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ASYMPTOTIC TRACKING OF A STATE REFERENCE FOR SYSTEMS WITH A FEEDFORWARD STRUCTURE

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Abstract

We are concerned with the problem of the asymptotic tracking of a given state reference for a system which can be written in a feedforward form. The state reference is assumed to be given as a particular bounded solution of the system. Our solution relies on the construction of a control Lyapunov function for the error system. It gives a time-varying state feedback which meets saturation constraints and ensures the global uniform asymptotic stability and the local exponential stability of the state reference. A practical example illustrates our central result.

1 Introduction

The basic problem we address is the following : Suppose that the system

$$\dot{x} = h(x, y, u) \quad , \qquad \dot{y} = f(y, u) \, . \tag{1}$$

admits $(x_r(t), y_r(t))$, a known bounded function, as a solution when it is driven by a known bounded input $u_r(t)$ and that $y_r(t)$ is a globally asymptotically stable solution of :

$$\dot{y} = f(y, u_r(t)) \, . \tag{2}$$

Then is it possible to design a saturated feedback $u(x, y, t)$ such that (x_r, y_r) is a globally asymptotically stable solution of (1) with $u(x, y, t)$ as input ?

In [14, Corollary 2.1], a solution is obtained for the chain of integrators subject to input saturation and it relies heavily on the linear structure.

For general non linear systems, the usual approaches to the tracking problem rely on the properties of global (resp. local) linearizability or of global (resp. local) partial linearizability. The input-output linearization theory, which leads to the celebrated normal form and to the notions of inverse dynamics and minimum phase systems in the non linear context, (see [5]), plays a central role in this type of works : for it may provide an efficient help for solving the problem of reproducing an output reference and next for determining a feedback law which asymptotically stabilizes the solution to be tracked. In this framework, it may be interesting to find appropriate output functions so that, may be with a dynamic extension, there is no inverse dynamics. This is possible for flat systems as they have been characterized in [2]. In this case all the solutions of the system can be completely parameterized in terms of these peculiar output functions and their derivatives. As a consequence, the problem of designing appropriate state references can be simpler. Also, those systems are dynamic feedback linearizable so that the asymptotic tracking problem is trivial. This approach has been applied in [10], for the design of a state tracker for the VTOL Aircraft. Unfortunately systems in the form (1) are typically not flat.

Here, we will proceed regardless of any kind of linearizability property. We center our efforts on the problem of stabilizing a state reference and consequently we round the problem of stability of the inverse dynamics. We do not address the problem of designing this state reference. Indeed, following [2, 9, 10] for instance, the tracking problem may be profitably split up into two steps :

1. Determination of a bounded state reference and of a bounded input which meet the desired control objective, for instance output reference tracking.
2. Stabilization of the state reference.

By offering a response to the second step, our technique may be seen as complementary to those which offer a response to the first. Moreover, the feedforward structure may also be exploited to construct the reference.

We solve the stabilization problem by a Lyapunov design, which extensively exploits the feedforward structure. As in the proof of [12, Theorem 3.1], Jurdjevic-Quinn approach (see [6]), higher order notion, changes of coordinates, are the clues of our demonstration. Our technique gives feedbacks meeting saturation constraints and guaranteeing global uniform and local exponential asymptotic stability of the reference.

Although not done in this paper, a recursive application of our main result is possible. It gives a response to the asymptotic tracking problem by saturated feedback for a system whose dynamics admit the following feedforward representation :

$$\left\{\begin{array}{rcl} \dot x_n &=& f_n(x_1,\dots,x_{n-1},x_n,u)\ , \\ &\vdots& \\ \dot x_2 &=& f_2(x_1,x_2,u)\ , \\ \dot x_1 &=& f_1(x_1,u)\ , \end{array}\right. \tag{3}$$

Due to space limitation, we can only sketch out the proof of our main result. For more details, we refer the reader to [11] or [13].

Notations and definitions.

