NONLINEAR RESCALING OF CONTROL LAWS WITH APPLICATION TO STABILIZATION IN THE PRESENCE OF MAGNITUDE SATURATION

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Abstract: Motivated by some recent results on the stabilization of homogeneous systems, we present a gain-scheduling approach for the stabilization of non-linear systems. Given a one-parameter family of stabilizing feedbacks and associated Lyapunov functions, we show how the parameter can be rescaled as a function of the state to give a new controller. We apply this approach to the problem of stabilization with magnitude limitations. For this problem, we develop a design method for single-input controllable systems with eigenvalues in the left closed plane.

Keywords: gain scheduling, global stabilization, bounded control, nonlinear feedback

1. INTRODUCTION

The problem of stabilization with control limitations is crucial in many applications while, even for otherwise linear systems, it cannot be solved with standard linear techniques. For controllable linear systems subject to magnitude limitations on the inputs, globally asymptotically stabilizing feedbacks exist if and only if the open loop system has no eigenvalues in the open right plane. Under this assumption, several design methods have recently been developed (see e.g. (Teel, 1992; Sussmann *et al.*, 1994; Teel, 1995; Megretski, 1996; Lin, 1996) and the references therein). Although different in their approach and characteristics, all these methods rely on some kind of gain scheduling, i.e. the control can be viewed as a linear feedback with gains converging to zero as the norm of the states tends to infinity.

Recently, in (M'Closkey and Murray, 1997), a rescaling method has been developed to transform a smooth feedback (stabilizing a driftless control system of homogeneous vector fields) into a homogeneous feedback yielding exponential stability (this latter property cannot be obtained with smooth feedback when the linearization of the system is not controllable). A similar approach was also developed independently in (Praly, 1997) for more general forms of homogeneity. In this paper, we extend this approach to general systems, i.e. not necessarily homogeneous. The main application that we consider is to the problem of stabilization with magnitude limitations. For single-input linear controllable systems, we design bounded feedbacks which ensure global stabilization of the controlled system. In this case, the controller is just a one-parameter family of linear

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controllers, with the parameter properly scaled as a function of the state. This gives a rather simple controller and requires very little on-line computation: only the scaling parameter is not explicitly defined as a function of the state. Our approach can be compared with (Teel, 1995; Megretski, 1996; Lin, 1996) in the sense that we also use a monotonic family of Lyapunov functions. Because we only require these functions to be non-increasing along the trajectories of the controlled system, we can find an explicit family of Lyapunov functions and more explicit control laws. As a counterpart, taking into account the magnitude limitations is much harder. Also, in the special case of a chain of integrators, our family of controllers is basically the same as that used by (Lauvdal and Murray, 1997). This suggests a way to extend (Lauvdal and Murray, 1997).

The paper is organized as follows. A motivating example is treated in Section 2. We present in Section 3 the main result on the rescaling of control law. In Section 4 we apply this result to the stabilization of single-input linear systems with control limitations. An example is treated in Section 5. The proofs, and other developments of this approach, can be found in (Morin *et al.*, 1997).

The following notation will be used. For any matrix M, M_i denotes the i-th row of M, and M^i the upper left minor of order i. For any vector $(d_1, \ldots, d_n)$ in $\mathbb{R}^n$, $\text{Diag}(d_i)$ denotes the diagonal matrix with d_i as (i,i)-th entry. Finally, $\mathbb{R}_+$ denotes the set of strictly positive real numbers.

2. MOTIVATING EXAMPLE

Consider the following system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \,, \end{cases} \tag{1}$$

and any linear stabilizing controller $u(x) = -a_1 x_1 - a_2 x_2 \quad (a_1, a_2 > 0)$. We want to find a bounded globally asymptotically stabilizing feedback for (1). For any $\lambda > 0$, the controller

$$u(\lambda, x) = -\frac{a_1}{\lambda^2} x_1 - \frac{a_2}{\lambda} x_2 \tag{2}$$

is a stabilizing feedback for (1), and the function

$$V(\lambda, x) = \frac{a_1}{\lambda^4} x_1^2 + \frac{1}{\lambda^2} x_2^2 \tag{3}$$

is non-increasing along the trajectories of the controlled system (1)-(2). We note that the rescaling of u and V is nonlinear in λ. Its particular form is due to the homogeneity properties of the system. Consider the equation $V(\lambda, x) = 1$. For any $x \neq 0$, this equation has a unique positive solution

$$\lambda = [\frac{x_2^2 + (x_2^4 + 4a_1 x_1^2)^{\frac{1}{2}}}{2}]^{\frac{1}{2}} . \tag{4}$$

