ON OBSERVERS WITH STATE INDEPENDENT ERROR LYAPUNOV FUNCTION

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Abstract: For systems which are linear up to nonlinear output injection or which have an open loop monotonicity property, there exists observers with the same dimension as the system and such that the error system admits a Lyapunov function depending only on the error. Here we investigate for general systems under which necessary or sufficient conditions such a property holds. This leads us to introduce the notions of Observer Lyapunov Function and State Independent Error Lyapunov Function.

Keywords: Non linear systems, Observers, Lyapunov functions, Output feedback.

1. INTRODUCTION

For the system :

$$\dot{x} \;=\; f(x) \quad , \qquad y \;=\; h(x) \;, \tag{1}$$

with x in $\mathbb{R}^n$ and y in $\mathbb{R}$ and where f and h are C^1, we are interested in the possibility of writing the global observer :

$$\dot{\mathcal{X}} \;=\; F(\mathcal{X},y) \quad , \qquad \widehat{x} \;=\; H(\mathcal{X},y) \;, \tag{2}$$

with input y, state $\mathcal{X}$ and output $\widehat{x}$, the estimation of x.

Global observer theory for nonlinear systems started mainly by exhibiting conditions under which there exist coordinates such that the system is linear up to injection of nonlinear functions of the output and the output is a linear map of these coordinates (see (Krener and Isidori, 1983)). This has been extended in (Lévine and Marino, 1986) to the case where the change of coordinates is actually an immersion and relaxed in (Kazantzis and Kravaris, 1998) where the hard constraint of linearity of the output map is removed. In all these cases a simple linear observer can be written (see for instance (Krstic, Kanellakopoulos and Kokotovic, 1995; Marino and Tomei, 1995)). Taking advantage of the robustness of such observers, one can handle also systems triangular dynamics involving globally Lipschitz nonlinearities (see (Gauthier, Hammouri and Othman, 1992) for instance).

These linear observers lead to an error system admitting a quadratic form in the error $e = \widehat{x} - x$ as Lyapunov function. Keeping this aspect as the main ingredient, another class of observers can be obtained as an application of the contraction analysis presented in (Lohmiller and Slotine, 1998) (see also (Hartman, 1982, Chapter XIV, Part III)) when the system has a monotonicity property. Then it is sufficient to copy the system to get a global observer. Monotonicity is also exploited in (Arcak and Kokotović, 1999) where a class of systems is exhibited for which this property can be obtained by output injection (see Proposition 11).

We investigate in this note under which conditions there exists an observer for x such that :

(1) $\widehat{x}$ can be taken as the state of the dynamical system involved in the observer, i.e. the system (2) takes the special :

$$\dot{\widehat{x}} \;=\; F(\widehat{x},y) \;. \tag{3}$$

(2) the error dynamics admit a Lyapunov function depending only on the error :

$$e = \widehat{x} - x ,$$

and not on the actual state or the observer state. In the following, this Lyapunov function is denoted V.

These two requirements are restrictive. For the first requirement, we know from the nonlinear filtering theory or from observability theory, with the injective map which associates the output path $h(X(x,t))$ to the initial condition x, that in whole generality an observer should have the form (2) with $\mathcal{x}$ an infinite dimensional state (e.g. the a posteriori density probability). This requirement excludes also things like the extended Kalman filter. The second requirement is a coordinate dependent property which imposes uniformity of the convergence of the error e with respect to x.

To be more specific, we introduce the definition :

Definition 1. We say that V is a State Independent Error Lyapunov Function (SIELF) if it is C^1, positive definite and radially unbounded and there exists a C^1 function F such that we have :

$$\frac{\partial V}{\partial e}(e)\,[F(x+e,h(x)) - f(x)] < 0\,\forall x,\,\forall e \neq 0\,. \quad (4)$$

Our problem here is to find sufficient and necessary conditions on f such that a SIELF V can be found. Only few publications are devoted to Lyapunov theory for observers. There is however the results of Tsinias (see (Tsinias, 1990) for instance). But they deal only with sufficient conditions, require a global Lipschitz property for f, impose a condition on $\frac{\partial h}{\partial x}$ and consider mainly only the local case or say the semi-global case (see Proposition 8).

