Semiglobal Stabilization in the Presence of Minimum-Phase Dynamic Input Uncertainties*

Laurent Praly
Centre Automatique et Systèmes
École des Mines
35 Rue Saint Honoré
77305 Fontainebleau, France
praly@cas.ensmp.fr

Zhong-Ping Jiang
Department of Electrical Engineering
University of Sydney
Sydney, NSW 2006
Australia
zjiang@ee.usyd.edu.au

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Abstract

This paper presents a dynamic state feedback approach to the semiglobal stabilization of nonlinear systems with minimum-phase dynamic input uncertainties. The assumption needed to get this new result is weaker in one direction than the assumption of input feedback passivity or that of nonlinear small gain.

Key-words: nonlinear robust control; dynamic input uncertainties; semiglobal stabilization; singular perturbations; output feedback stabilization.

1 Introduction

The last decade has seen important progress made in the field of global stabilization for classes of nonlinear systems. However, it has been pointed out in [20] (see also [17, 2]) that global stability may be lost in the presence of some dynamic input uncertainties. On the other hand, it is known [9, 11] that any control law which is optimal with respect to an appropriate performance index yields a global stability property which is robust to input strictly passive dynamic uncertainties. Also, in [10, 5], it is proved that such a robustness property holds when a small gain condition is satisfied.

In this paper, we intend to prove that the minimum-phase property of the dynamic uncertainties is already sufficient. This fact is very well known for linear systems. However, for this to be true, we also need information about the sign of the so-called high-frequency gain. Here we will see how this result can be extended to the nonlinear case. In the context of linear systems, a static high-gain state feedback is sufficient to achieve the robustness property. Unfortunately, for nonlinear systems, this may not be the case and *dynamic* feedback may be needed. Also, since the problem of asymptotic stabilization in the presence of dynamic uncertainties is actually a problem of stabilization with partial-state measurement, we know from [8] that global stabilization may be impossible to achieve. We show in this paper that a dynamic feedback law can be designed to solve the semiglobal stabilization problem in spite of dynamic input uncertainties.

The paper is organized as follows: In section 2, after introducing our notations and recalling some basic definitions, we state and discuss the main assumption which describes the class of uncertain

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systems to be controlled. In Section 3, we first show that the global stabilization problem in the present situation may be unachievable and the use of a static partial-state feedback is not sufficient to solve a less ambitious but practically meaningful problem of semiglobal stabilization. This is followed by the formulation of our main results. A second-order nonlinear system, for which there is no globally asymptotically stabilizing controller, is given in Section 4 to illustrate our semiglobal dynamic feedback approach.

The results presented here are reminiscent of those in [19]. However there are differences in the way the variables are observed and with the presence of the input in the unmodelled dynamics. So we have preferred to rederive a careful and complete analysis.

2 Definitions, Systems and Assumptions

2.1 Notation and definitions

I denotes all the identity matrices while `Id` stands for the identity function from $\mathbb{R}_+$ onto $\mathbb{R}_+$. `sat` $: \mathbb{R}^m \to \mathbb{R}^m$ is a function which equals the identity function inside some compact neighborhood of the origin and whose derivative is bounded. For any differentiable mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, $\frac{\partial f}{\partial x}(x)$ stands for the $m \times n$ matrix $(\frac{\partial f_i}{\partial x_j}(x))_{m\times n}$. The vertical bars $|\cdot|$ stands for the Euclidean norm of a vector or the induced norm of a matrix. For any measurable locally essentially bounded function $u : \mathbb{R}_+ \to \mathbb{R}^m$, $\|u\|$ stands for the L^m_∞ norm of u.

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and zero at zero. It is of class $\mathcal{K}_\infty$ if, moreover, γ is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed t, $\beta(\cdot, t)$ is a class-$\mathcal{K}$ function and, for each fixed r, $\beta(r, \cdot)$ is a decreasing function and goes to zero at infinity.

The equilibrium $x = 0$ of system $\dot{x} = f(x, u)$ is said to be semiglobally practically stabilized by a feedback law $\dot{\mathcal{X}} = \nu(\mathcal{X}, x, \pi)$, $u = \mu(\mathcal{X}, x, \pi)$ if, given any compact neighborhoods $\mathcal{S}_{x1}$, $\mathcal{S}_{x2}$ of $x = 0$ with the property that $\mathcal{S}_{x1} \subset \mathcal{S}_{x2}$, there exists a compact set $\mathcal{S}_{\mathcal{X}}$ and a parameter value π such that all the solutions of the closed-loop system starting from $\mathcal{S}_{x2} \times \mathcal{S}_{\mathcal{X}}$ reach $\mathcal{S}_{x1} \times \mathcal{S}_{\mathcal{X}}$ in finite time.

The equilibrium $x = 0$ of system $\dot{x} = f(x, u)$ is said to be semiglobally asymptotically stabilized by a feedback law $\dot{\mathcal{X}} = \nu(\mathcal{X}, x, \pi)$, $u = \mu(\mathcal{X}, x, \pi)$ if for each compact neighborhood $\mathcal{S}_x$ of $x = 0$, there exists a compact set $\mathcal{S}_{\mathcal{X}}$ and a parameter value π such that the origin of the closed-loop system is asymptotically stable with basin of attraction containing $\mathcal{S}_x \times \mathcal{S}_{\mathcal{X}}$.

2.2 System and assumptions

The class of systems we consider is of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)y \\ \dot{z} &= q(z, x, u) \\ y &= h(z, x, u) \end{aligned} \tag{1}$$

where x in $\mathbb{R}^n$ denotes the state of the certain part, z in $\mathbb{R}^p$ is the state of the uncertain part (i.e. not available for feedback design), u in $\mathbb{R}^m$ is the control input, y in $\mathbb{R}^m$ is the output of the uncertain z-subsystem and the input of the certain x-subsystem.