Consider now the feedback $u(\lambda(x), x)$ with $u(\lambda, x)$ defined by (2) and $\lambda(x)$ defined by

$$\lambda(x) = \begin{cases} 1 & \text{if } V(1,x) \leq 1 \,, \\ \text{the solution (4)} & \text{otherwise} . \end{cases} \tag{5}$$

We claim that this feedback is bounded and ensures global asymptotic stability of (1). The boundedness is easily verified from (2) and (4). The asymptotic stability of the closed loop system relies on the following fact. Since $\frac{\partial V}{\partial \lambda} < 0$ for any $x \neq 0$ and $\lambda > 0$, and since for $V(1,x) > 1$, $V(\lambda(x), x) = 1$, we obtain by differentiating this last equality that for $V(1,x) > 1$,

$$\frac{\partial \lambda}{\partial x} \dot{x} = -[\frac{\partial V}{\partial \lambda}]^{-1} \frac{\partial V}{\partial x} \dot{x} \leq 0 . \tag{6}$$

This implies that the proper function $\lambda(.)$ is non-increasing along the trajectories of the controlled system, and this is sufficient to imply asymptotic stability of the controlled system. Hence, by properly "scaling" the family of linear controllers (2), we obtain a bounded globally asymptotically stabilizing feedback for (1). Based on homogeneity properties, this approach was generalized to any chain of integrators in (Praly, 1997). The main contribution of this paper, is to show that we can generalize it to any single-input null-controllable system. The following section provides the general tool to do it.

3. RESCALING OF CONTROL LAWS

Consider a control system

$$\dot{x} = f(x, u) \qquad f \in C^1(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n) \,, \tag{7}$$

with a one-parameter family of control laws $u(\lambda, ., .)$ and Lyapunov functions candidates $V(\lambda, ., .)$, $(\lambda \in \mathbb{R})$.

Assumption: There exists an interval $\Lambda = [\lambda_0, +\infty)$ (or $(\lambda_0, +\infty)$) in $\mathbb{R}$ such that:

A. For any $\lambda \in \Lambda$, the feedback law $u(\lambda, x, t)$ makes the origin of the system (7) globally asymptotically stable.

B. For any $\lambda \in \Lambda$, the function $V(\lambda, x, t)$ is non-increasing along the trajectories of (7) controlled by $u(\lambda, x, t)$.

C. $u, V \in C^0(\Lambda \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$, $V(\lambda, ., .) \in C^1(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ for any $\lambda \in \Lambda$, and both u and V are T-periodic with respect to t, piecewise C^1 and everywhere left and right differentiable with respect to λ. For any $(\lambda, t) \in \Lambda \times \mathbb{R}$, $V(\lambda, ., t)$ is positive definite, proper, and vanishes at the origin. For any $(\lambda, t) \in \Lambda \times \mathbb{R}$, $u(\lambda, 0, t) = 0$.

With this assumption we shall define a function $\lambda(x,t)$ which is equal to λ_0 at $x = 0$, and is such that the feedback $u(\lambda(x,t), x, t)$ is still asymptotically stable for the system (7). More precisely, we have the following result which extends (M'Closkey and Murray, 1997, Th. 4).

