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A SELF-TUNING ROBUST NONLINEAR CONTROLLER

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Abstract. This paper deals with nonlinearly parameterized uncertain systems in the presence of input/state stable (ISS) dynamic uncertainties. A novel constructive control scheme is proposed to generate a minimal-order self-tuning globally stabilizing controller.

Keywords. Nonlinear systems, stabilizing controllers, robust control, uncertainty.

1. INTRODUCTION

The class of uncertain nonlinear systems studied in this paper is described by :

$$\begin{aligned} \dot{z} &= q(z, x_1) \\ \dot{x}_i &= a_i x_{i+1} + f_i(x, z, u, \theta^*), \quad 1 \le i \le n-1 \\ \dot{x}_n &= a_n u + f_n(x, z, u, \theta^*) \end{aligned} \tag{1}$$

where $u \in \mathbb{R}$ is the control input, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is the *measured* components of the system state and $z \in \mathbb{R}^l$ is the remaining *unmeasured* component of the system state, the a_i's are *unknown* nonzero constants and $\theta^* \in \mathbb{R}^p$ is a vector of *unknown* constant parameters. Assume that the f_i's and q are unknown, Lipschitz continuous functions.

Throughout the paper, the following assumptions are made on the system (1) :

(H1) For each $1 \le i \le n$, the sign of a_i is known. For notational simplicity, assume that $a_i > 0$ for all i.

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(H2) There exists an *unknown* positive constant ϑ^* such that, for all x in $\mathbb{R}^n$, z in $\mathbb{R}^l$ and all $1 \le i \le n$,

$$|f_i| \le \vartheta^* \phi_{i1}(|(x_1, \ldots, x_i)|) + \vartheta^* \phi_{i2}(|z|) \tag{2}$$

where ϕ_{i1} and ϕ_{i2} are two known nonnegative smooth functions with $\phi_{i1}(0) = \phi_{i2}(0) = 0$.

The class of uncertain nonlinear systems (1) to be controlled is motivated by recent studies on global stabilization of triangular systems in robust and adaptive control settings (see (Krstić *et al.*, 1995) and references therein). Comparing to these studies, broader classes of triangular systems with dynamic uncertainties have recently been considered in (Praly and Jiang, 1993; Jiang *et al.*, 1994). Robust output-feedback or partial-state feedback stabilizing controllers have been designed using a nonlinear small-gain argument (Jiang *et al.*, 1994). Related work on the similar problem within Lyapunov approach can be also found in (Tsinias, 1995).

The purpose of this paper is to relax the main assumption in (Praly and Jiang, 1993; Jiang *et al.*, 1994) where the bounding gain functions are exactly known. The control objective is to find a second-order dynamic feedback of the form $u = \mu(x, \chi)$, $\dot{\chi} = \varpi(\chi, x)$ with $\chi \in \mathbb{R}^2$ such that all solutions of the closed-loop system are bounded.

Furthermore,

$$\lim_{t\to\infty}(|x(t)|+|z(t)|) = 0 . \qquad (3)$$

To achieve this objective, additional assumptions on unmeasured dynamics z will be given in Section 2. The contributions of the paper are twofold: firstly, the uncertain systems under consideration are nonlinearly parameterized and subject to stable dynamic uncertainties. Neither the classical matching conditions nor any kind of growth conditions on the system nonlinearities as used in (Pomet and Praly, 1992; Praly *et al.*, 1991; Jiang and Praly, 1992; Jiang, 1995) are required here. Secondly, a new systematic design procedure is presented and it incorporates a minimal-order adaptive law and an introduction of certain scalar dynamic signal. It should be mentioned that the idea of using an available dynamic signal to inform about the size of dynamic uncertainties is well-known in adaptive linear control (Praly, 1990).

2. DEFINITION AND ASSUMPTIONS

The reader is referred to (Sontag, 1990) for basic definitions of class K, K_∞ and KL functions.

Motivated by the concept of input-to-state stability (ISS) and ISS-Lyapunov function introduced in (Sontag, 1990; Sontag and Wang, 1995), a notion of exp-ISS Lyapunov function is now needed.