A necessary condition is given in Section 2. It motivates the sufficient conditions given in Section 3. Due to space limitations all the proofs are omitted.

2. A NECESSARY CONDITION.

A first property given by the existence of a SIELF is the useful following simple remark :

Lemma 2. If V is a SIELF, then there exists a C^0 function K such that :

$$\begin{aligned} F(x+e,y) \;=\; & f(x+e) \qquad (5)\\ & + K(x+e,y)\;(h(x+e)-y) \end{aligned}$$

Then our key necessary condition is :

Proposition 3. If V is a SIELF, then we have :

$$\begin{aligned} &\{h(x+e) = h(x)\,,\; e\neq 0\} \;\Rightarrow \qquad (6)\\ &\qquad \frac{\partial V}{\partial e}(e)\,[f(x+e) - f(x)] \;<\; 0 \qquad \forall x\,. \end{aligned}$$

Moreover, if V is C^2 then the matrix $\frac{\partial^2 V}{\partial e^2}(0)$ is non negative and, for all x in $\mathbb{R}^n$, we have :

$$\frac{\partial h}{\partial x}(x)\,e \;=\; 0 \;\Rightarrow\; e^T \frac{\partial^2 V}{\partial e^2}(0) \frac{\partial f}{\partial x}(x)\,e \leq 0 \;\;\forall x\,. \quad (7)$$

When strict, the necessary condition (7) is nothing but the assumption A1 in (Tsinias, 1990) invoked to get a sufficient condition (see Proposition 8 below).

Proposition 3 gives an interesting tool to prove that there is no SIELF. In particular it can be used to emphasize that a SIELF is a coordinate dependent notion. What is specifically meant by "coordinate dependent" here concerns the system :

$$\dot{x} = f(x)\;,\quad \dot{e} = F(x+e,h(x)) - f(x)$$

Because the observer can depend only on $h(x)$ and we are interested in reducing the observation error, the change of coordinates under consideration are only those which do not mix x and $\widehat{x}$ and more particularly :

$$(x,e) \;\mapsto\; (\Phi(x)\,,\,\Phi(x+e) - \Phi(x))$$

where Φ is a global diffeomorphism.

Example 4. For the system :

$$\dot{x}_1 \;=\; x_2\;,\quad \dot{x}_2 \;=\; x_2^2\;,\quad y \;=\; x_1\;, \qquad (8)$$

an observer and a SIELF can be written when using the coordinates $(x_1, x_2\exp(-x_1))$. Writing this SIELF with the original coordinates, we get a Lyapunov function depending on (e_1,e_2) but also on y. Actually there is no SIELF in the original coordinates. Indeed, if V were a SIELF, we would have, for all x_2 and $e_2 \neq 0$,

$$\frac{\partial V}{\partial e_1}(0,e_2)\,e_2 \;+\; \frac{\partial V}{\partial e_2}(0,e_2)\,[2x_2e_2 + e_2^2] \;<\; 0\;.$$

This inequality is impossible since it is linear in x_2 which is arbitrary and $\frac{\partial V}{\partial e_2}(0,e_2)$ must be non zero for some $e_2 \neq 0$. △

Another use of Proposition 3 is to get some indication on the necessary structure of a SIELF.

Example 5. Consider the following system [1] :

$$\dot{x}_1 \;=\; x_1 x_2^2\;,\quad \dot{x}_2 \;=\; -x_2\;,\quad y \;=\; x_1\;. \qquad (9)$$

We know that $\dot{\widehat{x}}_2 = -\widehat{x}_2$ is a reduced order observer. But this is not our concern here.

Instead, assume the existence of a C^2 SIELF. Then, according to Proposition 3, by denoting $\begin{pmatrix} 1 & p \\ p & q \end{pmatrix}$ the second derivative of V at the origin, we should have $p^2 \leq q$ and, for all (x_1,x_2),

[1] This example was suggested to us by Elena Panteley.

$$2\,p\,x_1\,x_2\ -\ q\ \leq\ 0\ .$$

This is possible only if $p = 0$.