Theorem 1. Suppose that:

1. For any (x,t), $\lim V(\lambda, x, t) = 0$ as λ tends to $+\infty$, and $\lim V(\lambda, x, t)$ exists in $[0, +\infty]$ as λ tends to λ_0, so that we can define a partition (E_0, E_1) of $\mathbb{R}^n \times \mathbb{R}$ by:
$$E_0 = \{(x,t) : \lim_{\lambda \longrightarrow \lambda_0} V(\lambda, x, t) \leq 1\},$$
$$E_1 = \{(x,t) : \lim_{\lambda \longmapsto \lambda_0} V(\lambda, x, t) > 1\}.$$
2. $\lambda_0 \notin \Lambda \Longrightarrow E_0 = \{0\} \times \mathbb{R}$.
3. $V(\lambda, x, t) = 1 \Longrightarrow \frac{\partial^+}{\partial \lambda} V(\lambda, x, t) < 0$ and $\frac{\partial^-}{\partial \lambda} V(\lambda, x, t) < 0$, with $\frac{\partial^+}{\partial \lambda}$ and $\frac{\partial^-}{\partial \lambda}$ the right and left derivatives with respect to λ.

Then,

i) For any $(x,t) \in E_1$, the equation
$$V(\lambda, x, t) = 1 \qquad (8)$$
has a unique solution $\lambda \in \Lambda$.

ii) The function λ, with $\lambda(x,t)$ defined by
$$\begin{cases} \lambda_0 & (x,t) \in E_0\,, \\ \text{the solution of (8)} & (x,t) \in E_1\,, \end{cases} \qquad (9)$$
is C^0, Lipschitz continuous on $\mathbb{R}^n \times \mathbb{R}$ (resp. on $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}$) if $\lambda_0 \in \Lambda$ (resp. if $\lambda_0 \notin \Lambda$), and T-periodic with respect to t.

iii) If $\lambda_0 \in \Lambda$, the feedback law $u(x,t)$ defined by
$$u(x,t) = \begin{cases} 0 & x = 0\,, \\ u(\lambda(x,t), x, t) & x \neq 0\,, \end{cases} \qquad (10)$$
is C^0 and makes the origin of the system (7) globally asymptotically stable. If $\lambda_0 \notin \Lambda$, $u(x,t)$ makes the origin of (7) globally asymptotically stable provided that all solutions are well defined.

Remarks: 1. The main assumption in this Theorem is Assumption 3 introduced in (M'Closkey and Murray, 1997) in a different way as a "transversality condition".

2. If $\lambda_0 \in \Lambda$, the feedback law (10) is continuous since both $(x,t) \longmapsto \lambda(x,t)$ and $(\lambda, x, t) \longmapsto u(\lambda, x, t)$ are continuous, and since $u(\lambda, 0, t) \equiv 0$ (Assumption C). If $\lambda_0 \notin \Lambda$, we cannot guarantee in general that the feedback law (10) is continuous at $x = 0$. However, continuity can often be obtained. For instance, in this framework, Theorem 1 has nice applications to the stabilization of homogeneous systems. We shall not pursue here on this topic, but we refer the interested reader to (Morin *et al.*, 1997).

3. When $\lambda_0 \in \Lambda$, (9) and (10) imply that the the "λ-constant" feedback $u(\lambda_0, x, t)$ is applied in a neighborhood of the origin (more precisely, in $E_0 = \{(x,t) : V(\lambda_0, x, t) \leq 1\}$) whereas the "$\lambda$-varying" feedback $u(\lambda(x,t), x, t)$ is applied outside this set. In this case, a possible application of Theorem 1 is to the problem of stabilization with control limitations, where one wants to satisfy some nominal/optimal behavior close to the equilibrium point, and re-scale the controller when saturation problems may occur. This application is now discussed.

4. STABILIZATION WITH CONTROL LIMITATIONS

In this section, we consider the problem of stabilization with control limitations of the form $|u| \leq M$. We consider a single-input linear controllable system:
$$\dot{x} = Ax + bu\,. \qquad (11)$$

We assume throughout this section that A is in companion form and $b = (0, \ldots, 0, 1)^T$.