- The symbol c is used to denote generically a strictly positive real number.
- A function $\mathcal{F}(x,y)$ is said to be of order $p \geq 0$, if there exists a nonnegative continuous function $\widetilde{\mathcal{F}}$ satisfying :

$$|\mathcal{F}(x,y)| \;\leq\; \widetilde{\mathcal{F}}(x,y)\,|y|^p\ . \tag{4}$$

- For a real valued C^1 function k, we denote by k' its first derivative.
- A function $\alpha : [0,+\infty) \longrightarrow [0,+\infty)$ is said to be of class $\mathcal{K}$ if it is zero at zero and strictly increasing. If besides it goes to infinity when the argument goes to infinity, it is said to be of class $\mathcal{K}^\infty$.
- A function $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a saturation if it is continuous, bounded, differentiable at 0 and such that :

$$\sigma(s)\,s \;>\; 0\ ,\ \forall s\neq 0 \quad , \quad \sigma'(0) \;>\; 0 \tag{5}$$

$$\sigma|_{\mathbb{R}_+} \;\notin\; L^1(\mathbb{R}_+) \quad , \quad \sigma|_{\mathbb{R}_-} \;\notin\; L^1(\mathbb{R}_-)\ . \tag{6}$$

2 Main result.

We consider the system :

$$\left\{\begin{array}{rcl} \dot X &=& MX + H_1(Y) + H_2(Y,u)\,u\ , \\ \dot Y &=& F_0(Y) + F_2(Y,u)\,u\ , \end{array}\right. \tag{7}$$

where Y is in $\mathbb{R}^n$, X is in $\mathbb{R}^m$, u is in $\mathbb{R}^q$, all the functions are of class C^2. We introduce three assumptions.

A0 : *There exists a function $(X_r(t), Y_r(t), u_r(t))$ bounded on $[0,+\infty)$, of class C^2, and verifying :*

$$\left\{\begin{array}{rcl} \dot X_r(t) &=& MX_r(t) + H_1(Y_r(t)) \\ && \quad + H_2(Y_r(t),u_r(t))\,u_r(t)\ , \\ \dot Y_r(t) &=& F_0(Y_r(t)) + F_2(Y_r(t),u_r(t))\,u_r(t)\ . \end{array}\right. \tag{8}$$

This assumption guarantees that the matrix

$$A(t) \;=\; \left[\frac{\partial F_0}{\partial Y}(Y_r(t)) + \frac{\partial F_2}{\partial Y}(Y_r(t),u_r(t))u_r(t)\right] \tag{9}$$

is well-defined and of class C^1. Let $\Phi_A(t,t_0)$ be the transition matrix associated to this matrix.

A1 :

A11 : *The point $\tilde Y = 0$ is a globally uniformly asymptotically stable equilibrium point of the system :*

$$\dot{\tilde Y} \;=\; F_0(\tilde Y + Y_r(t)) - F_0(Y_r(t)) \tag{10}$$
$$+F_2(\tilde Y + Y_r(t),u_r(t))u_r(t) - F_2(Y_r(t),u_r(t))u_r(t)$$

A12 : *There exist a positive definite symmetric matrix Q and $c>0$, $\alpha>0$ such that :*

$$M^\top Q + QM \;=\; -R \;\leq\; 0\ , \tag{11}$$

$$|\exp(M(s-t))|\,|\Phi_A(t,s)| \;\leq\; c\,\exp(-\alpha(t-s))$$
$$\forall\, t>0\ ,\quad \forall\, s\in[0,t]\ . \tag{12}$$

Assumption A12 implies (see below) that the matrix :

$$P(t) \;=\; \int_t^{+\infty} \exp(M(t-s))C(s)\Phi_A(s,t)ds\ , \tag{13}$$

where :

$$C(t) \;=\; \frac{\partial H}{\partial \tilde Y}(0,t)\ , \tag{14}$$

with :

$$H(\tilde Y,t) = H_1(\tilde Y + Y_r(t)) + H_2(\tilde Y + Y_r(t),u_r(t))u_r(t) \tag{15}$$

is well-defined, bounded in norm and of class C^2.