So we look for a quadratic SIELF in the form :

$$V(e_1, e_2)\ =\ \tfrac{1}{2}\,e_1^2\ +\ \tfrac{q}{2}\,e_2^2\ .$$

We have a SIELF if there exist two functions k_1 and k_2 of (x_1, x_2, e_1) such that, for all (x_1, x_2) and $(e_1, e_2) \neq 0$, we have :

$$W := e_1\left(x_1 x_2^2 - [x_1 - e_1][x_2 - e_2]^2 + k_1 e_1\right) + q\,e_2\,(-e_2 + k_2 e_1)\ <\ 0\ .$$

But this W is strictly positive when e_2 is large and e_1 is such that $q + e_1[x_1 - e_1]$ is strictly negative.

However, we see that the difficulty could be overcome if we would have in W a term like $-e_2^4$. So our last try is for a SIELF in the form :

$$V(e_1, e_2)\ =\ \tfrac{1}{2}\,e_1^2\ +\ \tfrac{1}{2}\,e_2^2\ +\ \tfrac{1}{4}\,e_2^4\ .$$

In this case, we look for two functions k_1 and k_2 of (x_1, x_2, e_1) such that, for all (x_1, x_2) and $(e_1, e_2) \neq 0$, we have :

$$W := e_1\left(x_1 x_2^2 - [x_1 - e_1][x_2 - e_2]^2 + k_1 e_1\right) + e_2\left(1 + e_2^2\right)(-e_2 + k_2 e_1)\ <\ 0\ .$$

But, by using Young's inequality, we get :

$$W \leq -e_2^2 + (k_2 + 2x_2(x_1 - e_1))\,e_1 e_2 + \left([x_1 - e_1]^2 + x_2^2 + k_1 + k_2^4 e_1^3\right) e_1^2\ .$$

So our result follows by choosing :

$$k_2\ =\ -2x_2(x_1 - e_1)\ ,$$
$$k_1\ =\ -[x_1 - e_1]^2\ -\ x_2^2\ -\ k_2^4 e_1^3\ -\ 1\ . \qquad \triangle$$

To end this section, we remark that, if there exists a SIELF V, then, for each auxiliary system in the x-indexed family :

$$\dot{e}\ =\ F(x + e, h(x))\ -\ f(x)\ , \qquad (10)$$

we have global asymptotic stability of the origin $e = 0$. A trivial consequence is that all the known necessary conditions for global asymptotic stability can be invoked.

3. SUFFICIENT CONDITIONS

3.1 *General case*

In view of Proposition 3 and by analogy with the stabilization problem, we introduce the following notion :

Definition 6. A function V is an Observer Lyapunov Function (OLF) if it is C^1, positive definite, radially unbounded and satisfies (6).

In the case of stabilization, a CLF leads always to a locally bounded (at least away from the origin) controller guaranteeing global asymptotic stability in the sense of Krasovskii. We investigate here when the same holds for an OLF, i.e. when the converse of Proposition 3 is true.

An OLF is a SIELF if the implication (6) is sufficient to guarantee the existence, for each x and y, of a vector $K(x, y)$ such that, for all $e \neq 0$ satisfying $y = h(x - e)$, we have the inequality :

$$\frac{\partial V}{\partial e}(e)\,[f(x) - f(x - e) + K(x, y)\,(h(x) - y)] < 0$$

So, in particular, for each (x, e) where f does not satisfy :

$$\frac{\partial V}{\partial e}(e)\,[f(x) - f(x - e)]\ <\ 0\ ,$$

K must satisfy :

$$\frac{\partial V}{\partial e}(e)\,K(x, h(x - e))\ (h(x) - h(x - e))\ <\ 0\ .$$

This says that $\frac{\partial V}{\partial e}(e)\,K(x, h(x - e))$ must have the same sign as $(h(x - e) - h(x))$. This implies that as, e varies while x and $h(x - e)$ are kept fixed, the vector $\frac{\partial V}{\partial e}(e)^T$ must remain on the same side of the hyperplane orthogonal to $K(x, h(x - e))$. This fact is to be opposed to the property that, for each $r > 0$, the vector the vector $\frac{\partial V}{\partial e}(e)^T$ takes all the possible directions when e evolves on the sphere with radius r. So, given a function h, finding a function V such that the image by $\frac{\partial V}{\partial e}^T$ of the set $\{e : h(x - e) = y\}$ remains in a half space appears to be very difficult. In particular, going beyond the case where h is linear and V is quadratic seems to be quite involved as illustrated by the following example.