The design of the control laws is presented bellow. First, we construct a family of controllers [2] $u(\lambda, .)$ for the system (11) together with a family of functions $V(\lambda, .)$. In particular, these controllers and functions satisfy Assumptions A, B, and C of Section 3. They are also endowed with many degrees of freedom. Then, by fixing some of these degrees of freedom, we show how to fulfill the three assumptions of Theorem 1. Therefore we obtain a non-linear stabilizing feedback for (11). Finally, we show that this feedback is bounded, and how to modify this bound. Note that our family of function $u(\lambda, .)$ will be explicitly defined. Therefore, the sole on-line computation shall consist in solving the implicit equation (8). This is to be compared with the results in (Teel, 1995; Megretski, 1996; Lin, 1996) where heavier on-line computations have to be performed.

4.1 *The families $u(\lambda, .)$ and $V(\lambda, .)$*

The design of these families is based on the properties of the so-called "Schwartz matrices". Some of these properties are recalled here. The reader can consult (Morin and Samson, 1997) for additional properties and applications.

[2] Throughout the rest of this paper, and by contrast with the general result of Section 3, all functions will be autonomous

Definition 1. Let $s \in \mathbb{R}^n$. The "Schwartz matrix" associated with s is defined by

$$S(s) = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ -s_1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -s_2 & 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -s_{n-2} & 0 & 1 \\ 0 & 0 & 0 & 0 & -s_{n-1} & -s_n \end{pmatrix}$$

We recall some properties of Schwartz matrices.

Lemma 1. Let $u(x) = Kx$ be any linear stabilizing feedback for (11). Then, there exist a vector $s \in \mathbb{R}^n$ with $s_i > 0\,(i = 1, \dots, n)$, and a linear change of coordinates $x \longmapsto y = \xi(s)x$ which transforms the controlled system (11) into

$$\dot{y} = S(s)y\,. \tag{12}$$

Conversely, for any vector $s \in \mathbb{R}^n$ with $s_i > 0\,(i = 1, \dots, n)$, there exists a linear change of coordinates $y \longmapsto x = \psi(s)y$ which transforms the system (12) into the asymptotically stable system

$$\begin{aligned} \dot{x} &= Ax + bK(s)x\,, \\ K(s) &= \psi_n(s)S(s)\psi^{-1}(s) - A_n\,. \end{aligned} \tag{13}$$

Moreover,

i) The function $x^T\xi^T(s)D(s)\xi(s)x$, with

$$D(s) = \mathrm{Diag}(\prod_{k=i}^{n-1} s_k)\,, \tag{14}$$

is non-increasing, and tends to zero, along the trajectories of (13).

ii) $\psi(s)$ and $\xi(s)$ are lower triangular matrices such that $\psi_{i,j}(s) = \xi_{i,j}(s) = 0$ for any $j \notin I_i \triangleq \{j \in \mathbb{N} : j \leq i \text{ and } i - j \text{ is even}\}$ and,

$$\begin{cases} \psi_{i,i}(s) = 1\,, \\ \psi_{i,1}(s) = -s_1\psi_{i-1,2}(s) \quad (i \text{ odd}, i > 1)\,, \\ \psi_{i,j}(s) = \psi_{i-1,j-1}(s) - s_j\psi_{i-1,j+1}(s) \quad \text{otherwise}\,, \end{cases}$$

$$\begin{cases} \xi_{i,i}(s) = 1\,, \\ \xi_{i,1}(s) = s_{i-2}\xi_{i-2,1}(s) \quad (i \text{ odd}, i > 1)\,, \\ \xi_{i,j}(s) = \xi_{i-1,j-1}(s) + s_{i-2}\xi_{i-2,j}(s) \quad \text{otherwise}\,. \end{cases}$$

Remarks: 1. Given a stabilizing controller $u(x) = Kx$ for (11), there is a systematic way to compute the vector s such that the controlled system is transformed into (12): the components of s are obtained from the first column of the Routh table (see e.g. (Chen, 1984, Sec. 8-3,8-5), (Morin and Samson, 1997)).

2. It is shown in (Sarma *et al.*, 1968) that there exists a change of coordinates which transforms the control system (11) into a system in Schwartz form $\dot{y} = S(s)y + bu$. If A is not Hurwitz, the vector s is not unique in the sense that this transformation exists for any s such that the characteristic polynomial of $S(s)$ is equal to the characteristic polynomial of A.

