IGNORED INPUT DYNAMICS AND A NEW CHARACTERIZATION OF CONTROL LYAPUNOV FUNCTIONS

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Abstract

Our objective in this paper is to extend as much as possible the dissipativity approach for the study of robustness of stability in the presence of known/unknown but ignored input dynamics. This leads us to :

- give a new characterization of control Lyapunov functions (CLF) where L_fV is upper-bounded by a function of L_gV,
- define the dissipativity approach as :
  - assuming the ignored dynamics are dissipative with storage function W and (known) supply rate w,
  - analyzing closed-loop stability with the sum of the storage function W and a CLF for the nominal part.

Stability margin is given in terms of an inequality the supply rate should satisfy. Unfortunately this extension of the dissipativity approach cannot still cope with ignored dynamics which have non zero relative degree or are non minimum phase.

1 Introduction.

1.1 Problem statement

This last decade, various control designs have been proposed to deal with regulation of nonlinear systems. But they are mainly dedicated to systems of special kinds or having a peculiar structure. One way to meet such restrictions is to work with simplified model obtained for instance by neglecting well or poorly known input dynamics. This leads to the problem of (global) asymptotic stabilization of systems with ignored input dynamics.

The problem is to design a state feedback law $u = k(x)$ which globally asymptotically stabilizes the origin for the system whose dynamics are described by a nominal part :

$$\dot{x} \;=\; f(x) \;+\; g(x)\,y \tag{1}$$

with state x in $\mathbb{R}^n$. Its input y, in $\mathbb{R}^m$, may be accessed only through the system :

$$\left\{ \begin{array}{rcl} \dot{z} & = & j(z,x,u) \\ y & = & h(z,x,u) \end{array} \right. \tag{2}$$

In the control law (but not in its design), this system is ignored because it is unknown or it is known but its state z is unavailable or its dynamics are too complicated or irrelevant to the control objective.

In the nonlinear framework, besides the recent approach via disturbance estimation proposed in [10, 11], two main ways of tackling with this problem have been proposed : the dissipativity approach (see [7, 2, 14, 12] for instance) and the non linear small gain approach (see [5, 8, 4] for instance). In this paper we concentrate our attention on the former trying to extend it as much as possible.

1.2 Motivation

From the dissipativity approach, we retain :

1. the characterization of the ignored systems (2) as those for which there exists a positive definite, proper

and C^1 function W, the storage function, such that :

$$\frac{\partial W}{\partial z}(z)j(z,x,u) \leq w(u,y) \; - \; \alpha(|z|) \qquad \forall (z,x,u) \tag{3}$$

with $y = h(z,x,u)$, α a non negative continuous function and w a continuous function, called the supply rate, and the only known data on the ignored input dynamics.

2. the idea of studying the stability of the overall system via a Lyapunov function U which is the sum of W and of V, a control Lyapunov function (CLF) for the nominal part. Namely we assume the data of a positive definite, proper and C^1 function V such that :

$$\{L_gV(x) \;=\; 0\,,\; x \neq 0\} \qquad \Rightarrow \qquad L_fV(x) \;<\; 0 \tag{4}$$

and we pick :

$$U(x,z) \;=\; \psi(V(x)) \;+\; W(z) \tag{5}$$

where ψ is to be chosen as a positive definite, proper and C^1 function.

From the above, the problem studied in this paper reduces to find ψ and $u = k(x)$ so that the right hand side of :

$$\dot{U}(x,z) \;\leq\; L_f\psi(V(x)) \;+\; L_g\psi(V(x))\,y \;+\; w(u,y) \tag{6}$$

with $y = h(z,x,u)$ is non positive for all (z,x). From this we see readily that if γ is the function defined as (when it makes sense) :

$$\gamma(s) \;=\; -\inf_u \sup_y \{sy + w(u,y)\}\ , \tag{7}$$

which depends only on w, then ψ should be chosen such that :

$$L_f\psi(V(x)) \;<\; \gamma(L_g\psi(V(x))) \qquad \forall x \neq 0\ . \tag{8}$$

In section 2, we shall observe that V is a CLF if and only if for any function γ in an appropriate class, there exists a function ψ so that (8) holds. So there is no loss of generality in considering (8). With such a result, the class of admissible supply rates w is simply the one giving γ, by (7), in this appropriate class. This will be stated in Theorem 3.1 in section 3. Following our arguments, it is the broadest class we can expect by following the dissipativity approach as defined above. But we shall see that (3) and the non negativeness of γ in (7) imply that $h(0,x,u)$ must depend on u, this is a zero relative degree requirement. Also, at least in the case where u is in $\mathbb{R}$, there must exist a function λ such that :