In this paper, we restrict ourselves with the case where the relative degree of y with respect to u is zero. More precisely,

Assumption 1 *(Relative degree zero) We have*[1]*:*

$$h(z, x, u) = u - h_1(z, x, u) \tag{2}$$

[1] May be after rescaling g.

where there is a constant $0 < \varepsilon < 1$ *so that*

$$\left|\frac{\partial h_1}{\partial u}(z,x,u)\right| \;\le\; 1 \;-\; \varepsilon\ , \quad \forall\,(z,x,u) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m\ . \tag{3}$$

Thanks to this Assumption 1, we know that the relative degree from u to $y = u - h_1(z,x,u)$ is zero since

$$\frac{\partial y}{\partial u} \;=\; I \;-\; \frac{\partial h_1}{\partial u}(z,x,u) \tag{4}$$

is a globally invertible matrix. In the linear case, $\frac{\partial y}{\partial u}$ is often referred to as the high-frequency gain. From (3), its dominant part is the identity matrix and therefore we know the "sign" of this high-frequency gain.

Note that we have not assumed that h_1 is zero at the equilibrium point of interest. This implies that an integral action may be needed to achieve regulation.

Assumption 2 *(Minimum-phase) There exist a* C^1 *positive definite, radially unbounded function* W*, a class-*$\mathcal{K}_\infty$ *function* α *and a class-*$\mathcal{K}$ *function* γ *such that*

$$\frac{\partial W}{\partial z}(z)q(z,x,u) \;\le\; -\alpha(|z|) \;+\; \gamma(|(x,y)^T|) \tag{5}$$

According to the terminology in [15], this Assumption 2 says that the z-system is input/output-to-state stable (IOSS) when x is considered as its input and y as its output. Another interpretation of this condition is that if u is computed as the solution of

$$y \;=\; u \;-\; h_1(z,x,u)$$

which is guaranteed to exist from Assumption 1, then the z-system in (1) with this u as input is input-to-state stable (ISS) (see [12]) whenever instead (x,y) is viewed as input. This means that we assume the inverse dynamics of the z-system with output y in (1) is input-to-state stable.

The known results on global stabilization in spite of dynamic input uncertainties are of two kinds:

- those characterized in terms of input feedback passivity.
- those characterized in terms of small gain.

2.2.1 Input feedback passivity

Lemma 1 ([11]) *Consider the system in (1). Assume there is a positive definite and radially unbounded function* W*, a positive definite function* α *and a strictly positive real number* λ *such that*

$$\frac{\partial W}{\partial z}(z)q(z,x,u) \;\le\; -\alpha(|z|) \;+\; y^T u \;-\; \lambda u^T u \tag{6}$$

Then, if the x*-system with* y *as input is globally asymptotically stabilizable, then the whole system (1) is globally practically*[2] *stabilizable by a feedback depending only on* x*.*

If we strengthen (6) by imposing that α be a class-$\mathcal{K}_\infty$ function, then the input feedback passivity condition (6) implies Assumption 2 since

$$\frac{\partial W}{\partial z}(z)q(z,x,u) \;\le\; -\alpha(|z|) \;+\; \frac{1}{4\lambda}y^T y \tag{7}$$

Example 1 *[11]* Consider the system

$$\begin{aligned} \dot z &= -z \;+\; z^2 u \\ y &= u \;+\; c_1 z^3\ , \end{aligned} \tag{8}$$

The input feedback passivity condition (6) and Assumption 2 hold when $c_1 > 0$.

[2] To get actually asymptotic stability an extra smoothness property at the origin is needed (cf. [11, Section 3.5.3]).

2.2.2 Nonlinear small gain

Lemma 2 *[10] Consider the system in (1). Assume the existence of class-$\mathcal{KL}$ functions β_z and β_y and class-$\mathcal{K}$ functions γ_{zx}, γ_{zu}, γ_{yx} and γ_{yu} such that, for any initial condition $z(0)$, we have, for all t,*

$$
\begin{aligned}
|z(t)| &\leq \beta_z(|z(0)|, t) + \gamma_{zx}(\|x\|) + \gamma_{zu}(\|u\|) \qquad (9)\\
|y(t) - u(t)| &\leq \beta_y(|z(0)|, t) + \gamma_{yx}(\|x\|) + \gamma_{yu}(\|u\|) \qquad (10)
\end{aligned}
$$

where

$$\mathtt{Id} - \gamma_{yu} \in \mathcal{K}_\infty . \qquad (11)$$

Then if the x-system with y as input is globally asymptotically stabilizable, then the whole system (1) is globally asymptotically stabilizable by a feedback depending only on x.

Like the input feedback passivity condition (6), the small gain conditions (9), (10) and (11) imply (5). This can be seen by considering the interconnection of a dynamic system with input (u, x), state z and output $y - u$ with a static system with input $(y, y - u)$ and output u. This interconnection is well posed if there exists a unique solution $u = u^*(x, y, z)$ to equation $y = h(z, x, u)$. Then the condition $\mathtt{Id} - \gamma_{yu}$ being of class-$\mathcal{K}_\infty$ guarantees that the small gain theorem [6, Theorem 2.1] applies. It follows that the z-system with (x, y) as input is ISS. The existence of a function W satisfying (5) is then a consequence of [14, Theorem 1].

Example 2 For the following system

$$
\begin{aligned}
\dot{z}_1 &= -z_1 + u \\
\dot{z}_2 &= z_1^3 - z_2^3 \qquad (12)\\
y &= u + c_1 z_2
\end{aligned}
$$

the small gain condition (9), (10) and (11) is satisfied only for $|c_1| < 1$ and the input feedback passivity condition (6) does not hold[3] if $c_1 \neq 0$. But Assumptions 1 and 2 hold for $c_1 > -1$ since

$$
\begin{aligned}
\dot{z}_1 &= -z_1 - c_1 z_2 + y \\
\dot{z}_2 &= z_1^3 - z_2^3 \qquad (13)\\
y &= u + c_1 z_2
\end{aligned}
$$

and (5) holds with

$$W(z_1, z_2) = \frac{a}{4} z_1^4 + z_2^2 \qquad (14)$$

where $a = 3$ for $c_1 \in (-1, 5/3)$ and $a = 1/c_1$ for $c_1 \in [5/3, +\infty)$.

Note that the small gain condition (9) does not hold for system (8) which is not ISS.