Lemma 1 provides us with families $u(\lambda, .)$ and $V(\lambda, .)$. Indeed, let $\Lambda = [1, +\infty)$ and consider any vector valued function $s : \Lambda \longmapsto \mathbb{R}^n_+$ and any function $k : \Lambda \longmapsto \mathbb{R}_+$ with s and k being C^0, piecewise C^1 and everywhere left and right differentiable. We define the family $u(\lambda, .)$ by

$$\begin{aligned} u(\lambda, x) &\triangleq K(\lambda)x \\ K(\lambda) &\triangleq \psi_n(s(\lambda))S(s(\lambda))\xi(s(\lambda)) - A_n\,, \end{aligned} \tag{15}$$

and the family $V(\lambda, .)$ by

$$V(\lambda, x) \triangleq k(\lambda)\, x^T\xi^T(s(\lambda))D(s(\lambda))\xi(s(\lambda))x\,.$$

Then, it immediately follows from Lemma 1 that Assumptions A, B, and C of Section 3 are satisfied for the system (11). Therefore, for any choice of s and k such that the assumptions of Theorem 1 are satisfied, the controller $u(\lambda(x), x)$ with $u(\lambda, x)$ defined by (15), and $\lambda(x)$ defined by

$$\begin{cases} 1 & V(1, x) \leq 1\,, \\ \text{the solution of } V(\lambda, x) = 1 & \text{otherwise}\,, \end{cases} \tag{16}$$

ensures global asymptotic stability of (11).

4.2 *The stability conditions*

In this section we provide a specific vector-valued function s and a function k in order to fulfill the assumptions of Theorem 1.

First, (see Remark 2 in Section 4.1) the system (11) can be rewritten, after some possible change of coordinates $x \longmapsto \tilde{x}$ into the Schwartz representation:

$$\dot{\tilde{x}} = S(c_0)\tilde{x} + bu\,, \tag{17}$$

with $c_0^T =$

$$(\underbrace{0, \dots, 0}_{m}, \underbrace{\omega_1^2, 0, \dots, \omega_p^2, 0}_{2p}, \underbrace{c_{0,m+2p+1}, \dots, c_{0,n}}_{q})$$

m is the number of 0 eigenvalues, $2p$ is the number of pure imaginary eigenvalues, and $q = n - m - 2p$ is the number of eigenvalues with strictly negative real part. Each $\pm j\omega_i$ is an imaginary eigenvalue of A, and the $c_{0,k} > 0\,(k = m + 2p + 1, \dots, n)$ are associated with the stable part of A.

The functions k and s_i are defined by:

$$k(\lambda) = \begin{cases} \eta\lambda^{-2\beta} & \lambda \in [1, \lambda_1] \\ \eta\lambda_1^{2(1-\beta)}\lambda^{-2} & \lambda > \lambda_1 , \end{cases} \quad (18)$$

$$s_i(\lambda) = \begin{cases} \frac{c_{1,i}}{\lambda^{2\tau_{1,i}}} & \lambda \in [1, \lambda_1] \\ \frac{c_{2,i}}{\lambda^{2\tau_{2,i}}}(\frac{\lambda_1}{\lambda_2})^{2\tau_{2,i}} & \lambda > \lambda_1 . \end{cases} \quad (19)$$

The various parameters in (18) and (19) are to be chosen as follows:

$\eta > 0$ is used to modify the bound satisfied by the control law.

$\lambda_1 > 1$.

$c_{1,i} > 0$ ($i = 1, \ldots, n$). These coefficients define the (linear) controller applied in a neighborhood of the origin.