Definition 2.1 A C^1 function V is said to be an *exp-ISS Lyapunov function* for system $\dot{x}=f(x,u)$ if

- there exist functions ψ_1, ψ_2 of class K_∞ such that

$$\psi_1(|x|)\le V(x)\le \psi_2(|x|),\ \forall x\in \mathbb{R}^n \qquad (4)$$

- there exist a constant $c>0$ and a K_∞-function γ such that

$$\frac{\partial V}{\partial x}(x)\,f(x,u) \;\le\; -cV(x) + \gamma(|u|) \qquad (5)$$

It was shown in (Sontag and Wang, 1995, Proof of Theorem 1) that a control system $\dot{x}=f(x,u)$ is ISS iff it has an ISS-Lyapunov function. Further, it was shown in (Praly and Wang, 1994, Proof of Lemma 3) that a control system $\dot{x}=f(x,u)$ has an ISS-Lyapunov function iff it has an exp-ISS Lyapunov function.

Lemma 2.1 *If V is an exp-ISS Lyapunov function for a control system $\dot{z}=q(z,u)$, i.e. (4) and (5) hold, then, for any constants $\bar{c}$ in $(0,c)$, $r^o>0$ and any initial condition z^o, for any function $\bar{\gamma}$ such that $\bar{\gamma}(u)\ge\gamma(|u|)$, there exist a finite $T^o\ge 0$, a nonngeative constant $D(t)$ defined for all $t\ge 0$ and a signal described by :*

$$\dot{r} \;=\; -\bar{c}r \;+\; \bar{\gamma}(u(t))\,,\quad r(0)=r^o \qquad (6)$$

such that $D(t)=0$ for all $t\ge T^o$ and :

$$V(z(t)) \;\le\; r(t) \;+\; D(t) \qquad (7)$$

for all $t\ge 0$ where the solutions are defined.

Proof: follows from Gronwall's lemma.

The following assumptions are relative to the unmeasured dynamics z in system (1).

(H3) The z-system in (1) has an exp-ISS Lyapunov function V_z in the sense of Definition 2.1, i.e., there exist K_∞-functions ψ_1, ψ_2, a positive constant c and a K_∞-function γ such that

$$\psi_1(|z|)\;\le\; V_z(z)\;\le\;\psi_2(|z|),\quad \forall z\in\mathbb{R}^l \qquad (8)$$

$$\frac{\partial V_z}{\partial z}(z)\,q(x,z,u)\;\le\; -cV_z(z)\;+\;\gamma(|x_1|) \qquad (9)$$

Moreover, $\bar{c}\in(0,\,c)$, γ and ψ_1 are known.

(H4) γ is of class C^2 whose first-order derivative is zero at zero, i.e. $\partial\gamma/\partial s(0)=0$. There exist class K_∞-functions ρ_i $(1\le i\le n)$ such that

$$\lim_{r\to 0}\phi_{i2}\circ\psi_1^{-1}\circ(\mathrm{Id}+\rho_i)(r)/\sqrt{r}\;<\;+\infty \qquad (10)$$

with ϕ_{i2} as introduced in (H2).

Remark 2.1 Upon specializing (1) to linear systems, (H3) is checked if the linear system $\dot{z}=q(0,z)$ is asymptotically stable with a known stability margin. In this case, it is easy to see that (H4) is also satisfied.

3. CONTROL DESIGN PROCEDURE

First notice that, thanks to Assumption (H4), there exists a smooth nonnegative function φ_0 such that

$$\gamma(|x_1|)\;\le\; x_1^2\varphi_0(x_1) \qquad (11)$$

So, by Lemma 2.1, available signal r defined by :

$$\dot{r}\;=\;-\bar{c}\,r\;+\;x_1^2\varphi_0(x_1)\,,\quad r(0)=r^o>0 \qquad (12)$$

possesses the property

$$V_z(z(t))\;\le\; r(t)\;+\;D(t) \qquad (13)$$

for all t where the solutions are defined, with $D(t)$ defined for each $t\ge 0$ and $D(t)=0$ for all $t\ge T^o$ ($T^o\ge 0$ being finite and depending continuously on the initial conditions r^o, z^o).

Step 1 : Let ϑ be a positive constant satisfying :

$$\vartheta \;\geq\; \max\left\{\frac{\vartheta^*}{a_1}\,,\;\frac{\vartheta^{*2}}{a_1^2}\right\} \;:=\; b_1 \qquad (14)$$

and let $\widehat{\vartheta}(t)$ be an update estimate of ϑ. Consider the positive function

$$V_1 \;=\; \frac{1}{2a_1}x_1^2 \;+\; r \;+\; \frac{1}{2\lambda}(\widehat{\vartheta}-\vartheta)^2 \qquad (15)$$

with $\lambda > 0$ an adaptation gain and r defined as in (12).