2.2.3 Minimum phase

The above examples show that Assumption 2 is more general than the input feedback passivity hypothesis (6) or the small gain condition (9), (10) and (11). In fact, in some sense, Assumption 2 is the weakest assumption that can be imposed to the uncertain dynamics irrespective of the certain part. This can be seen with the following example inspired from the one studied in [20].

[3] Pick $u = \eta z_2$, with η small and opposite sign to c_1. This gives $yu = \eta(\eta + c_1)\, z_2^2$. And observe that z_2 converges so slowly that $\int_0^\infty z_2(t)^2 dt = +\infty$. This can be seen from the fact that there is a center manifold $z_1 = \eta z_2 + (\eta - \eta^4) z_2^3 + O(z_2^4)$ (see [1, Theorem 2.3]), i.e. there exists initial conditions $(z_1(0), z_2(0))$ so that $\dot{z}_2 = -(1 - \eta^3) z_2^3 + O(z_2^4)$.

Example 3 Consider the system

$$\begin{aligned} \dot{x} &= c_1|x| + y \\ \dot{z} &= -z + u \\ y &= u - c_2 z \end{aligned} \tag{15}$$

For $|c_2| < 1$, the small gain condition (9), (10) and (11) holds. For $c_2 < 1$, the input feedback passivity condition (6) and Assumption 2 hold. But for $c_2 \geq 1$ the origin is not asymptotically reachable in the case where we have the following coupling condition between the certain and the uncertain parts

$$c_1 \geq c_2 - 1 \ . \tag{16}$$

Indeed, we have

$$\dot{\overbrace{x - z}} = [c_1|x| - (c_2 - 1)x] + [c_2 - 1]\,[x - z] \tag{17}$$

So $c_2 \geq 1$ and $c_1 \geq c_2 - 1$ imply that the set $x - z \geq 1$ for instance is invariant.

On the other hand, if the input feedback passivity hypothesis and the small gain condition imply Assumption 2, they are already sufficient to get the stabilizability property, whereas here we shall require also Assumption 1.

Example 4 Consider the system

$$\begin{aligned} \dot{z} &= -z + u \\ y &= u + c_1 z \sin(u^2) \end{aligned} \tag{18}$$

For $|c_1| < 1$, the input feedback passivity hypothesis, the small gain condition and Assumption 2 hold. But Assumption 1 does not hold.

3 Counterexamples and Main Results

3.1 Static feedback is not sufficient

As mentioned earlier, the problem of global asymptotic stabilization for the system (1) is a problem of stabilization by output feedback, the state z being unmeasured. Restricting ourselves with static output feedback is already a severe limitation for linear systems. Nevertheless, for linear systems with minimum-phase linear dynamic input uncertainties, asymptotic stabilization can be achieved by means of high-gain static output feedback when Assumption 1 holds – see [16].

Such a result does not hold in our more general nonlinear context as shown by the following system:

$$\begin{aligned} \dot{z} &= z^3 - 2uz^2 \\ \dot{x} &= y \\ y &= u - z \end{aligned} \tag{19}$$

Assumption 1 holds and since we have

$$\dot{z} = -z^3 - 2z^2 y \tag{20}$$

so does Assumption 2 with $W(z) = \frac{1}{2}z^2$, $\alpha(r) = \frac{1}{4}r^4$ and $\gamma(r) = 4r^4$. However it is shown in Appendix A that semiglobal stabilization is impossible if we restrict ourselves with feedback of the form

$$u = -k(x) \ . \tag{21}$$

We conclude that within the context of Assumptions 1 and 2 the class of static time invariant feedbacks of partial-state x is not rich enough. From this point we could proceed with time-varying static feedback. Indeed it is known from linear systems theory that such class of feedback laws allows us to round some difficulties. Here instead, we shall propose dynamic feedback.

3.2 Global asymptotic stabilization may be impossible.

Although the above example gives a hope for global asymptotic stabilization via different types of controllers other than static feedback, the following example shows that global asymptotic stability may be unachievable due to an intrinsic obstruction of finite escape time. More specifically, as shown in [8], in the case of output feedback stabilization, global asymptotic stability may be impossible when the escape in very short time for the z-component is possible. For instance, this is the case for the system

$$\begin{array}{rcl} \dot{z} & = & -z \,+\, z^4 u \\ \dot{x} & = & y \\ y & = & u + z \end{array} \tag{22}$$

Assumptions 1 and 2 hold with $W(z) = \frac{1}{2}z^2$, $\alpha(r) = r^2 + \frac{1}{6}r^6$ and $\gamma(r) = \frac{1}{6}r^6$. However, the problem of unobservable unboundedness may occur since the solutions of $\dot{z} = z^4$ blow up so fast that the solutions of $\dot{x} = z$ remain bounded. We can even draw a stronger conclusion that there is no *time-varying* dynamic feedback depending on x allowing us to guarantee global asymptotic stability of the origin for system (22) (see Appendix B for a proof). Nevertheless, we show in this paper that system (22) is semiglobally stabilizable – see Section 4 below.

We are now ready to state the problem we solve in this paper.

Problem : For the system (1), design a dynamic state feedback depending on x and guaranteeing the semiglobal asymptotic stability of an equilibrium of interest.

3.3 Main results

For our solution to the semiglobal stabilization problem stated above, we need two other assumptions:

Assumption 3 *(Stabilizability) There is a continuous function ϑ such that the origin is a globally asymptotically stable equilibrium point of $\dot{x} = f(x) + g(x)\vartheta(x)$.*

This Assumption says that we know how to stabilize the system (1) whenever there is no uncertainty. With this information in mind, it is natural to choose the controller as

$$u \;=\; \vartheta(x) \;+\; v$$

where v remains to be designed to counteract the effect of input uncertainty. Noticing that the system (1) can be written as

$$\dot{x} \;=\; f(x) \;+\; g(x)u \;-\; g(x)h_1(z,x,u)\;, \tag{23}$$

our idea is to construct an observer to approximate the uncertain nonlinearity h_1 and to use this observation $\widehat{h}_1$ to design the additional control effort v, that is, $v = \widehat{h}_1$.