Example 7. Consider the linear system :

$$\dot{x}_1\ =\ x_2 \quad , \qquad \dot{x}_2\ =\ 0 \quad , \qquad y\ =\ x_1\ .$$

This is a linear observable system. So we know of course a linear observer with quadratic SIELF. But let us consider another possibility.

The function [2] :

$$V(e_1, e_2)\ =\ e_1^2\left(1 + e_2^4\right)\ -\ e_1\,e_2^3\ +\ e_2^2$$

is C^1, positive definite and radially unbounded. It is an OLF since we have :

$$\{e_1 = 0\,,\ e_2 \neq 0\} \quad \Rightarrow \quad \tfrac{\partial V}{\partial e_1}(0, e_2)\,e_2\ =\ -e_2^4\ <\ 0\ .$$

If V is also a SIELF, then from Lemma 2, there must exist two functions k_1 and k_2 of $((x_1 + e_1), (x_2 + e_2), e_1)$ such that, for all (x_1, x_2) and $(e_1, e_2) \neq 0$, we have :

$$W\ :=\ \frac{\partial V}{\partial e_1}\,(e_2 + k_1 e_1)\ +\ \frac{\partial V}{\partial e_2} k_2 e_1\ <\ 0\ .$$

[2] Suggestion of Jean-Michel Coron.

This function W defined this way is a polynomial of degree 5 in e_2. So once (x_1+e_1, x_2+e_2, e_1), i.e. (k_1, k_2, e_1), are fixed, we can always find e_2 making W positive. This implies that V is not a SIELF. $\triangle$

Although we have seen that the system (9) has a SIELF which cannot be quadratic, we restrict now our attention to the case of quadratic SIELF's, i.e.

$$V(e) \;=\; \tfrac{1}{2} e^T P e \tag{11}$$

and linear output maps, i.e.

$$h(x) \;=\; C^T x \; . \tag{12}$$

In this case, the necessary condition (6), when strict, turns out to be sufficient for getting a semi-global result :

Proposition 8. If the output map is linear (see (12), V is quadratic (see (11)) and we have :

$$\{C^T e = 0\,,\; e \neq 0\} \;\Rightarrow \tag{13}$$
$$e^T P \tfrac{\partial f}{\partial x}(x)\, e \;<\; 0 \quad \forall x \in \mathbb{R}^n \;,$$

then V is a quadratic SIELF on any compact subset of $\mathbb{R}^n$, i.e. for each compact subset E of $\mathbb{R}^n$, we can find a function F_E such that (4) holds for all $x \in \mathbb{R}^n$ and $e \in E$.

A very similar result is given in (Tsinias, 1990, Theorem 2) but there the assumption is more restrictively written :

$$\{C^T e = 0\,,\; e \neq 0\} \;\Rightarrow$$
$$e^T P \tfrac{\partial f}{\partial x}(x)\, e \;\leq\; -k\,|e|^2 \;<\; 0 \quad \forall x \in \mathbb{R}^n \;.$$

Also, it turns out that the linearity of the output map can be relaxed as in (Tsinias, 1990, Theorem 2), namely $y = h(x)$ with h satisfying :

$$h(x+e) - h(x) \neq 0 \;\Rightarrow$$
$$e^T \tfrac{\partial h}{\partial x}(x+e)\,[h(x+e) - h(x)] \;>\; c|e|^2 \;.$$

Unfortunately, in general, we cannot go beyond the semi-global result stated in Proposition 8. The system (9) is an example of a system admitting a quadratic OLF but no quadratic SIELF. To get globalization, we need some extra assumption. It can be boundedness of $\frac{\partial f}{\partial x}$, e.g. (Tsinias, 1993, Theorem 2.2) or homogeneity, e.g. linear systems. In the following, we shall study three different types of such assumption.