By Assumptions (H1) and (H2), differentiating V_1 along the solutions of (1)-(12) gives :

$$\dot{V}_1 \leq x_1x_2 + \frac{\vartheta^*}{a_1}\left(|x_1|\phi_{11}(|x_1|) + |x_1|\phi_{12}(|z|)\right) \\ -\bar{c}r + x_1^2\varphi_0(x_1) + \frac{1}{\lambda}(\widehat{\vartheta}-\vartheta)\dot{\widehat{\vartheta}} \qquad (16)$$

Since ϕ_{11} is smooth and is zero at zero, there exists a smooth function φ_{11} such that

$$|x_1|\phi_{11}(|x_1|) \;\leq\; x_1^2\varphi_{11}(x_1)\,, \quad \forall x_1 \in \mathbb{R} \qquad (17)$$

From (13), (14), assumptions (H3) and (H4), by completing the squares and using (Jiang *et al.*, 1994, eq. (6)), it follows successively :

$$\frac{\vartheta^*}{a_1}|x_1|\phi_{12}(|z|) \;\leq\; \frac{\vartheta^*}{a_1}|x_1|\phi_{12}\circ\psi_1^{-1}(r+D(t)) \\ \leq \frac{\vartheta^*}{a_1}|x_1|\phi_{12}\circ\psi_1^{-1}\circ(\mathrm{Id}+\rho_1)(r) \\ + \frac{\vartheta^*}{a_1}|x_1|\phi_{12}\circ\psi_1^{-1}\circ(\mathrm{Id}+\rho_1^{-1})(D(t)) \\ \leq \frac{\vartheta^*}{a_1}|x_1|\phi_{12}\circ\psi_1^{-1}\circ(\mathrm{Id}+\rho_1)(r) \\ \leq \frac{\bar{c}}{2}r + \vartheta\frac{1}{2\bar{c}}x_1^2\varphi_{12}(r)^2 + \frac{1}{4}x_1^2 + d_1(t) \qquad (18)$$

where φ_{12} is some nonnegative smooth function and $d_1(t)$ is defined by :

$$d_1(t) \;=\; \left(\frac{\vartheta^*}{a_1}\phi_{12}\circ\psi_1^{-1}\circ(\mathrm{Id}+\rho_1^{-1})(D(t))\right)^2 \qquad (19)$$

Notice that $d_1(t) = 0$ for all $t \geq T^o$.

Consequently, in view of (17) and (18), (16) implies :

$$\dot{V}_1 \leq x_1\left[x_2 + \frac{1}{4}x_1 + \vartheta x_1\varphi_{11}(x_1) + \vartheta\frac{1}{2\bar{c}}x_1\varphi_{12}(r)^2\right. \\ \left. + x_1\varphi_0(x_1)\right] - \frac{\bar{c}}{2}r + d_1(t) + \frac{1}{\lambda}(\widehat{\vartheta}-\vartheta)\dot{\widehat{\vartheta}} \qquad (20)$$

Introducing the following notation :

$$\tau_1(x_1,r) = \lambda x_1^2\varphi_{11}(x_1) + \frac{\lambda}{2\bar{c}}x_1^2\varphi_{12}(r)^2 \qquad (21)$$

$$w_1(x_1,r,\widehat{\vartheta}) = -(k_1+\frac{1}{4})x_1 - \widehat{\vartheta}x_1\varphi_{11}(x_1) \\ -\widehat{\vartheta}\frac{1}{2\bar{c}}x_1\varphi_{12}(r)^2 - x_1\varphi_0(x_1) \qquad (22)$$

$$\overline{x}_2 = x_2 - w_1(x_1,r,\widehat{\vartheta}) \qquad (23)$$

where $k_1 > n-1$, (20) gives :

$$\dot{V}_1 \leq -k_1x_1^2 + x_1\overline{x}_2 - \frac{\bar{c}}{2}r + d_1(t) \\ + \frac{1}{\lambda}(\widehat{\vartheta}-\vartheta)(\dot{\widehat{\vartheta}} - \tau_1) \qquad (24)$$

Note that (24) holds as long as ϑ satisfies (14). Also note that $w_1(0,r,\widehat{\vartheta}) = 0$ for all r and $\widehat{\vartheta}$ and that for any r and $\widehat{\vartheta}$, $(x_1,x_2)\mapsto(x_1,\overline{x}_2)$ is a global diffeomorphism preserving the origin.