To design such an observer, we need an extra assumption concerning the fact that g can be globally rectified in a general sense. Namely,

Assumption 4 *(Rectifiability) We know a C^1 mapping $l : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\frac{\partial l}{\partial x}(x)g(x) \;=\; a(x)I \tag{24}$$

where $a(x) \geq \delta_a > 0$.

With Assumption 4 we are ready to introduce our observer. Let

$$\sigma = h_1(z,x,u) + Ll(x) \tag{25}$$

with L a design parameter to be tuned. Noticing that the time derivative of σ satisfies

$$\dot{\sigma} \;=\; \dot{h}_1 + L\frac{\partial l}{\partial x}(x)f(x) + La(x)u - La(x)(\sigma - Ll(x)) \tag{26}$$

we choose the following observer (where $\dot{h}_1$, unknown, is not introduced):

$$\dot{\widehat{\sigma}} \;=\; L\frac{\partial l}{\partial x}(x)f(x) + La(x)u - La(x)(\widehat{\sigma} - Ll(x)) \tag{27}$$

Denote

$$\widehat{h}_1(\widehat{\sigma},x) = \widehat{\sigma} - Ll(x)\ . \tag{28}$$

Based on this observation $\widehat{h}_1$ that approximates h_1, our first choice of controller should be as (see (33) below):

$$u \;=\; \vartheta(x) \;+\; \widehat{h}_1(\widehat{\sigma},x) \tag{29}$$

Letting

$$e = \widehat{\sigma} - \sigma\ , \tag{30}$$

in view of (26) and (27), we have:

$$\dot{e} \;=\; -La(x)e \;-\; \dot{h}_1 \tag{31}$$

Since a is lower bounded by a strictly positive constant, the idea is to choose L large enough so that the effect of the unknown function $\dot{h}_1$ can be neglected. Unfortunately, in doing so, we observe that the initial value

$$\widehat{h}_1(0) = \widehat{\sigma}(0) - Ll(x(0)) \tag{32}$$

grows with L. This means that $\widehat{h}_1(0)$ makes no sense. To disregard the effect of such bad initial condition, we follow the lines of [3] (see also [19]) and introduce a saturation. This leads to the final form of our feedback law:

$$\begin{aligned} \dot{\widehat{\sigma}} &= L\frac{\partial l}{\partial x}(x)f(x) + La(x)u - La(x)(\widehat{\sigma} - Ll(x)) \\ u &= \vartheta(x) \;+\; \mathtt{sat}(\widehat{\sigma} - Ll(x)) \end{aligned} \tag{33}$$

Our semiglobal results are stated as follows.

Theorem 1 (Practical stabilization) *Under Assumptions 1 to 4, for any compact set Ω in $\mathbb{R}^{n+p+m}$, there is a function* `sat` *such that, for all sufficiently large L, the closed-loop system (1) and (33) admits the origin as a practically stable equilibrium point with basin of attraction containing Ω.*

Theorem 2 (Asymptotic stabilization) *Under the conditions of Theorem 1, if the matrices*

$$\frac{\partial f}{\partial x}(0) + g(0)\frac{\partial \vartheta}{\partial x}(0) \quad \text{and} \quad \frac{\partial q}{\partial z}(0) + \frac{\partial q}{\partial u}(0)\left(I - \frac{\partial h_1}{\partial u}(0)\right)^{-1}\frac{\partial h_1}{\partial z}(0) \tag{34}$$

are asymptotically stable, then the system (1) is semiglobally asymptotically stabilized by (33).

Remark. An interpretation of stability of the matrices in (34) is that the nominal x-system is locally exponentially stable (LES) and the inverse dynamics $\dot{z} = q(z,0,u)$ with u defined by $u = h_1(z,0,u)$ is LES at $z = 0$.

Before proving these theorems, we first write out the closed-loop dynamics via appropriate coordinates. A good candidate for this purpose is the coordinates (z, x, e). Unfortunately, when working with the system of coordinates (z, x, e), instead of $(z, x, \widehat{\sigma})$, we find that the control input u is now implicitly defined as (see (25), (30) and (33))

$$u = \vartheta(x) + \mathtt{sat}(e + h_1(z, x, u)) \tag{35}$$

This problem can be overcome if the function $\mathtt{sat}$ is chosen appropriately, as shown in the following.

Lemma 3 *Under Assumption 1, for any function* $\mathtt{sat}$ *whose derivative is dominated by one, i.e.*

$$\left|\frac{\partial \mathtt{sat}}{\partial s}(s)\right| \leq 1 , \quad \forall s \in \mathbb{R}^m \tag{36}$$

there exists a unique C^1 *function* $\Phi_{\mathtt{sat}}(e, \theta, x, z)$ *solving the equation*

$$u = \theta + \mathtt{sat}(e + h_1(z, x, u)) \tag{37}$$

Proof. For each fixed (e, θ, x, z), consider the mapping

$$\mathcal{T}(u) = \theta + \mathtt{sat}(e + h_1(z, x, u)) \tag{38}$$

From (3) and (36), it follows that $\mathcal{T}$ is a contraction. The proof of Lemma 3 is completed by application of the Contraction Mapping Theorem and by noticing that $\mathcal{T}$ is C^1 with respect to (e, θ, x, z). □

Thanks to this Lemma 3, the control law (33) can be rewritten as:

$$u = \Phi_{\mathtt{sat}}(e, \vartheta(x), x, z) \tag{39}$$

Denote $h_{\mathtt{sat}}$ the function defined as

$$h_{\mathtt{sat}}(e, x, z) = h_1(z, x, \Phi_{\mathtt{sat}}(e, \vartheta(x), x, z)) \tag{40}$$

Then the closed-loop system (1), (33) can be expressed as

$$\begin{aligned} \dot{z} &= q(z, x, \Phi_{\mathtt{sat}}(e, \vartheta(x), x, z)) \\ \dot{x} &= f(x) + g(x)\left[\vartheta(x) + \mathtt{sat}\left(h_{\mathtt{sat}}(e, x, z) + e\right) - h_{\mathtt{sat}}(e, x, z)\right] \end{aligned} \tag{41}$$

$$\dot{e} = -La(x)e - \dot{h}_{\mathtt{sat}} \tag{42}$$

where

$$\dot{h}_{\mathtt{sat}} = \frac{\partial h_{\mathtt{sat}}}{\partial x}\dot{x} + \frac{\partial h_{\mathtt{sat}}}{\partial z}\dot{z} + \frac{\partial h_{\mathtt{sat}}}{\partial e}\dot{e} \tag{43}$$