$$w(u,y) \leq \lambda(u)y\ . \tag{9}$$

This implies that the ignored dynamics must be minimum phase. (9) is reminiscent from the input feedback passivity assumption invoked in [12] where :

$$\lambda(u) = u\ . \tag{10}$$

So an inherent limitation in the dissipativity approach is that, when the nominal part is not already open loop stable, one can handle only those ignored dynamics which are minimum phase and with zero relative degree.

Our paper is organized as follows. In Section 2, we state our necessary and sufficient condition for a Lyapunov function to be a CLF. The problem of robust stabilization in the presence of ignored input dynamics is studied in section 3.

Due to space limitation, the proofs cannot be included. They can be found in the longer version of this paper.

2 Another Characterization of Control Lyapunov Functions

Since [1] (see also [13]), it is known that the existence of a control Lyapunov function (CLF) for systems of the form

$$\dot{x} \;=\; f(x) \;+\; g(x)\,u \tag{11}$$

with x in $\mathbb{R}^n$ and u in $\mathbb{R}^m$, is equivalent to the existence of a global asymptotic stabilizer $u = k(x)$, which is C^0 on $\mathbb{R}^n \setminus \{0\}$. Here we propose another way of characterizing such CLF's.

Theorem 2.1 *Let V be a C^1, positive definite and proper function. V is a CLF for (11) if and only if, for any $\varepsilon > 0$ and for any function $\gamma \in C^0(\mathbb{R}^p, \mathbb{R}_+)$, such that :*
(i) $\gamma(0) = 0$,
(ii) for all $s \in \mathbb{R}^p \setminus \{0\}$, $\gamma(\varphi\, s)/\varphi$ is an increasing function of φ,
(iii) for all $s \in \mathbb{R}^p \setminus \{0\}$, $\lim_{\varphi \to +\infty} \gamma(\varphi\, s)/\varphi = +\infty$,
there exists a positive definite and radially unbounded function $\psi_\varepsilon \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that :
(a) the derivative ψ'_ε is strictly positive on $\mathbb{R}_+ \setminus \{0\}$,
(b) we have :

$$L_f\psi_\varepsilon(V(x)) \;<\; \gamma(L_g\psi_\varepsilon(V(x))) \qquad \forall |x| \geq \varepsilon\ . \tag{12}$$

Moreover, we can take ψ_ε independent of ε if and only if γ is such that :

$$\exists\, k \;>\; 0 \;: \qquad \limsup_{\substack{x \to 0 \\ L_gV(x) \neq 0}} \; \frac{L_fV(x)}{\gamma(k\, L_gV(x))} \;<\; \frac{1}{k}\ . \tag{13}$$

Remark : The case $\gamma(s) = k|s|^2$ was already known. It was established indirectly invoking the relation between CLF's and optimal value functions. Indeed in [12, 6], the authors prove that if V is a CLF and satisfies a local condition at the origin, discussed below, then there exists a C^1 function ψ such that $\psi(V(x))$ is the optimal value function associated to the cost functional

$$J(x) \;=\; \int_0^{\infty} [\ell(X(x,t)) + \frac{1}{4k}|u(t)|^2]dt \qquad (14)$$

with ℓ being positive definite. More precisely, $\psi(V(x))$ is a solution of the following Hamilton-Jacobi-Bellman (HJB) equation :

$$\ell(x) \;+\; L_f\psi(V(x)) \;-\; k\,|L_g\psi(V(x))|^2 \;=\; 0\;. \qquad (15)$$

Our result with $\gamma(s) = k|s|^2$ follows since, the function ℓ being positive definite, we get readily :

$$L_f\psi(V(x)) \;<\; k\,|L_g\psi(V(x))|^2 \qquad \forall x \neq 0\;. \qquad (16)$$

The local condition, mentioned above, has been stated in [6] (see also [9]) as :

$$\limsup_{x\to 0} \frac{L_fV(x)}{|L_gV(x)|^2} \;<\; +\infty\;. \qquad (17)$$

This is nothing but (13) for the case $\gamma(s) = k|s|^2$.