A2 : *There exists a function $K(t)$, bounded, of class C^2 and such that the solution $x = 0$ of*

$$\dot x \;=\; -\,(M + D(t)K(t))^\top\, x\ , \tag{16}$$

is exponentially stable with

$$\begin{array}{rcl} D(t) &=& \frac{\partial\varphi}{\partial u}(Y_r(t),u_r(t),t)u_r(t) \\ && +\varphi(Y_r(t),u_r(t),t)\ , \\ \varphi(Y,u,t) &=& H_2(Y,u) + P(t)F_2(Y,u)\ . \end{array} \tag{17}$$

Theorem 2.1 *If the system (7) satisfies the assumptions A0, A1 and A2 then, for all $\overline{u}$ in $(0, +\infty]$, there exists a C^1 feedback law $\overline{u}(X, Y, t)$ verifying $|\overline{u}(X, Y, t) - u_r(t)| \leq \overline{u}$ and such that the closed-loop system admits (X_r, Y_r) as a globally asymptotically stable solution. Moreover, if the pair $\left(M, \begin{pmatrix} D(t)^\top Q \\ R^{\frac{1}{2}} \end{pmatrix}\right)$ is uniformly observable, then $\overline{u}(X, Y, t)$ may be chosen so that (X_r, Y_r) is a globally uniformly asymptotically and a locally exponentially stable solution of the corresponding closed-loop system.*

2.1 Discussion on A1 and A2

Assumption A1. Since M is stable, for $s \leq t$, $|\exp(M(s-t))|$ is bounded away from 0. So Assumption A12 ensures the local exponential stability of the solution Y_r of the Y subsystem of (7) with the input set to zero.

On the other hand A12 implies that P given by (13) is well-defined, bounded, of class C^2 and solution on $[0, +\infty)$ of :

$$\dot{\mathcal{P}}(t) = M\mathcal{P}(t) - \mathcal{P}(t)A(t) - C(t) \ . \tag{18}$$

Indeed, the functions F_0, F_2, u_r and Y_r being of class C^2, the function $A(t)$ is of class C^1. The functions H_1, H_2, u_r and Y_r being of class C^2, the function $C(t)$ is of class C^1. So, the function $\exp(M(t-s))C(s)\Phi_A(s,t)$ is of class C^1. Then, from Assumption A0, the function Y_r is bounded on $[0, +\infty)$. It follows that the function $|C(t)|$ is bounded by a positive real number c. With inequality (12), this implies :

$$\left|\int_t^{+\infty} \exp(M(t-s))C(s)\Phi_A(s,t)ds\right| \tag{19}$$

$$\leq c\left|\int_t^{+\infty} \exp(-\alpha(s-t))ds\right| = \frac{c}{\alpha} < +\infty \ .$$

It follows readily that $P(t)$ is well-defined and of class C^1. Next, by simply evaluating the derivative of P, it follows that P satisfies (18), which in turn implies that P is of class C^2.

We will exploit the properties of P mentioned above in the proof of Theorem 2.1 for designing a change of coordinates which facilitates the construction of a control Lyapunov function for (7). At last, let us notice that the conditions (11), (12) are closely linked with the properties of exponential or ordinary dichotomy introduced in [1]. When $A(t)$ is constant, the requirement (12) imposes on the largest real part of the eigenvalues of A to be smaller than the smallest real part of the eigenvalues of M. So, when Theorem 2.1 is recursively applied to a system of the form (3), the closed loop system thus obtained admits a time-scale decomposition which is analogous to the one obtained when a stable linear equation is put in a triangular form.

Assumption A2. According to [4, Definition 4.2], Assumption A2 holds if the pair $(-M^\top, D(t)^\top)$ is completely uniformly detectable. According to [4, Corollary 4.1], when $R = 0$ in (11), the pair $(-M^\top, D(t)^\top)$ is completely uniformly detectable if it is completely uniformly observable. This last property (see [4, Definition 2.6]) means that there exist strictly positive numbers σ, α_1 and α_2 such that :

$$0 < \alpha_1 I \leq W(t, t+\sigma) \leq \alpha_2 I \tag{20}$$

with

$$W(t, s) = \int_t^s w(t, \tau)d\tau \ . \tag{21}$$

$$w(t, \tau) = \exp[M(t-\tau)]D(\tau)^\top D(\tau)\exp[M^\top(t-\tau)] \tag{22}$$