To get an explicit expression in (42), we note that (40) and (35) give:

$$\frac{\partial h_{\mathtt{sat}}}{\partial e} = \frac{\partial h_1}{\partial u}\frac{\partial \Phi_{\mathtt{sat}}}{\partial e} \tag{44}$$

and

$$\frac{\partial \Phi_{\mathtt{sat}}}{\partial e} = \frac{\partial \mathtt{sat}}{\partial s}\left(I + \frac{\partial h_{\mathtt{sat}}}{\partial e}\right) \tag{45}$$

It follows that

$$\frac{\partial h_{\mathtt{sat}}}{\partial e} = \left(I - \frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right)^{-1}\frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s} \tag{46}$$

Note that the matrix inversion makes sense because of Assumption 1 and (36). Similarly, we establish

$$\frac{\partial h_{\mathtt{sat}}}{\partial x} = \left(I - \frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right)^{-1}\left(\frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial u}\frac{\partial \vartheta}{\partial x}\right) \tag{47}$$

$$\frac{\partial h_{\mathtt{sat}}}{\partial z} = \left(I - \frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right)^{-1}\frac{\partial h_1}{\partial z} \tag{48}$$

In these expressions, it follows from (40) that the partial derivatives of h_1 are evaluated at $(z, x, \Phi_{\mathtt{sat}})$. Since $\frac{\partial \mathtt{sat}}{\partial s}$ is bounded, it follows from (35) and (39) that, if sat is also a bounded function, the partial derivatives of h_1 in (46), (47) and (48) are bounded functions of variable e. This fact will be used in getting (70).

Now we can write the $\dot{e}$-equation (42) in explicit form as:

$$\begin{aligned}\dot{e} \;=\; & -La(x)\left(I - \frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right)e \\ & - \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial u}\frac{\partial \vartheta}{\partial x}\right)\left(f(x) + g(x)\left[\vartheta(x) + \mathtt{sat}\left(h_{\mathtt{sat}}(e,x,z) + e\right) - h_{\mathtt{sat}}(e,x,z)\right]\right) \\ & + \frac{\partial h_1}{\partial z} q(z, x, \Phi_{\mathtt{sat}}(e, \vartheta(x), x, z))\end{aligned} \tag{49}$$

Finally we recall that we have to study the solutions of the closed-loop system (1), (33) with initial condition $(x(0), z(0), \widehat{\sigma}(0))$ in the given compact set Ω. Since we have

$$e = \widehat{\sigma} - h_1(x, z, [\vartheta(x) + \mathtt{sat}(\widehat{\sigma} - Ll(x))]) - Ll(x)\ , \tag{50}$$

if sat is a bounded function, then there exists some positive real number b such that

$$|x(0)| \leq b \quad , \qquad |z(0)| \leq b \quad , \qquad |e(0)| \leq b(1+L) \tag{51}$$

So we are led to study the solutions of the system (41), (49) with initial conditions satisfying (51).

Proof of Theorem 1. To study the system (41), (49), we could try to check if the assumptions of [19, Lemma 2.4] are satisfied. We prefer a more direct and simple path. For this we follow the lines proposed by Alberto Isidori [4] for another proof of [18]. Our proof is comprised of two steps. In the first step, we do a Lyapunov analysis, define the function sat and derive some useful inequalities. In the second step, we prove that the solutions are ultimately bounded.

Step a. – Lyapunov analysis : Let us first consider the system (41) when the identity Id is used as a sat function. In this case, we have

$$u = \Phi_{\mathtt{Id}}(e, \vartheta(x), x, z) \tag{52}$$

and, when $e = 0$, this gives

$$y = \vartheta(x) \tag{53}$$

This leads to

$$\{\mathtt{sat} = \mathtt{Id}\ ,\ e = 0\} \quad \Longrightarrow \quad \left\{\begin{array}{rcl} \dot{z} & = & q(z, x, \Phi_{\mathtt{Id}}(0, \vartheta(x), x, z)) \\ \dot{x} & = & f(x) \;+\; g(x)\vartheta(x) \end{array}\right. \tag{54}$$

where from Assumption 2 we have

$$\frac{\partial W}{\partial z}(z) q(z, x, \Phi_{\mathtt{Id}}(0, \vartheta(x), x, z)) \;\leq\; -\alpha(|z|) \;+\; \gamma(|(x, \vartheta(x))^T|) \tag{55}$$

Since the x-subsystem is globally asymptotically stable, it follows from [13], that the origin is a global asymptotically stable equilibrium point of (54). From a classical Lyapunov Converse Theorem (see, e.g., [7], [10, Prop. 13]), we deduce the existence of a C^1 positive definite and radially unbounded function V such that its time derivative along the solutions of (54) satisfies:

$$\dot{V}_{(54)}(x,z) \;\leq\; -V(x,z) \tag{56}$$

This function V will be the main tool of our analysis.

First we remark that there exists a strictly positive real number c such that

$$\{|x| \leq b\,,\ |z| \leq b\} \qquad \Longrightarrow \qquad (x,z) \in \{(x,z):\ V(x,z) \leq c\} \tag{57}$$

This leads us to define the set

$$\Delta := \{(e,x,z):\ |e| \leq 1\,,\ V(x,z) \leq c+2\} \tag{58}$$

on which we shall do some estimation. Before this, let us define the function $\mathtt{sat}$:
Let h_1^* be the positive real number given by

$$h_1^* \;=\; \sup_{(z,x,z)\in\Delta} |h_{\mathtt{Id}}(e,x,z)+e| \tag{59}$$

where $h_{\mathtt{Id}}$ is obtained from $h_{\mathtt{sat}}$ by choosing the identity as a $\mathtt{sat}$ function. With this in hand, the function $\mathtt{sat}$ which we will use in the sequel can be any C^1 bounded function such that

$$\mathtt{sat}(s) \;=\; s \quad \text{if} \quad |s| \leq h_1^*; \quad \text{and} \quad \left|\frac{\partial \mathtt{sat}}{\partial s}(s)\right| \leq 1 \quad \forall\, s \in \mathbb{R}^m \tag{60}$$