The parameters $\tau_{1,i}$ are defined in order to ensure the continuity of the functions s_i at $\lambda = \lambda_1$. More precisely, we define

$$\tau_{1,i} = \frac{1}{2\ln\lambda_1}(\ln c_{1,i} - \ln c_{2,i} + 2\tau_{2,i}\ln\lambda_2) .$$

The parameters $\tau_{2,i}$ are chosen to guarantee the boundedness of the controller, and also to ensure in part that Assumption 3 of Theorem 1 is satisfied,

$$\tau_{2,i} = \begin{cases} 0 & c_{0,i} \neq 0 \\ \frac{1}{2} & c_{0,i} = 0, \, i = n \\ 1 + \sum_{k=i+1}^{n-1} \tau_{2,k} & c_{0,i} = 0, \\ & m - 1 \leq i \leq n - 1 \\ \tau_{2,m-1} & c_{0,i} = 0, \, i \leq m - 2 . \end{cases} \quad (20)$$

The parameters $c_{2,i}$ are also chosen in order to guarantee Assumption 3 (for $\lambda \geq \lambda_1$). One can show that this assumption is equivalent to

$$D(\lambda)[-R + \lambda\frac{\partial\xi}{\partial\lambda}(\lambda)\psi(\lambda)] < 0 , \quad (21)$$

with $R = \mathrm{Diag}(r_i)$, $r_i = 1 + \sum_{k=i}^{n-1} \tau_{2,i}$. To get (21), we choose:

$$\begin{cases} c_{2,i} = c_{0,i} & \text{if } c_{0,i} \neq 0 \\ c_{2,i} > 0 & \text{if } c_{0,i} = 0 \text{ and } m - 1 < i , \end{cases} \quad (22)$$

and for $i = 1, \ldots, m-1$, $c_{2,i}$ is recursively defined as any solution of the following LMI in $\frac{1}{c_{2,i}}$:

$$C^{i+1}G^{i+1} = \begin{pmatrix} C^iG^i & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{c_{2,i}}\begin{pmatrix} 0 & 0 & 0 \\ C_{i,i}G_{i+1,1} & \ldots & C_{i,i}G_{i+1,i+1} \end{pmatrix} < 0 \quad (23)$$

with

$$C = \mathrm{Diag}(\prod_{k=1}^{i-1} \frac{1}{c_{2,k}}) , \quad G = -R + [\frac{\partial\xi(\bar{s}(\bar{\lambda}))}{\partial\bar{\lambda}}\psi(\bar{s}(\bar{\lambda}))]_{|\bar{\lambda}=1} , \quad \bar{s}(\bar{\lambda}) = (\frac{c_{2,1}}{\bar{\lambda}^{2\tau_{2,1}}}, \ldots, \frac{c_{2,n}}{\bar{\lambda}^{2\tau_{2,n}}}) . \quad (24)$$

All parameters have been specified but β and λ_2. These must be chosen in accordance with the following proposition.

Proposition 1. For any choice of the above parameters, Assumptions 1 and 2 of Theorem 1 are satisfied and,

i) There exists $\bar{\lambda}_2$ such that for $\lambda_2 > \bar{\lambda}_2$, Assumption 3 of Theorem 1 is satisfied for $\lambda \in [\lambda_1, +\infty)$. In particular, if all eigenvalues of A are zero (i.e., for a chain of integrators), Assumption 3 is satisfied for any $\lambda_2 > 0$.

ii) For any λ_2, there exists $\beta(\lambda_2)$ such that for $\beta > \beta(\lambda_2)$, Assumption 3 of Theorem 1 is satisfied for $\lambda \in [1, \lambda_1]$.

Remarks: 1. When the control parameters are chosen as indicated in Proposition 1, Theorem 1 applies to yield the stabilizing controller (15). This controller has the following characteristics. On the set $E_0 = \{x : V(1, x) \leq 1\}$, the function $\lambda(.)$ is identically equal to 1. Therefore, the controller (15) is linear on this set. Moreover, in view of (19), $s_i(1) = c_{1,i}$. In view of Lemma 1, it follows that any linear controller can be applied by choosing the corresponding $c_{1,i}$. On the set $\{x : V(\lambda_1, x) \geq 1\} = \{x : \lambda(x) \geq \lambda_1\}$, the function x is unbounded. Therefore, on this set the control limitations will become predominant. In particular, note that in view of (19), (20), and (22), each function $s_i(\lambda)$ tends to the coefficient $c_{0,i}$ of the open-loop system (17) as λ tends to $+\infty$.

2. If one is only interested in *semi-global* stability instead of *global* stability, one can basically neglect the definition of the function s on the interval $[\lambda_1, +\infty)$, and most of the design complexity is then avoided.