Step i $(2 \leq i \leq n)$: Assume that, at Step $(i-1)$, there exist smooth functions τ_j and w_j $(1 \leq j \leq i-1)$, $\tau_j(0,\ldots,0,r,\widehat{\vartheta}) = w_j(0,\ldots,0,r,\widehat{\vartheta}) = 0\ \forall(r,\widehat{\vartheta})$, so that, with :

$$\overline{x}_1 = x_1\,, \quad \overline{x}_{j+1} = x_{j+1} - w_j(x_1,\ldots,x_j,r,\widehat{\vartheta}) \quad (25)$$

the time derivative of the positive function

$$V_{i-1} \;=\; \sum_{j=1}^{i-1}\frac{1}{2a_j}\overline{x}_j^2 \;+\; r \;+\; \frac{1}{2\lambda}(\widehat{\vartheta}-\vartheta)^2 \qquad (26)$$

along the solutions of (1)-(12) satisfies :

$$\dot{V}_{i-1} \;\leq\; -\sum_{j=1}^{i-1}(k_j - i + 1 + j)\overline{x}_j^2 + \overline{x}_{i-1}\overline{x}_i - \frac{\bar{c}}{2^{i-1}}r \\ + d_{i-1}(t) + \frac{1}{\lambda}(\widehat{\vartheta}-\vartheta)(\dot{\widehat{\vartheta}} - \tau_{i-1}) + (i-2)(\dot{\widehat{\vartheta}} - \tau_{i-1})^2 \\ + 2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\dot{\widehat{\vartheta}} - \tau_{i-1}) \qquad (27)$$

where $k_j > n-j$, $d_{i-1}(t) \geq 0$ and $= 0$ for all $t \geq T^o$. Assume further that there exists an unknown $b_{i-1} > 0$ such that (27) holds as long as $\vartheta \geq b_{i-1}$.

It is proven in the sequel that (27) holds also for i as long as $\vartheta \geq b_i$ for certain positive (unknown) real number b_i. In case $i = n$, set $x_{n+1} = u$ and in this case, $\overline{x}_{n+1} = 0$ and $\dot{\widehat{\vartheta}} = \tau_n$.

Consider the positive function

$$V_i \;=\; V_{i-1} \;+\; \frac{1}{2a_i}\overline{x}_i^2 \qquad (28)$$

By assumption, its time derivative along the solutions of (1)-(12) satisfies :

$$\dot{V}_i \leq -\sum_{j=1}^{i-1}(k_j - i + 1 + j)\bar{x}_j^2 + \bar{x}_{i-1}\bar{x}_i - \frac{\bar{c}}{2^{i-1}}r$$
$$+d_{i-1}(t) + \frac{1}{\lambda}(\hat{\vartheta} - \vartheta)(\dot{\hat{\vartheta}} - \tau_{i-1}) + (i-2)(\dot{\hat{\vartheta}} - \tau_{i-1})^2$$
$$+ 2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\dot{\hat{\vartheta}} - \tau_{i-1})$$
$$+\bar{x}_i\left[x_{i+1} + \frac{1}{a_i}f_i - \sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}(\frac{a_j}{a_i}x_{j+1} + \frac{1}{a_i}f_j)\right.$$
$$\left.-\frac{1}{a_i}\frac{\partial w_{i-1}}{\partial r}(-\bar{c}r + x_1^2\varphi_0) - \frac{1}{a_i}\frac{\partial w_{i-1}}{\partial \hat{\vartheta}}\dot{\hat{\vartheta}}\right] \quad (29)$$

Let ϑ be a real number satisfying :

$$\vartheta \geq \max\left\{b_{i-1}, \frac{\vartheta^*}{a_i}, \frac{\vartheta^{*2}}{a_i^2}, \frac{a_{i-1}}{a_i}, \frac{a_1^2}{a_i^2}, \ldots, \frac{a_{i-1}^2}{a_i^2}, \frac{1}{a_i^2}\right\}$$
$$:= b_i \quad (30)$$