For instance, we can take[4]

$$\mathtt{sat}(s) \;=\; \begin{cases} s & \text{if} \quad |s| \leq h_1^* \\ \dfrac{s}{|s|}\left((|s|-h_1^*)(1+h_1^*-|s|)+h_1^*\right) & \text{if} \quad h_1^* \leq |s| \leq h_1^*+0.5 \\ \dfrac{s}{|s|}\left(0.25+h_1^*\right) & \text{if} \quad |s| \geq h_1^*+0.5 \end{cases} \tag{61}$$

We make the following important observation:

$$(e,x,z) \in \Delta \quad \Longrightarrow \quad |h_{\mathtt{sat}}(e,x,z)+e| \;\leq\; h_1^* \tag{62}$$

Indeed, letting $u_{\mathtt{Id}} = \Phi_{\mathtt{Id}}(e,\vartheta(x),x,z)$, it follows from the definitions of $h_{\mathtt{Id}}$ and h_1^* that

$$(e,x,z) \in \Delta \quad \Longrightarrow \quad |h_1(z,x,u_{\mathtt{Id}})+e| \leq h_1^* \tag{63}$$

Thus, $\mathtt{sat}(h_1(z,x,u_{\mathtt{Id}})+e) = h_1(z,x,u_{\mathtt{Id}})+e$. As a consequence of uniqueness, $u_{\mathtt{Id}}$ is the solution of equation (35) when $(e,x,z) \in \Delta$. Therefore, $h_{\mathtt{sat}}(e,x,z) = h_1(z,x,u_{\mathtt{Id}})$ on Δ and the claim (62) follows from (63).

With the definition we have proposed for $\mathtt{sat}$, we have, from (41) and (62),

$$(e,x,z) \in \Delta \quad \Longrightarrow \quad \begin{cases} \dot{z} &= \; q(z,x,\Phi_{\mathtt{Id}}(e,\vartheta(x),x,z)) \\ \dot{x} &= \; f(x) \;+\; g(x)(\vartheta(x)+e) \end{cases} \tag{64}$$

[4] From a practical point of view it is important to remark that the function $\mathtt{sat}$ is completely defined by the data of a single parameter.

Then by comparing (54) and (64) and using the fact that the functions in question are C^1, with (56), we obtain

$$\dot{V}_{(41)} \;\leq\; -V(x,z) \;+\; \eta(e,x,z)|e| \tag{65}$$

where η is some continuous nonnegative function. In particular, by defining

$$\eta_1 \;=\; \sup_{(e,x,z)\in\Delta} \eta(e,x,z)$$

which is independent of L, we have

$$(e,x,z)\in\Delta \quad\Longrightarrow\quad \dot{V}_{(41)} \;\leq\; -V \;+\; \eta_1|e| \tag{66}$$

This inequality is valid in particular only if $|e|\leq 1$. But thanks to the fact that $\mathtt{sat}$ is a bounded function another estimation can be obtained for $\dot{V}_{(41)}$. Indeed, it follows from Lemma 3 and (40) that the functions $\Phi_{\mathtt{sat}}$ and $h_{\mathtt{sat}}$ are bounded functions of e. Hence, from (41) and the properness of V, there exists a positive constant η^* such that

$$V(x,z)\leq c+2 \quad\Longrightarrow\quad |\dot{x}|+|\dot{z}| \;\leq\; \eta^* \;, \quad \forall\, e\in \mathbb{R}^m \tag{67}$$

We conclude that there exists a positive real number η_2 (independent of L) such that

$$V(x,z)\leq c+2 \quad\Longrightarrow\quad \dot{V}_{(41)} \;\leq\; \eta_2 \;. \tag{68}$$

Let us finally consider the $\dot{e}$-equation (49). From (60) and Assumptions 1 and 4, it holds:

$$a(x)\left(I-\frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right) \;+\; a(x)\left(I-\frac{\partial h_1}{\partial u}\frac{\partial \mathtt{sat}}{\partial s}\right)^T \;\geq\; 2\delta_a\varepsilon \tag{69}$$

Therefore, from (49), the boundedness of $\mathtt{sat}$ and by completing the squares, we get the existence of two positive real numbers η_3 and η_4 (independent of L) such that

$$V(x,z)\leq c+2 \quad\Longrightarrow\quad \dot{\overbrace{e^T e}}_{(49)} \;\leq\; -2(\delta_a\varepsilon L-\eta_3)e^Te \;+\; \eta_4 \tag{70}$$

Step b. – Practical stability : Consider any solution of (41), (49) defined on the open set $\{(e,x,z):\ e\in\mathbb{R}^m\,,\ V(x,z)<c+2\}$ with the initial condition satisfying (since $\mathtt{sat}$ is bounded)

$$V(x(0),z(0))\leq c \quad\text{and}\quad |e(0)|\leq b(1+L)\;. \tag{71}$$

Such a solution is well defined on a right maximal interval $[0,T_c)$. We show in the sequel that $T_c=+\infty$.

If T_c were finite, (70) would imply that

$$\lim_{t\to T_c} V(x(t),z(t)) = c+2\;. \tag{72}$$

Let us show that this is impossible when L is large enough. Denote $T_1=1/\eta_2$. From (68) and (71), we know that

$$V(x(t),z(t)) \;\leq\; c \;+\; 1 \quad \forall t\in[0,T_1] \tag{73}$$

It follows that $T_c>T_1$. Using (70) and (71), we have

$$|e(t)| \;\leq\; \exp(-(\delta_a\varepsilon L-\eta_3)t)b(1+L) \;+\; \sqrt{\frac{\eta_4}{2(\delta_a\varepsilon L-\eta_3)}} \qquad \forall t\in[0,T_c) \tag{74}$$

So, there exists a positive constant L_1^* such that

$$L \geq L_1^* \quad \Longrightarrow \quad |e(t)| \ \leq \ \min\left\{1 \ , \ \frac{c+1}{\eta_1}\right\} \quad \forall t \in [T_1, T_c) \tag{75}$$

with η_1 involved in (66). For such values of L, the solution is in Δ for t in $[T_1, T_c)$. So it follows from (66) and (73) that

$$V(t) \ \leq \ c \ + \ 1 \quad \forall t \in [T_1, T_c) \tag{76}$$

This together with (74) implies that $T_c = +\infty$ and that the closed-loop solution is bounded and remains in the open set $\{(e, x, z) : e \in \mathbb{R}^m \ , \ V(x, z) < c + 2\}$.

