization process may not be valid on the entire state space, and because the physical meaning of some of the linearizing coordinates is abstruse, the design of trajectories under constraints for some models, such that walking cycles for the RABBIT, may be more difficult than in the original coordinates. So, as suggested by a reviewer for a preliminary version of this communication, an important question to be addressed now is the selection of the various matrices involved in the change of coordinates in order to push the linearization singularities as far as possible from the domain of interest.

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