A way to check if (20) holds is given in [15, 2.31 Theorem] : when $D(t)$ is a column i.e. in the single-input case, the inequalities (20) hold when the matrix :

$$\Gamma(t) = [q_0, q_1, \ldots, q_{n-1}] \tag{23}$$

is invertible for all t and Γ, Γ^{-1}, $\dot{\Gamma}$ are continuous and bounded functions on $[0, +\infty)$ with the function q_k defined recursively as :

$$q_{k+1} = -Mq_k + \dot{q}_k \ ; \ q_0 = D(t)^\top \tag{24}$$

3 Sketch of proof.

First step : Change of coordinates.
In order to transform the asymptotic tracking problem into a stabilization one and to make the coupling term H_1 of second order, we rewrite the system (7) in new coordinates. Let :

$$\left\{\begin{array}{lcl} x &=& [X - X_r(t)] + P(t)\,[Y - Y_r(t)] \\ y &=& Y - Y_r(t) \quad , \qquad v \;=\; u - u_r(t) \ . \end{array}\right. \tag{25}$$

We get :

$$\left\{\begin{array}{lcl} \dot{x} &=& Mx + h_1(y,t) + h_2(y,v,t)v \ , \\ \dot{y} &=& f_0(y,t) + f_2(y,v,t)v \ , \end{array}\right. \tag{26}$$

where all the functions are of class C^1, bounded with respect to t and h_1 is such that :

$$|h_1(y,t)| \leq \gamma(|y|)\,|y|^2 \qquad \forall (y,t) \tag{27}$$

for some continuous function γ. This second order property plays a crucial role in the second step.

Remark. As in [12], we can prove that, under our assumptions, appropriate coordinates (x, y) allowing us to rewrite (7) in the form :

$$\left\{\begin{array}{lcl} \dot{x} &=& Mx + h_2(y,v,t)\,v \ , \\ \dot{y} &=& f_0(y,t) + f_2(y,v,t)\,v \ , \end{array}\right. \tag{28}$$

exist. Then the stabilization problem could possibly be solved by applying a generalization of the Jurdjevic and Quinn's approach i.e. by choosing v solution of :

$$\left(x^\top Q h_2(y,v,t) + \frac{\partial V}{\partial y}(y,t) f_2(y,v,t)\right) v \;<\; 0 \qquad (29)$$

for all $(x,y) \neq 0$. However, we pursue our proof with the system (26) because the coordinates which give the simpler form (28) are, from a practical point of view, much more difficult to determine than are those given in (25).

Second step : Lyapunov design.
Thanks to Assumption A1, we may prove :

Lemma 3.1 *There exists a Lyapunov function $V(y,t)$ of class C^1 and functions α_1, α_2, α_3 and α_4 of class $\mathcal{K}^\infty$ such that, for all y such that $|y| \leq c$,*

$$\begin{array}{c} c|y|^2 \;\leq\; V(y,t) \;\leq\; c|y|^2 \\ \left|\frac{\partial V}{\partial y}(y,t)\right| \;\leq\; c|y| \;, \quad c|y|^2 \;\leq\; \alpha_3(|y|) \;, \end{array} \qquad (30)$$

and, for $v = 0$ and for all y,

$$\begin{array}{r} \alpha_1(|y|) \;\leq\; V(y,t) \;\leq\; \alpha_2(|y|) \;, \\ \left|\frac{\partial V}{\partial y}(y,t)\right| \;\leq\; \alpha_4(|y|) \;, \\ \dot{\overbrace{V(y,t)}}_{(26)} \;\leq\; -\,\alpha_3(|y|) \;. \end{array} \qquad (31)$$

This result is obtained by combining the Lyapunov functions given by the converse Lyapunov theorems for equilibrium points which are locally exponentially stable or globally uniformly asymptotically stable. Details can be found in [11, Annexe G].