3 Robustness to Input Dynamics.

In this section we use the CLF characterization given in section 2 to solve the stabilization design problem stated in Introduction. More specifically, we consider the class of systems of the following form :

$$\left\{\begin{array}{rcl} \dot{x} &=& f(x) \;+\; g(x)\,y \\ \dot{z} &=& j(z,x,u) \\ y &=& h(z,x,u) \end{array}\right. \qquad (18)$$

where $x \in \mathbb{R}^n$ represents the state of the system to be controlled, $z \in \mathbb{R}^p$ represents the state of the ignored part and is not available for feedback, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ is the output of the uncertain z-subsystem and the input of the x-subsystem.

For nonlinear systems and within the dissipativity approach, the study of the margin of stability of systems in the presence of input uncertainties began by exploiting the properties of optimal controllers. Precisely, it has been established that if $u = k(x)$ is a minimizer of the cost functional :

$$J(x) \;=\; \int_0^{\infty} [\ell(X(x,t)) + r(u(t))]dt \qquad (19)$$

with r and ℓ being positive definite functions, then this control law guarantees global asymptotic stability in presence of ignored input dynamics for which the supply rate w in (3) satisfies :

$$w(u,y) \;\leq\; (y-u)r'(u) \;+\; r(u)\;. \qquad (20)$$

This is established, for instance, in [7] when r is quadratic and in [2, 14] for general r. Since it is sufficient to have an optimal control to get such a property, this leads to the question of when a control law is optimal. Such a question is addressed and solved in the nonlinear context in [7, 12] under the constraint of a quadratic r, i.e. (20) takes the form :

$$w(u,y) \;\leq\; u\,y \;-\; k\,u^2 \qquad (21)$$

and the corresponding ignored dynamics are said input feedforward passive. In particular in [12], it is established that the knowledge of a CLF satisfying (17) is sufficient to derive an optimal control law. This proves that optimal synthesis is not necessary to design a robust control law.

In this section we consider the case of a general supply rate and propose a controller design adapted to it and providing global asymptotic stability for the system (18).

3.1 Main Results

To design a control law for the system (18), we assume :

1. We know a CLF V for (11).
2. The z-subsystem satisfies the following dissipativity inequality :

$$\frac{\partial W}{\partial z}(z)j(z,x,u) \leq w(u,y) \;-\; \alpha(|z|) \qquad \forall (z,x,u) \qquad (22)$$

with $y = h(z,x,u)$, W a positive definite, proper and C^1 function, α a non negative continuous function and w a continuous function which is known for the design.

Theorem 3.1 *Assume that, for the supply rate w, there exists a continuous function π such that a function γ satisfying :*

$$\gamma(s) \;\leq\; -\sup_y\{w(\pi(s),y) + s\,y\} \qquad (23)$$

meets the properties (i) to (iii) in Theorem 2.1. Under this condition, if condition (13) in Theorem 2.1 holds, there exists[1] a function ψ and a controller :

$$u \;=\; \pi(L_g\psi(V(x))) \qquad (24)$$

which guarantees global stability of the origin for (18) and :

$$\lim_{t\to+\infty} x(t) \;=\; 0\;. \qquad (25)$$

Moreover this controller is globally asymptotically stabilizing if α is positive definite.

[1]The procedure for getting u in (24) is :
– to find γ and π satisfying (23). They depend only on w.
– to find ψ satisfying (12) with the above γ. It depends only on f, g, V and γ.

Example : Consider the following system, which is not input feedforward passive in the sense of [12],

$$\left\{ \begin{array}{rcl} \dot{x} &=& x+y \\ \dot{z} &=& -z^3+u^3 \\ y &=& u \;+\; \frac{1}{2}(u^3+z^3)^{\frac{1}{3}} \end{array} \right. \tag{26}$$

$V(x) = \frac{1}{2}x^2$ is a CLF for the nominal system $\dot{x} = x+y$ and the z-subsystem is dissipative with :

$$\overbrace{\tfrac{1}{2}z^2}^{\cdot} \;=\; -z^4 \;+\; u^3\,(8[y-u]^3-u^3)^{\frac{1}{3}} \; . \tag{27}$$

So here, a possible supply rate :

$$w(u,y) \;=\; u^3\,(8[y-u]^3-u^3)^{\frac{1}{3}} \tag{28}$$

is not in the form $(y-u)r'(u)+r(u)$ as in (20). Nevertheless, for such a function w, we have :

$$\begin{array}{l} \sup_y(sy+w(u(s),y)) \qquad\qquad (29) \\ \qquad = \;\; -2^{\frac{1}{3}}\,s^{\frac{4}{3}} \quad \text{if} \quad s+2u(s)^3=0 \; , \\ \qquad = \;\; +\infty \qquad \text{if} \quad s+2u(s)^3 \neq 0 \; . \end{array}$$