In fact, for any $\rho > 0$, there exist $T_2 > 0$ and $L_2^* > 0$ so large that

$$L \geq L_2^* \quad \Longrightarrow \quad \max\{V(x(t), z(t)) \ , \ |e(t)|\} \ \leq \ \rho \quad \forall \, t \in [T_2, +\infty) \tag{77}$$

Indeed, let

$$\rho^* \ = \ \min\left\{\rho \ , \ \frac{\rho}{2\eta_1}\right\} \quad , \quad L_2^* \ = \ \max\left\{L_1^* \ , \ \frac{(\eta_4/{\rho^*}^2) + \eta_3}{\varepsilon \delta_a}\right\} \tag{78}$$

From (74), it is seen that, for all $L \geq L_2^*$, there exists $T_3 > 0$ such that

$$|e(t)| \ \leq \ \rho^* \quad \forall \, t \geq T_3 \tag{79}$$

Hence, in view of (66) and (76), we deduce the existence of some time instant $T_2 \geq T_3$ such that

$$|V(x(t), z(t))| \ \leq \ \rho \quad \forall \, t \geq T_2 \tag{80}$$

which completes the proof of Theorem 1. □

Next we prove Theorem 2 on the asymptotic stability of the closed-loop system (41),(49) at the origin.

Proof of Theorem 2. Around the origin the system (41),(49) can be written as[5]

$$\begin{aligned} \dot{z} &= Q_z z \ + \ Q_x x \ + \ Q_e e \ + \ O_z(e, x, z) \\ \dot{x} &= F x \ + \ G(\Theta x + e) \ + \ O_x(e, x, z) \\ \dot{e} &= -L a(x) \left(I - \frac{\partial h_1}{\partial u} \frac{\partial \mathtt{sat}}{\partial s}\right) e \ + \ H_x x \ + \ H_z z \ + \ H_e e \ + \ O_e(e, x, z) \end{aligned} \tag{81}$$

where O_z, O_x and O_e are continuous functions independent of L and satisfying

$$\limsup_{|e|+|x|+|z| \to 0} \frac{|O(e, x, z)|}{|e|^2 + |x|^2 + |z|^2} < \infty \tag{82}$$

and, in particular,

$$F = \frac{\partial f}{\partial x}(0) \ , \ G = g(0) \ , \ \Theta = \frac{\partial \vartheta}{\partial x}(0) \ , \ Q_z = \frac{\partial q}{\partial z}(0) + \frac{\partial q}{\partial u}(0) \frac{\Phi_{\mathtt{sat}}}{\partial z}(0) \tag{83}$$

with using $\frac{\partial \mathtt{sat}}{\partial s}(0) = I$,

$$\frac{\partial \Phi_{\mathtt{sat}}}{\partial z}(0) \ = \ \frac{\partial h_1}{\partial z}(0) \ + \ \frac{\partial h_1}{\partial u}(0) \frac{\partial \Phi_{\mathtt{sat}}}{\partial z}(0) \tag{84}$$

[5]We do not look simply at the linearization since its domain of validity may vanish with L going to infinity.

It follows by assumption that Q_z and $F + G\Theta$ are asymptotically stable matrices satisfying

$$P_z Q_z \;+\; Q_z^T P_z \;=\; -I \tag{85}$$

$$P_x(F + G\Theta) \;+\; (F + G\Theta)^T P_x \;=\; -I \tag{86}$$

for some positive definite symmetric matrices P_z and P_x. We have also (69).

To establish the asymptotic stability of the null solution of system (81), consider the positive definite function

$$V_l(e, x, z) \;=\; c_z z^T P_z z \;+\; x^T P_x x \;+\; \frac{1}{2} e^T e \tag{87}$$

where c_z is a positive constant to be made precise later on. From (82), there exist strictly positive real numbers d_1 and d_2 (independent of L) such that

$$V_l(e, x, z) \leq d_1 \quad \Longrightarrow \quad \frac{\partial V_l}{\partial e} O_e + \frac{\partial V_l}{\partial x} O_x + \frac{\partial V_l}{\partial z} O_z \leq d_2\, V_l^{\frac{3}{2}} \tag{88}$$

Then, by completing the squares, the time derivative of V_l along the solutions of (81) satisfies, for $V_l \leq d_1$,

$$\dot{V}_l \;\leq\; -\frac{1}{8} c_z |z|^2 \;-\; \left(\frac{1}{4} - 2c_z |P_z Q_x|^2\right) |x|^2 \tag{89}$$

$$-\; \left(\varepsilon \delta_a L - 4c_z |P_z Q_e|^2 - 2|P_x G|^2 - |H_x|^2 - 2|H_z|^2 - |H_e|\right) |e|^2 \;+\; d_2\, V_l^{\frac{3}{2}}$$

As a result, by picking $0 < c_z < 1/(8|P_z Q_z|^2)$ and $L_3^* > 0$ sufficiently large, there exists $d_3 > 0$ such that, for all $L \geq L_3^*$, we have

$$V_l(e, x, z) \leq d_1 \quad \Longrightarrow \quad \dot{V}_l \;\leq\; -\left[d_3 - d_2\, V_l^{\frac{1}{2}}\right] V_l \tag{90}$$

It follows that the origin is asymptotically stable with basin of attraction containing the set

$$\{(e, x, z) :\; V_l(e, x, z) \leq \min\{d_1 \,,\, (d_3/2d_2)^2\}\}$$

But by picking ρ small enough in (77), we know that for all $L \geq L_2^*$ all the solutions issued from Ω reaches in finite time and remain hereafter in this set. This proves that the origin is asymptotically stable with domain of attraction containing Ω. □

4 An Example

We use the elementary example (22) to illustrate our semiglobal approach.