With the help of Assumptions (H1) and (H2), by virtue of the fact that mapping $(x_1, \ldots, x_i) \mapsto (\bar{x}_1, \ldots, \bar{x}_i)$ is a global diffeomorphism preserving the origin, lengthy but simple calculations imply the existence of some non-negative smooth functions φ_{i1} and φ_{i2} such that

$$\bar{x}_i\frac{1}{a_i}f_i \leq \frac{1}{4}\sum_{j=1}^{i}\bar{x}_j^2 + \frac{\bar{c}}{6\times 2^{i-1}}r$$
$$+\vartheta\bar{x}_i^2\varphi_{i1}(x_1, \ldots, x_i, r, \hat{\vartheta}) + d_{i1}(t) \quad (31)$$
$$-\bar{x}_i\sum_{j=1}^{i-1}\frac{\partial w_{i-1}}{\partial x_j}(\frac{a_j}{a_i}x_{j+1} + \frac{1}{a_i}f_j) \leq \frac{1}{4}\sum_{j=1}^{i-1}\bar{x}_j^2$$
$$+\vartheta\bar{x}_i^2\varphi_{i2}(x_1, \ldots, x_i, r, \hat{\vartheta}) + \frac{\bar{c}}{6\times 2^{i-1}}r$$
$$+\frac{1}{4}\bar{x}_i^2\sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2 + d_{i2}(t) \quad (32)$$

where $d_{i1}(t)$ and $d_{i2}(t)$ are defined by :

$$d_{i1}(t) = \left(\frac{\vartheta^*}{a_i}\phi_{i2}\circ\psi_1^{-1}\circ(\mathrm{Id} + \rho_i^{-1})(D(t))\right)^2$$
$$d_{i2}(t) = \sum_{j=1}^{i-1}\left(\frac{\vartheta^*}{a_i}\phi_{j2}\circ\psi_1^{-1}\circ(\mathrm{Id} + \rho_j^{-1})(D(t))\right)^2$$

Notice that $d_{i1}(t) = d_{i2}(t) = 0$ for all $t \geq T^o$.

On the other hand, completing the squares yields :

$$-\bar{x}_i\frac{1}{a_i}\frac{\partial w_{i-1}}{\partial r}(-\bar{c}r + x_1^2\varphi_0) \leq \frac{\bar{c}}{6\times 2^{i-1}}r$$
$$+\frac{1}{4}x_1^2 + \vartheta\bar{x}_i^2(\frac{\partial w_{i-1}}{\partial r})^2\left(3\bar{c}2^{i-2}r + x_1^2\varphi_0^2\right) \quad (33)$$

It remains to examine the last term in (29). Observing

$$|\tau_{i-1}| \leq |(\bar{x}_1, \ldots, \bar{x}_{i-1})|\hat{\tau}_{i-1}(x_1, \ldots, x_{i-1}, r, \hat{\vartheta}) \quad (34)$$

where $\hat{\tau}_{i-1}$ is a nonnegative smooth function, with (30) and by completing the squares,

$$-\bar{x}_i\frac{1}{a_i}\frac{\partial w_{i-1}}{\partial \hat{\vartheta}}\dot{\hat{\vartheta}} \leq \frac{1}{4}\sum_{j=1}^{i-1}\bar{x}_j^2 + \vartheta\bar{x}_i^2(\frac{\partial w_{i-1}}{\partial \hat{\vartheta}})^2(1 + \hat{\tau}_{i-1}^2)$$
$$+(\tau_{i-1} - \dot{\hat{\vartheta}})^2 \quad (35)$$

Accordingly, in view of (31), (32), (33) and (35), (29) implies :

$$\dot{V}_i \leq -\sum_{j=1}^{i-1}(k_j - i + j)\bar{x}_j^2 - \frac{\bar{c}}{2^i}r + d_i(t)$$
$$+\frac{1}{\lambda}(\hat{\vartheta} - \vartheta)(\dot{\hat{\vartheta}} - \tau_{i-1}) + (i-2)(\dot{\hat{\vartheta}} - \tau_{i-1})^2$$
$$+2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\dot{\hat{\vartheta}} - \tau_{i-1})$$
$$+\bar{x}_i\left[x_{i+1} + \bar{x}_{i-1} + \frac{1}{4}\bar{x}_i + \frac{1}{4}\bar{x}_i\sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2\right.$$
$$+\vartheta\bar{x}_i(\varphi_{i1} + \varphi_{i2}) + \vartheta\bar{x}_i(\frac{\partial w_{i-1}}{\partial r})^2\left(3\bar{c}2^{i-2}r + x_1^2\varphi_0^2\right)$$
$$\left.+\vartheta\bar{x}_i(\frac{\partial w_{i-1}}{\partial \hat{\vartheta}})^2(1 + \hat{\tau}_{i-1}^2)\right] + (\tau_{i-1} - \dot{\hat{\vartheta}})^2 \quad (36)$$

where $d_i(t) = d_{i1}(t) + d_{i2}(t)$. Notice that $d_i(t) = 0$ for all $t \geq T^o$.