So according to the statement of Theorem 3.1, we let :

$$\pi(s) \;\;=\;\; -\left(\frac{s}{2}\right)^{\frac{1}{3}} \; , \tag{30}$$

$$\gamma(s) \;\;=\;\; 2^{\frac{1}{3}}\,|s|^{\frac{4}{3}} \; . \tag{31}$$

Then, since we have :

$$L_fV(x) \;\;=\;\; x^2 \;=\; 2V \; , \tag{32}$$

$$L_gV(x) \;\;=\;\; x \;=\; \text{sign}(x)\,\sqrt{2V} \; , \tag{33}$$

we look for a function ψ so that, for $V \neq 0$,

$$\begin{array}{l} L_f\psi(V) \;=\; 2V\psi'(V) \qquad\qquad (34) \\ \qquad < \;\gamma(L_g\psi(V)) \;=\; 2^{\frac{1}{3}}\,\psi'(V)^{\frac{4}{3}}\,(2V)^{\frac{2}{3}} \end{array}$$

This yields, for $V \neq 0$,

$$\psi'(V) \;>\; V \; . \tag{35}$$

So for instance, we choose :

$$\psi(V) \;=\; V^2 \; . \tag{36}$$

Then, according to (24), a control law is :

$$u(x) \;=\; -\left[\tfrac{1}{2}\right]^{\frac{1}{3}}\,x \; . \tag{37}$$

It provides global asymptotic stability for the system (26) but not for the nominal system

$$\dot{x} \;=\; x \;+\; u \; . \tag{38}$$

Note that the small gain design of [8] applies also for the system (26).

3.2 Discussion

3.2.1 Known results with specific $w(u,y)$ in (22)

As already mentioned the result of Theorem 3.1 is not new at least for the following two specific expressions of the supply rate w.

- $w(u,y) = uy - ku^2$, $k > 0$. This is is the case considered in [12]. For such a supply rate, $\sup_y\{w(\pi(s),y) + s\,y\}$ is finite if and only if $\pi(s) = -s$, and γ can be chosen as $\gamma(s) = k\,s^2$.
- $w(u,y) = (y-u)r'(u) + r(u)$, with r defining the cost functional (19) (see [14]). For such a supply rate, we get $\pi(s)$ as the solution of

  $$r'(\pi(s)) \;+\; s \;=\; 0 \; . \tag{39}$$

  Then (23) defines γ as :

  $$\gamma(s) \;\;=\;\; -r(\pi(s)) \;+\; \pi(s)\,r'(\pi(s)) \; . \tag{40}$$

  We observe that if $r(u) - u\,r'(u)$ is not positive, such a function γ is not appropriate for Theorem 3.1. This restriction on r is not present in [14]. It follows in our case from the fact that we adopt a worst case approach and we do not take into account that the CLF for the nominal system could be such that $L_fV(x)$ is negative for some x.

3.2.2 Stabilization of the nominal system (11)

In general, the control (24) does not stabilize the nominal system. This is definitely not a drawback in particular for the case where the ignored dynamics are well known but we do not want to take them into account in the control law.

If we insist on having (24) to stabilize the nominal system it is sufficient to have, for $x \neq 0$,

$$L_fV(x) + L_gV(x)\,\pi(L_g\psi(V(x))) \;<\; 0 \; . \tag{41}$$

Since, according to Theorem 2.1, we have :

$$\begin{array}{l} L_fV(x) \;+\; L_g\,V(x)\,\pi(L_g\psi(V(x))) \qquad\qquad (42) \\ \qquad < \;\frac{\gamma(\psi' L_g\,V(x))}{\psi'(V(x))} \;+\; L_g\,V(x)\,\pi(L_g\psi(V(x))) \; , \end{array}$$

a sufficient condition for the stability of the nominal system is that γ and π satisfy :

$$\gamma(s) \;\leq\; -s\,\pi(s) \; . \tag{43}$$

3.2.3 Zero relative degree and minimum phase

Following our approach, (23) characterizes the class of allowed supply rates for the ignored dynamics. We state here that, at least in the single input case and with a minor extra assumption, this class is contained in the class of systems with a zero relative degree and minimum phase.