Next, let us focus our attention on the Lyapunov function :

$$U(x,y,t) \;=\; K(V(y,t)) \;+\; \int_0^{\sqrt{x^\top Q x}} \sigma(s) ds \;, \qquad (32)$$

where K is to be chosen as a function of class $\mathcal{K}^\infty$ and of class C^2 with a strictly positive derivative and where σ is a saturation. $U(x,y,t)$ is a positive definite and radially unbounded function. By using [12, Lemma B.2], the properties (30), (31) and (27), we can show that K can be chosen such that :

$$\begin{array}{l} \dot{\overbrace{U(x,y,t)}}_{(26)} \;\leq\; \Gamma(x,y,v,t)v \\ \qquad - \frac{1}{2}K'(V(y,t))\alpha_3(|y|) \;-\; \sigma(\sqrt{x^\top Q x})\frac{x^\top R x}{\sqrt{x^\top Q x}} \end{array} \qquad (33)$$

with :

$$\begin{array}{l} \Gamma(x,y,v,t) \;=\; K'(V(y,t))\frac{\partial V}{\partial y}(y,t) f_2(y,v,t) \\ \qquad + \,\sigma(\sqrt{x^\top Q x})\frac{x^\top Q}{\sqrt{x^\top Q x}} h_2(y,v,t) \;. \end{array} \qquad (34)$$

Since f_2 and h_2 are of class C^1, it is possible to prove (see [12, Appendix A]) that there exists a strictly positive function λ of class C^1 such that the right hand side of (33) is negative when the input is :

$$v(y,x,t) \;=\; -\,\lambda(y,x,t)\Gamma(x,y,0,t)^\top \;, \qquad (35)$$

and this function is in norm bounded by $\overline{u}$. This result implies the global uniform stability of $(X_r(t), Y_r(t))$.

To prove the asymptotic stability, we integrate inequality (33). We deduce that the integrals

$$\int_0^{+\infty} K'(V(y(s),s))\alpha_3(|y(s)|)ds \;,$$
$$\int_0^{+\infty} \sigma(\sqrt{x^\top Q x})\frac{x(s)^\top R x(s)}{\sqrt{x^\top Q x}} ds \;,$$
$$\int_0^{+\infty} \lambda(y(s),x(s),s)|\Gamma(x,y,0,s)|^2 ds$$

are finite. It follows that the function $x(t)^\top R x(t)$ belongs to $L^1([0,+\infty))$ and that (see [7, Lemma 4.4]) :

$$\lim_{t\to+\infty} y(t) \;=\; 0 \;. \qquad (36)$$

This in turn with (34) implies that the function $x(t)^\top Q h_2(0,0,t)$ belongs to $L^2([0,+\infty))$. At last, the conclusion can be obtained by noticing that $P(t)$ is bounded and by establishing the following lemma:

Lemma 3.2 *Let $D(t)$ be a bounded continuous function and M be a matrix such that*

$$M^\top Q \,+\, QM \;=\; -\,R \;\leq\; 0 \qquad (37)$$

for some positive definite symmetric matrix Q. Suppose that there exists a bounded continuous function $K(t)$ such that

$$\dot{\mathcal{X}} \;=\; -\,(M + D(t)K(t))^\top \mathcal{X} \qquad (38)$$

is exponentially stable. Then, if φ_1 and φ_2 are in $L^2([0,+\infty))$ and if φ_3 is in $L^1([0,+\infty))$, any solution of the system :

$$\begin{array}{rcl} \dot{x}(t) &=& M\,x(t) \;+\; \varphi_1(t) \\ x(t)^\top Q\,D(t) &=& \varphi_2(t) \\ x(t)^\top R\,x(t) &=& \varphi_3(t) \end{array} \qquad (39)$$

converges to zero as the time goes to infinity.