4.3 *Boundedness of the controller*

We now assume that the functions s_i defined by (19) have been chosen as indicated in the previous section. We consider the nonlinear globally asymptotically stabilizing feedback (15) with $\lambda = \lambda(x)$ defined by (16). We have:

Proposition 2. The stabilizing feedback $u(\lambda(x), x)$ defined by (15)-(16) is bounded, and the bound is proportional to $\eta^{-\frac{1}{2}}$ (with η defined in (18)).

5. AN ILLUSTRATIVE EXAMPLE

We illustrate our design method on the system

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & -x_3 + u\,, \end{array}\right. \tag{25}$$

for which we assume a magnitude limitation $|u| \leq 1$. From (15),

$$\begin{array}{rl} u(\lambda, x) = & -s_1 s_3(\lambda) x_1 - (s_1 + s_2) s_4(\lambda) x_2 \\ & -(s_1 + s_2 + s_3)(\lambda) x_3 - s_4(\lambda) x_4 + x_3\,, \end{array} \tag{26}$$

and $\lambda(x)$ is defined by (16) via

$$\begin{array}{l} V(\lambda, x) = k(\lambda)[(s_1 s_2 s_3)(\lambda) x_1^2 + (s_2 s_3)(\lambda) x_2^2 \\ +s_3(\lambda)(x_3 + s_1(\lambda) x_1)^2 + (x_4 + (s_1 + s_2)(\lambda) x_2)^2] \end{array}$$

The various parameters which define the functions k and $s_i\,(i = 1, \ldots, 4)$ have been chosen in accordance with Section 4.2:

$\eta = 20$. This value has been obtained by simulation in order to satisfy the constraint $|u| \leq 1$.

$\lambda_1 = 2$.

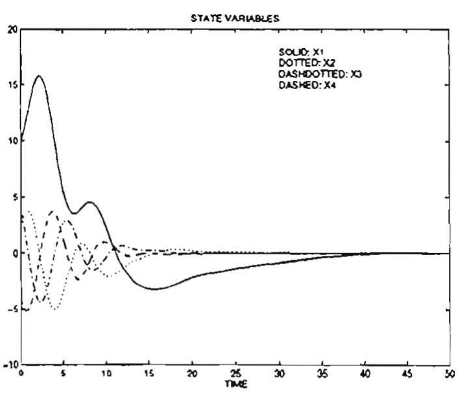
$c_1 = (1/5, 4/5, 5, 4)^T$. This choice has been made to set all the eigenvalues equal to -1 in the domain where u is linear (i.e., in $\{x : V(1, x) \leq 1\}$).

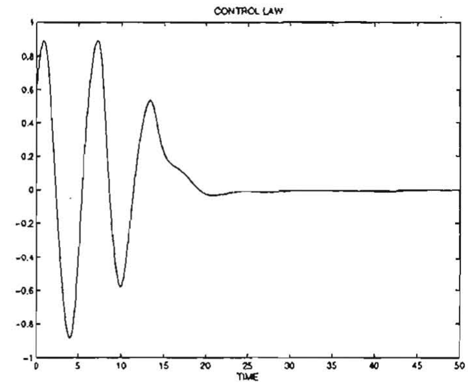
The vectors τ_1 and τ_2 are defined, in view of the definition of the other parameters, by $\tau_2 = (2, 1, 0, 1/2)^T$, and $\tau_1 = (0, 0, (\ln 5)/(2 \ln 2), 1/2)^T$.

$c_2 = (1/5, 4/5, 1, 2)^T$. Since $m = 2$, $c_{2,1}$ has to be chosen to ensure (23). A simple computation shows that (23) is satisfied for any $c_{2,1} > 0$ (because, for $i = 1$, $C^{i+1} G^{i+1}$ is diagonal).

Finally, $\beta = \lambda_2 = 1$.

The following simulation result, with initial conditions $x(0) = (10, 2, 4, -4)^T$, illustrates the behavior of the controlled system (25)-(26). The implicit equation $V(\lambda, x) = 1$ has been solved online by bisection.





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