3.2 $D^T f(x)$ *is globally Lipschitz.*

In order to exhibit an extra condition implying that an OLF is a SIELF, let us make the following observation :

Let K_1 be an arbitrary vector satisfying :

$$C^T K_1 \;=\; 1 \; . \tag{14}$$

Then, on the one hand, we have, for any e in $\mathbb{R}^n$,

$$C^T \left(e - K_1 C^T e\right) \;=\; 0$$

and, on the other hand, we have the following identity :

$$f(x - K_1 C^T e) - f(x-e) \tag{15}$$
$$= \int_0^1 \frac{\partial f}{\partial x}(x - e + s(e - K_1 C^T e))ds \left(I - K_1 C^T\right) e$$

It follows that V in (11) is an OLF if and only if, for all x and $e \neq 0$ in $\mathbb{R}^n$, we have :

$$\left(e - K_1 C^T e\right)^T P \left(f(x - K_1 C^T e) - f(x-e)\right) < 0 \tag{16}$$

Then V is also a SIELF if we can find a function K_2 such that, for all x and $e \neq 0$ in $\mathbb{R}^n$, we have :

$$e^T P \left(f(x - K_1 C^T e) - f(x-e)\right) \tag{17}$$
$$+\, e^T P K_2(x, C^T e) C^T e \;<\; 0 \;.$$

To study how the OLF property (16) could imply this SIELF property, let us strengthen (16) so that we have, for some strictly positive real number ρ and all x and e in $\mathbb{R}^n$,

$$\left(e - K_1 C^T e\right)^T P \left(f(x - K_1 C^T e) - f(x-e)\right)$$
$$\leq\; -\rho \left|e - K_1 C^T e\right|^2 \;.$$

Then in (17), we restrict our choice of K_2 to :

$$K_2 \;=\; -\lambda P^{-1} C$$

with λ a strictly positive real number. This yields :

$$e^T P \left(f(x - K_1 C^T e) + K_2 C^T e - f(x-e)\right) \tag{18}$$
$$\leq\; |C^T e| \left|K_1^T P \left(f(x - K_1 C^T e) - f(x-e)\right)\right|$$
$$-\; \rho \left|e - K_1 C^T e\right|^2 \;-\; \lambda (C^T e)^2 \;.$$

It follows that (17) can be obtained by picking λ large enough if there exists a real number σ such that, for all x and e, we have :

$$\left|K_1^T P \left(f(x - K_1 C^T e) - f(x-e)\right)\right|$$
$$\leq\; \sigma \left|e - K_1 C^T e\right| \;.$$

We formalize this as follows :

Proposition 9. Assume the output map is linear, (see (12)). If we can find [3] a positive definite matrix P, a vector D and strictly positive real numbers σ and ρ such that we have :

$$C^T P^{-1} D \neq 0 \;, \quad \left|D^T \frac{\partial f}{\partial x}(x)\right| \;\leq\; \sigma \quad \forall x \in \mathbb{R}^n \;,$$

[3] Actually C, D, ρ and σ may depend on y.

and :

$$C^T e = 0 \qquad \Rightarrow \qquad e^T P \frac{\partial f}{\partial x}(x)\, e \;\leq\; -\rho\, |e|^2 \ .$$

then V in (11) is a SIELF for the system (1).

Example 10. Consider the system :

$$\dot{x}_1 = -f(x_1, x_2) \ , \quad \dot{x}_2 = f(x_1, x_2) \ , \quad y = x_1 \ ,$$

where f is a C^1 function satisfying :

$$\frac{\partial f}{\partial x_2}(x_1, x_2) \;\leq\; -\rho \;<\; 0 \quad \forall (x_1, x_2) \ .$$

Under this condition there exists a quadratic OLF. Indeed, let q satisfy $1 > q^2$. We get :

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} \begin{pmatrix} -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \; (1-q)\, \tfrac{\partial f}{\partial x_2} \;\leq\; -(1-q)\, \rho \;<\; 0 \ .$$

Then, with $D^T \;=\; \begin{pmatrix} 1 & -1 \end{pmatrix}$, we have :

$$C^T P^{-1} D \;=\; -\frac{1}{1-q} \ , \qquad \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \end{pmatrix} D \;=\; 0$$