Third step : Uniform asymptotic stability.
If the reference $(X_r(t), Y_r(t))$ and the feedback law $u_r(t)$ are periodic functions, then, by invoking for instance [16, Theorem 11.3], we can conclude immediately that the system (7) in closed loop with the feedback we have designed admits $(X_r(t), Y_r(t))$ as a globally *uniformly* asymptotically stable solution. If we do not assume that $(X_r(t), Y_r(t))$ and $u_r(t)$ are periodic, then we assume now that the pair $\left(M, \begin{pmatrix} D(t)^\top Q \\ R^{\frac{1}{2}} \end{pmatrix}\right)$ is uniformly observable. To prove that, in this case, we get uniform asymptotic stability, we proceed as follows :

1. By borrowing an idea from [8, Proof of Lemma2], we construct a Lyapunov function $\mathcal{Q}$ such that :

$$\frac{\partial \mathcal{Q}}{\partial t}(x,t) + \frac{\partial \mathcal{Q}}{\partial x}(x,t)\,[Mx + D(t)v(0,x,t)] \leq -\tfrac{1}{2}|x|^2 \ . \tag{40}$$

2. We prove that there exists a positive continuous function Ξ such that, for all (x,y) and all $t \geq 0$,

$$|h_2(y,v(y,x,t),t)v(y,x,t) - h_2(0,0,t)v(0,x,t)| \leq \Xi(|y|)|y| + c|v(0,x,t)|^2 \ . \tag{41}$$

3. Thanks to this inequality, we show the existence of a positive continuous function Γ, such that, by letting :

$$\bar{l}(q) = \int_0^q \frac{1}{1+s^2}ds \ , \tag{42}$$

we have :

$$\dot{\overline{\bar{l}(\mathcal{Q}(x,t))}}_{(26),(35)} \leq -\tfrac{1}{4}|x|^2\,\bar{l}'(\mathcal{Q}(x,t)) + \Gamma(|y|)^2|y|^2 \ . \tag{43}$$

4. Since $\Gamma(|y|)^2|y|^2$ is of order two, by using [12, Lemma B.2], we prove the existence of $\overline{K}$, of class $\mathcal{K}^\infty$ satisfying :

$$\dot{\overline{\bar{l}(\mathcal{Q}(x,t)) + \overline{K}(U(x,y,t))}}|_{(26),(35)} \leq -\tfrac{1}{4}\,\alpha_3(|y|) - c\,\frac{|x|^2}{1+|x|^4} \ . \tag{44}$$

5. To get the property of global uniform asymptotic stability, we check that [3, Chapter 10, Theorem 3.1] applies.

Fourth step : Local exponential stability.
From inequality (44), from the properties (30), from the definition of $\bar{l}$ in (42) and from [7, Corollary 3.4], we deduce that the origin of the system (26) with the control (35) is locally exponentially stable.

4 Application : periodic oscillations of the cart-pendulum system

The Lyapunov design proposed this paper can be used to stabilize a periodic solution for the cart-pendulum system. After preliminary changes of feedback, coordinates and time the dynamics of this system are :

$$\left\{\begin{array}{c} \dot{x}_0 = s_0 \ , \quad \dot{s}_0 = u_0 \ , \quad \dot{\theta}_0 = \omega_0 \\ \dot{\omega}_0 = \sin(\theta_0) - u_0\cos(\theta_0) \end{array}\right. \tag{45}$$

We want to find a bounded feedback law such that, for any initial condition with the angle θ_0 in $]-\frac{\pi}{2},\frac{\pi}{2}[$, we have :

$$\lim_{t\to+\infty} |\tan(\theta_0(t)) - \cos(t)| = 0 \ . \tag{46}$$

To meet this requirement, we proceed in three steps.

First step : change of coordinates and feedback.
A change of coordinates and feedback maps $\mathbb{R}\times\mathbb{R}\times(-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R}$ into $\mathbb{R}^4$. and transforms (45) into :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& s_1 - t_1 \ , \\ \dot{s}_1 &=& -u_1 \ , \\ \dot{t}_1 &=& r_1 \ , \\ \dot{r}_1 &=& -(t_1+r_1)\sqrt{1+t_1^2} - u_1\sqrt{1+t_1^2} \ , \end{array}\right. \tag{47}$$