To any supply rate w satisfying (23), we can associate a set-valued map $\mathcal{U}_w : \mathbb{R}^m \setminus \{0\} \rightsquigarrow \mathbb{R}^m$ defined as

$$\mathcal{U}_w(s) \;=\; \left\{ u \;:\; \sup_y \{ s\, y + w(u,y)\} \;<\; 0 \right\} . \tag{44}$$

$\mathcal{U}_w$ is said to have a continuous selection when there exists a continuous function $\pi(s) : \mathbb{R}^m \to \mathbb{R}^m$ such that $\pi(s) \in \mathcal{U}_w(s)$ for $s \neq 0$.

Theorem 3.2 *Given w, if $\mathcal{U}_w$ has a continuous selection, then the systems which admits w as supply rate:*
- are such that, for all non zero (s,x), there exist u such that $sh(0,x,u) < 0$ (a zero relative degree property),
- and, when $m = 1$, have globally stable zero dynamics (when they exist).

3.2.4 About the condition (13)

In the statement of Theorem 3.1, we impose that γ satisfies also the condition (13) concerning its behavior around 0. We illustrate why this restriction is imposed by considering the following nominal system studied in [3] :

$$\dot{x} \;=\; x^3 \;+\; x^2\, u \;. \tag{45}$$

It admits a CLF but if we pick $\gamma(s) = s^2$, there is no C^1 function V such that :

$$\frac{L_f V(x)}{\gamma(L_g V(x))} \;=\; \frac{L_f V(x)}{(L_g V(x))^2} \;=\; \frac{1}{V'(x) x} \tag{46}$$

is bounded on a neighborhood of zero, i.e. (13) cannot hold. Although the problem is only when x is small, this opens the possibility of getting unbounded solutions with ignored dynamics satisfying (22). Indeed, let :

$$\left\{ \begin{array}{rcl} \dot{z} &=& -z\, f(z) \;+\; u \\ y &=& z \;+\; u \end{array} \right. \tag{47}$$

with $f(z)$ non negative. This system satisfies (22) with the supply rate :

$$w(u,y) \;=\; u\, y \;-\; u^2 \;. \tag{48}$$

Specifically, we have :

$$\dot{\overbrace{\tfrac{1}{2} z^2}} \;=\; -z^2 f(z) \;+\; u\, y \;-\; u^2 \;. \tag{49}$$

Also, we have :

$$\inf_u \sup_y \{ w(u,y) + sy\} \;=\; -s^2 \;=\; -\gamma(s) \tag{50}$$

where $\inf_u$ is given by :

$$u \;=\; \pi(s) \;=\; -s \;. \tag{51}$$

This establishes that the condition (23) of Theorem 3.1 holds.

So all the assumptions of Theorem 3.1 and (43) are satisfied, except (13). But, it can be shown that, for instance when :

$$f(z) \;=\; \exp(z^3) \;, \tag{52}$$

there is no static feedback depending only on x which guarantees both global asymptotic stability for the nominal system and boundedness of the z-components of the solution of the overall system :

$$\left\{ \begin{array}{rcl} \dot{x} &=& x^3 \;+\; x^2\, y \\ \dot{z} &=& -z\, f(z) \;+\; u \\ y &=& z \;+\; u \end{array} \right. \tag{53}$$

It is interesting to observe however that when :

$$\gamma(s) \;=\; |s|^{(1+c)} \;, \tag{54}$$

with $c > 0$, then for a C^1 function V we have :

$$\frac{L_f V(x)}{\gamma(L_g V(x))} \;=\; \frac{L_f V(x)}{|L_g V(x)|^{(1+c)}} \;=\; |V'(x)|^{-c}\, |x|^{1-2c} \;. \tag{55}$$

There exists a C^1 function V making this ratio bounded on a neighborhood of 0 iff $c \in (0, \frac{1}{2}]$. For such c's, i.e. such γ's, Theorem 3.1 applies and gives a stability margin but which is not for supply rates in the form (48).

We end this section by noting that, when condition (13) is not satisfied, boundedness of all the solutions is achievable when the ignored dynamics have a stronger stability property, i.e. α is a class $\mathcal{K}^\infty$ function.

Theorem 3.3 *Under the conditions of Theorem 3.1, if α is a class $\mathcal{K}^\infty$ function, then there exists a controller which guarantees boundedness of all the solutions.*

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