If follows that the assumptions of Proposition 9 are met. To check here that :

$$V(e_1, e_2) \;=\; \frac{1}{2} \left(e_1^2 \;+\; 2q e_1 e_2 \;+\; e_2^2 \right)$$

is a SIELF, we consider the observer given by (actually it is constructively given in the proof of Proposition 9) :

$$F(x_1, x_2, y) = \begin{pmatrix} -f(x_1, x_2 - x_1 + y) - (x_1 - y) \\ f(x_1, x_2 - x_1 + y) + q(x_1 - y) \end{pmatrix}$$

We get successively :

$$\tfrac{1}{1-q} \begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} \times$$
$$\times \begin{pmatrix} -f(x_1, x_2 + e_2 - e_1) + f(x_1, x_2) - e_1 \\ f(x_1, x_2 + e_2 - e_1) - f(x_1, x_2) + q e_1 \end{pmatrix}$$
$$= (e_1 - e_2) \left(f(x_1, x_2 + e_2 - e_1) - f(x_1, x_2) \right) \\ -(1+q) e_1^2 \ ,$$
$$\leq \; -\rho (e_1 - e_2)^2 - (1+q) e_1^2 \ .$$

Since the right hand side is negative definite, we conclude that V is a SIELF. △

3.3 *Decomposition into scalar monotonic nonlinearities*

Following (Arcak and Kokotović, 1999) [4] , we consider the case of a linear output map (see (12)) and where f can be decomposed as :

$$f(x) \;=\; A\, x \;+\; \sum_{i=1}^{m} G_i \gamma_i (H_i^T x) \ , \qquad (19)$$

where the G_i's and H_i's are constant vectors and the γ_i's are C^1 functions. With such a decomposition, if V in (11) is a quadratic SIELF, then we have, for all x,

$$C^T e \;=\; 0 \;\Rightarrow \\ e^T P \left[A + \sum_{i=1}^{m} \gamma_i'(H_i^T x)\, G_i H_i^T \right] e \;\leq\; 0 \ ,$$

or equivalently :

$$C^T e \;=\; 0 \;\Rightarrow \\ e^T P A\, e \;+\; \sum_{i=1}^{m} \frac{\gamma_i'(H_i^T x)}{4} \left| (PG_i + H_i)^T e \right|^2 \\ - \sum_{i=1}^{m} \frac{\gamma_i'(H_i^T x)}{4} \left| (PG_i - H_i)^T e \right|^2 \;\leq\; 0 \ .$$

Conversely, we have :

Proposition 11. (Arcak and Kokotović, 1999) Assume that f and h can be written as in (19) with the derivative of the γ_i's satisfying [5] :

$$-\infty \;<\; a_i \;\leq\; \gamma_i'(s) \;<\; b_i \;\leq\; +\infty \qquad \forall s \ .$$

If we can find a positive definite matrix P, satisfying the implication [6] :

$$\left\{ C^T e = 0 \ , \ e \neq 0 \right\} \;\Rightarrow \\ e^T P A\, e \;+\; \sum_{i=1}^{m} \frac{b_i}{4} \left| (PG_i + H_i)^T e \right|^2 \qquad (20) \\ - \sum_{i=1}^{m} \frac{a_i}{4} \left| (PG_i - H_i)^T e \right|^2 \;<\; 0 \ .$$

then V in (11) is a SIELF for the system (1).

3.4 *Triangular systems*

Let us finally consider the case where the output can be taken as one state component, i.e. :

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[4] The point of view adopted here is different from the one in (Arcak and Kokotović, 1999). There the observer design is considered as an absolute stability problem.