Second step : reference state trajectory and corresponding input.
The functions :

$$\begin{array}{rcl} x_{1r}(t) &=& -\cos(t) - \displaystyle\int_0^t \arcsin\left(\frac{1}{\sqrt{2}}\sin(s)\right)ds \ , \\ s_{1r}(t) &=& \cos(t) + \sin(t) - \arcsin\left(\frac{1}{\sqrt{2}}\sin(t)\right) \ , \\ t_{1r}(t) &=& \cos(t) \ , \\ r_{1r}(t) &=& -\sin(t) \ , \\ u_{1r}(t) &=& \frac{\cos(t)}{\sqrt{1+\cos^2(t)}} - \cos(t) + \sin(t) \end{array} \tag{48}$$

are periodic and give a particular solution of (47) such that the output $t_{1r}(t)$ behaves exactly as desired, i.e.

$$t_{1r}(t) = \cos(t) \ . \tag{49}$$

Third step : design of a stabilizing feedback.
With the notations :

$$\begin{array}{c} \tilde{x}_1 = x_1 - x_{1r} \ , \quad \tilde{s}_1 = s_1 - s_{1r} \ , \\ \tilde{t}_1 = t_1 - t_{1r} \ , \quad \tilde{r}_1 = r_1 - r_{1r} \ , \\ u_1 = u_{1r}(t) - u_2 \ , \end{array} \tag{50}$$

we obtain the error system :

$$\left\{\begin{array}{rcl} \dot{\tilde{x}}_1 &=& \tilde{s}_1 - \tilde{t}_1 \ , \\ \dot{\tilde{s}}_1 &=& u_2 \ , \\ \dot{\tilde{t}}_1 &=& \tilde{r}_1 \ , \\ \dot{\tilde{r}}_1 &=& -(\tilde{t}_1+\tilde{r}_1)\sqrt{1+t_1^2} + \dot{r}_{1r}\frac{\tilde{t}_1^2+2\tilde{t}_1 t_{1r}}{\zeta(t_1,t_{1r}(t))} \\ && +\sqrt{1+t_1^2}\,u_2 \ , \end{array}\right. \tag{51}$$

where :

$$\zeta(t_1,t_{1r}(t)) = 1 + t_{1r}^2(t) + \sqrt{1+t_{1r}^2(t)}\sqrt{1+t_1^2} \ . \tag{52}$$

For this system, we first design a feedback which globally asymptotically and locally exponentially stabilizes the $(\tilde{t}_1,\tilde{r}_1)$-subsystem of (51). Then by using recursively the technique introduced in the proof of Theorem 2.1, we design in two steps a feedback which guarantees the uniform asymptotic tracking of the state reference (for more details see [11, 13]). This feedback for the system (47) is :

$$
\begin{aligned}
u_1 \;=\;& \tfrac{\cos(t)}{\sqrt{1+\cos^2(t)}} - \cos(t) + \sin(t) \\
& - \cos(t)\tfrac{\tilde{t}_1^2}{\sqrt{1+t_1^2}\zeta(t_1,t_{1r}(t))} \\
& - 2(\tilde{t}_1+\tilde{r}_1)\tfrac{\cos(t)\tilde{t}_1}{1+t_1^2+\sqrt{1+t_1^2}\sqrt{1+\tilde{t}_1^2}} \\
& + \tfrac{1}{50}\left(\tfrac{\tilde{s}_1}{\sqrt{1+\tilde{s}_1^2}} - \tilde{t}_1\right) + \tfrac{\tilde{x}_2}{\sqrt{1+\tilde{x}_2^2}} \\
& + \left[4 + \tfrac{101}{25}(Q_2(\tilde{s}_1,\tilde{t}_1,\tilde{r}_1)+1)\right] \times \\
& \quad \left[3(\tilde{t}_1+\tilde{r}_1)\sqrt{1+t_1^2} + \tfrac{\tilde{s}_1}{\sqrt{1+\tilde{s}_1^2}}\right]
\end{aligned}
\qquad (53)
$$