Example 5

$$\begin{aligned} \dot{z} &= -z \;+\; z^4 u \\ \dot{x} &= y \\ y &= u + z \end{aligned} \tag{91}$$

As shown in Appendix B, this system (91) fails to be globally asymptotically stabilizable by dynamic partial-state feedback.

Nevertheless, the conditions of Theorem 2 hold for system (91). Following the control design scheme presented in this paper, system (91) is semiglobally asymptotically stabilized by a dynamic feedback of partial-state x of the form

$$\begin{aligned} \dot{\widehat{\sigma}} &= -L\widehat{\sigma} \;+\; Lu \;+\; L^2 x \\ u &= -x \;+\; \mathtt{sat}(\widehat{\sigma} - Lx) \end{aligned} \tag{92}$$

where $\mathtt{sat}$ is a C^1 bounded function satisfying (60).

5 Concluding Remarks

We addressed the robust stabilization problem for nonlinear systems in the presence of minimum-phase dynamic input uncertainties. We first showed that a system in our class may be impossible to be globally asymptotically stabilizable. Then we proposed a dynamic feedback semiglobal method to achieve semiglobal practical stabilization and, under additional conditions, semiglobal asymptotic stabilization. The present framework extends former work on the basis of passivity and nonlinear small-gain arguments.

In this paper the z-dynamics are presented as uncertain. But the same hold if these dynamics are known but either z is unmeasured or its dynamics is too complicated to be taken into account explicitly in the control law.

One drawback of the present result is that the "high frequency gain", or at least a good approximation of it, is supposed to be known. We are working on relaxing this assumption.

Appendix

A A counterexample to semiglobal stabilization by static time invariant feedback

Consider the system

$$\begin{aligned} \dot z &= z^3 - 2uz^2 \\ \dot x &= y \\ y &= u - z \end{aligned} \tag{93}$$

Assume, for each $c > 0$, the existence of a continuous static time invariant feedback $k_c(x)$ such that

$$u = -k_c(x) \tag{94}$$

makes the origin in $\mathbb{R}^2$ asymptotically stable with basin of attraction containing the set

$$\{(x,z): \ |x| \le c \,, \ |z| \le c\} \ .$$

We first observe that $z = 0$ is an invariant manifold. So, looking at the $\dot x$-equation, we conclude that $k_c(x)$ must satisfy

$$xk_c(x) \ > \ 0 \,, \quad 0 < |x| \le c \ . \tag{95}$$

Then, we have

$$\begin{aligned} \dot{\overbrace{zx}} &= -k_c(x)z - z^2 + xz^3 + 2k_c(x)z^2x \\ &= z\Big(-k_c(x) + xz^2 + 2k_c(x)zx\Big) - z^2 \end{aligned} \tag{96}$$

It follows from (95) that

$$\{zx \ge 2 \,, \ |x| \le c\} \quad \Longrightarrow \quad \dot{\overbrace{zx}} \ \ge \ zx\left(3\frac{|k_c(x)|}{|x|} + z^2\right) \ - \ z^2 \ > \ 0 \tag{97}$$

and

$$\{zx \ge 2 \,, \ x = c\} \quad \Longrightarrow \quad \dot x \ = \ -k(c) - z \ < \ -\frac{2}{c} \tag{98}$$

This proves that $\{(x,z): \ zx \ge 2 \,, \ x \le c\}$ is an invariant set. Therefore, all solutions issued from this set cannot converge to the origin. In other words, the possibility of achieving semiglobal stabilization via a continuous static time invariant feedback $k_c(x)$ is excluded because for $c \ge \sqrt{2}$, the set

$$\{(x,z): \ |x| \le c \,, \ |z| \le c\}$$

cannot be contained in the domain of attraction of the closed-loop system (93) with $u = k_c(x)$.

B A counterexample to global stabilization

To prove that the system (22) is not globally asymptotically stabilizable by *any* continuous dynamic feedback depending on x, it suffices to prove the following fact.

Lemma 4 *Consider the system*

$$\begin{aligned} \dot z &= -z \ + \ z^4\mu(x,\chi) \\ \dot x &= z + \mu(x,\chi) \\ \dot \chi &= \nu(x,\chi) \end{aligned} \tag{99}$$

where μ and ν are two continuous functions. For any μ and ν, there are solutions of (99) that do not converge to the origin.

Proof. We may assume that there exists a point $(x^*, \mathcal{X}^*)$ such that $\mu(x^*, \mathcal{X}^*) > 0$. Otherwise, pick $x(0) < 0$ and $z(0) = 0$. Then, $z(t) = 0$ and $x(t) \leq x(0)$ for all $t \geq 0$ which proves the Lemma.

Let $\mathcal{C}$ be a compact neighborhood of $(x^*, \mathcal{X}^*)$ with a non empty interior set $\text{int}(\mathcal{C})$ and such that

$$0 \ < \ \mu_{\min} \ \leq \ \mu(x, \mathcal{X}) \ \leq \ \mu_{\max} \ , \quad \forall (x, \mathcal{X}) \in \mathcal{C} \tag{100}$$

We have:

$$(x, \mathcal{X}) \in \mathcal{C} \quad \Longrightarrow \quad -z + z^4 \mu_{\min} \ \leq \ \dot{z} \ \leq \ -z + z^4 \mu_{\max} \tag{101}$$

and therefore

$$\left\{(x, \mathcal{X}) \in \mathcal{C} \ , \quad z \geq \max\left\{(2/\mu_{\min})^{\frac{1}{3}}, \mu_{\max}\right\}\right\} \quad \Longrightarrow \quad \left\{ \begin{array}{l} z^4 \dfrac{\mu_{\min}}{2} \ \leq \ \dot{z} \ \leq \ z^4 \mu_{\max} \\ z + \mu(x, \mathcal{X}) \ \leq \ 2z \end{array} \right. \tag{102}$$

It follows that [8, Lemma 3] applies. This implies the existence of z^* such that solutions issued from $(z^*, x^*, \mathcal{X}^*)$ escape in finite time. □

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