[5] As remarked in (Arcak and Kokotović, 1999), the result still holds if γ_i depends also on $y = C^T x$ and we have :

$$-\infty \;<\; a_i \;\leq\; \frac{\partial \gamma_i}{\partial s}(s, y) \;<\; b_i \;\leq\; +\infty \qquad \forall (s, y) \ .$$

[6] (20) is to be understood as, if $b_i \;=\; +\infty$, then $(PG_i + H_i)^T e = 0$.

$$\dot{x}_1 = f_1(x_1, x_2)\ ,\ \dot{x}_2 = f_2(x_1, x_2)\ ,\ y = x_1\ , \quad (21)$$

with x_2 in $\mathbb{R}^{n-1}$. In this case, if the function :

$$V(e_1, e_2) \ = \ \frac{r}{2}\, e_1^2 \ + \ e_1\, Q^T e_2 \ + \ \frac{1}{2}\, e_2^T P_2 e_2\ , \quad (22)$$

is a quadratic OLF, then we have for all (x_1, x_2) and $e_2 \neq 0$,

$$\begin{aligned} e_2^T\, (Q[f_1(x_1, x_2 + e_2) - f(x_1, x_2)] \quad & (23) \\ +P_2[f_2(x_1, x_2 + e_2) - f_2(x_1, x_2)]) \ < \ 0 \end{aligned}$$

But following (Besançon, 2000, Corollary 3.1), we observe that, by replacing the coordinate x_2 by :

$$\chi_2 \ = \ x_2 \ + \ P_2^{-1} Q\, y\ ,$$

we get :

$$\begin{aligned} \dot{\chi} &= \overline{f}(y, \chi_2) \\ &:= f_2(y, \chi_2 - P_2^{-1} Q\, y) \quad (24) \\ &\quad + P_2^{-1} Q\, f_1(y, \chi_2 - P_2^{-1} Q\, y)\ . \end{aligned}$$

Then, the condition (23) is saying nothing but that by copying the reduced-order system (24), we get a global reduced order observer.

The condition (23) implies also :

$$e_2^T \left(Q \frac{\partial f_1}{\partial x_2}(x_1, x_2) + P_2 \frac{\partial f_2}{\partial x_2}(x_1, x_2) \right) e_2 \leq 0\ . \quad (25)$$

To obtain a converse statement, we restrict further the structure (21) in :

$$\dot{x}_1 = C_1^T x_2 + f_1(x_1)\ ,\ \dot{x}_2 = f_2(x_1, x_2)\ ,\ y = x_1\ , \quad (26)$$

and the function (22) in :

$$V(e_1, e_2) = \frac{r}{2}\, e_1^2 - q\, e_1\, C_1^T e_2 + \frac{1}{2}\, e_2^T P_2 e_2\ . \quad (27)$$

We have :

Proposition 12. For the system (26), if there exists a real number q and a positive definite matrix P_2 such that, for all (x_1, x_2), we have, for all $e_2 \neq 0$,

$$e_2^T P_2 \frac{\partial f_2}{\partial x_2}(x_1, x_2)\, e_2 \ < \ q\, (C_1^T e_2)^2\ , \quad (28)$$

then there exists a strictly positive real number r such that the function (27) is a quadratic SIELF.

A very similar result can be obtained from (Shim and Seo, 2000*b*, Theorem 1). However in this case it is also required that f_2 be globally Lipschitz. This follows from the fact that, ultimately [7], Shim and Seo restrict their attention to a classical observer in the form :

$$\begin{cases} \dot{\widehat{x}}_1 \ = \ C_1^T \widehat{x}_2 \ + \ f_1(x_1) \ + \ K_1\, (\widehat{x}_1 \ - \ x_1) \\ \dot{\widehat{x}}_2 \ = \ f_2(x_1, \widehat{x}_2) \ + \ K_2\, (\widehat{x}_1 \ - \ x_1) \end{cases}$$

[7] See (8) and the expression of v at the end of the proof of Theorem 1 in (Shim and Seo, 2000*b*).

with K_1 and K_2 constant and not functions of $(\widehat{x}_1, \widehat{x}_2, x_1)$ as considered here.

By combining Propositions 3 and 12, we get :

Corollary 13. A necessary and sufficient condition for the second order system :

$$\dot{x}_1 = x_2 + f_1(x_1)\ ,\ \dot{x}_2 = f_2(x_1, x_2)\ ,\ y = x_1\ ,$$

to admit a SIELF is that there exists a real number q such that, for all (x_1, x_2), we have :

$$\frac{\partial f_2}{\partial x_2}(x_1, x_2) \ \leq \ q\ .$$

Moreover, if this condition holds, we can always find a quadratic SIELF.

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