where

$$
\begin{aligned}
\tilde{x}_1 &= x_1 + \cos(t) \\
&\quad + \int_0^t \arcsin\left(\frac{1}{\sqrt{2}}\sin(s)\right) ds \ , \\
\tilde{x}_2 &= \tilde{x}_1 + 50\tilde{s}_1 \ , \\
\tilde{s}_1 &= s_1 - \cos(t) - \sin(t) \\
&\quad + \arcsin\left(\tfrac{1}{\sqrt{2}}\sin(t)\right) \ , \\
\tilde{t}_1 &= t_1 - \cos(t) \\
\tilde{r}_1 &= r_1 + \sin(t) \ , \\
Q_2(\tilde{s}_1,\tilde{t}_1,\tilde{r}_1)) &= 3\left((1+\tilde{t}_1^2)^{\frac{3}{2}} - 1 + \tilde{t}_1\tilde{r}_1 + \tilde{r}_1^2\right) \\
&\quad + \sqrt{1+\tilde{s}_1^2} - 1 \ , \\
\zeta(t_1, t_{1r}(t)) &= 1 + \cos^2(t) \\
&\quad + \sqrt{1+\cos^2(t)}\sqrt{1+t_1^2}
\end{aligned}
\qquad (54)
$$

5 Conclusion.

We have constructed saturated feedback laws which globally uniformly asymptotically and locally exponentially stabilize a given state reference of a non linear system obtained by adding one integration. A recursive application of this design is possible. It gives a new technique for dealing with tracking problems for non linear systems, in feedforward form.

References

[1] W.A. Coppel, *Dichotomies in Stability Theory.* Springer-Verlag Berlin Heidelberg New York (1978).

[2] M. Fliess, J. Lévine, P. Martin, P. Rouchon, Flatness and defect of nonlinear systems : introductory theory and examples. *Int. J. Control*, (1995), **61 (6)**, 1327-1361

[3] J.K. Hale : *Ordinary differential equations.* Robert E. Krieger publishing company, Malabar, Florida (1980)

[4] M. Ikeda, H. Maeda, S. Kodama, Estimation and feedback in linear time-varying systems : a deterministic theory. *SIAM J. Control and Optimization*, **13 (2)**, (1975).

[5] A. Isidori : *Nonlinear Control System.* Third Edition. Springer Verlag (1995).

[6] V. Jurdjevic, J.P. Quinn, Controllability and stability. *Journal of differential equations.* **4** (1978) pp. 381-389

[7] H. Khalil : *Nonlinear Systems.* Macmillan Publishing Company New York, Maxwell Macmillan Canada Toronto, Maxwell Macmillan international New York Oxford Singapore Sydney. (1992)

[8] W. Liu, Y. Chitour, E. Sontag, On finite-gain stabilizability of linear systems subject to input saturation. *Siam J. Control and Optimization* **34 (4)** pp.1190-1219, July 1996

[9] R. Marino, I. Kanellakopoulos, P. Kokotovic, Adaptative Tracking for Feedback Linearizable SISO Systems. *Proceedings of the 28th Conference on Decision and Control.* Tampa, Florida. December (1989).

[10] P. Martin, S. Devasia and B. Paden, A different look at output tracking : control of a VTOL aircraft. *Automatica*, **32 (1)**, pp 101-107, 1996.

[11] F. Mazenc, *Stabilisation de trajectoires, Ajout d'intégration, Commande saturées.* Thèse en Mathématiques et Automatique. École des Mines de Paris. (1996).

[12] F. Mazenc, L. Praly, Adding integrations, saturated controls and global asymptotic stabilization for feedforward systems, *IEEE Transactions on Automatic Control,* **41 (11)**, November (1996).

[13] F. Mazenc, L. Praly, Asymptotic tracking of a state reference for systems with a feedforward structure. Submitted for publication in *Automatica.* October (1996)

[14] A. Teel, Feedback stabilization : nonlinear solutions to inherently nonlinear problems. Memorandum No. UCB/ERL M92/65. 12 June (1992)

[15] K.S. Tsakalis, P.A. Ioannou : *Linear Time-varying Systems, Control and Adaptation.* Prentice Hall, Engelwood Cliffs, New Jersey 07632.

[16] T. Yoshizawa : *Stability theory by Lyapunov's second method.* The mathematical Society of Japan, (1966).