Introducing the following notation :

$$\tau_i = \tau_{i-1} + \lambda\bar{x}_i^2\left[\varphi_{i1} + \varphi_{i2} + (\frac{\partial w_{i-1}}{\partial r})^2\times\right.$$
$$\left.(3\bar{c}2^{i-2}r + x_1^2\varphi_0^2) + (\frac{\partial w_{i-1}}{\partial \hat{\vartheta}})^2(1 + \hat{\tau}_{i-1}^2)\right]$$
$$:= \tau_{i-1} + \bar{x}_i^2\tau_{(i-1)i}(x_1, \ldots, x_i, r, \hat{\vartheta}) \quad (37)$$
$$w_i = -(k_i + \frac{1}{4})\bar{x}_i - \kappa_i(x_1, \ldots, x_i, r, \hat{\vartheta})\bar{x}_i - \bar{x}_{i-1}$$
$$-\frac{1}{4}\bar{x}_i\sum_{j=1}^{i-1}\left(\frac{\partial w_{i-1}}{\partial x_j}\right)^2 - \hat{\vartheta}\bar{x}_i\left[(\frac{\partial w_{i-1}}{\partial \hat{\vartheta}})^2(1 + \hat{\tau}_{i-1}^2)\right.$$
$$\left.+(\frac{\partial w_{i-1}}{\partial r})^2\left(3\bar{c}2^{i-2}r + x_1^2\varphi_0^2\right) + \varphi_{i1} + \varphi_{i2}\right]$$
$$\bar{x}_{i+1} := x_{i+1} - w_i(x_1, \ldots, x_i, r, \hat{\vartheta})$$

where $k_i > n - i$ and κ_i is any nonnegative smooth function, (36) implies :

$$\dot{V}_i \le -\sum_{j=1}^{i}(k_j - i + j)\overline{x}_j^2 + \overline{x}_i\overline{x}_{i+1} - \frac{\bar{c}}{2^i}r + d_i(t)$$
$$- \kappa_i\overline{x}_i^2 + 2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\dot{\widehat{\vartheta}} - \tau_{i-1}) \quad (38)$$
$$+\frac{(\widehat{\vartheta} - \vartheta)(\dot{\widehat{\vartheta}} - \tau_i)}{\lambda} + (i-2)(\dot{\widehat{\vartheta}} - \tau_{i-1})^2 + (\tau_{i-1} - \dot{\widehat{\vartheta}})^2$$

Direct calculations yield :

$$2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\dot{\widehat{\vartheta}} - \tau_{i-1}) + (i-2)(\dot{\widehat{\vartheta}} - \tau_{i-1})^2$$
$$+(\tau_{i-1} - \dot{\widehat{\vartheta}})^2 = 2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})(\tau_i - \tau_{i-1})$$
$$+(i-1)(\tau_i - \tau_{i-1})^2 + 2\sum_{j=2}^{i}(j-1)(\tau_j - \tau_{j-1})(\dot{\widehat{\vartheta}} - \tau_i)$$
$$+ (i-1)(\dot{\widehat{\vartheta}} - \tau_i)^2 . \quad (39)$$

Using the important fact in (37) :

$$\tau_i - \tau_{i-1} = \overline{x}_i^2\tau_{(i-1)i}(x_1, \ldots, x_i, r, \widehat{\vartheta}) \quad (40)$$

by choosing κ_i such that

$$\kappa_i \ge \left|2\sum_{j=2}^{i-1}(j-1)(\tau_j - \tau_{j-1})\tau_{(i-1)i} + (i-1)\overline{x}_i^2\tau_{(i-1)i}^2\right| , \quad (41)$$

from (38), it follows :

$$\dot{V}_i \le -\sum_{j=1}^{i}(k_j - i + j)\overline{x}_j^2 + \overline{x}_i\overline{x}_{i+1} - \frac{\bar{c}}{2^i}r$$
$$+d_i(t) + \frac{1}{\lambda}(\widehat{\vartheta} - \vartheta)(\dot{\widehat{\vartheta}} - \tau_i) + (i-1)(\dot{\widehat{\vartheta}} - \tau_i)^2$$
$$+2\sum_{j=2}^{i}(j-1)(\tau_j - \tau_{j-1})(\dot{\widehat{\vartheta}} - \tau_i) \quad (42)$$

This means that (27) holds again for i as long as (30) is satisfied.

Therefore, by induction, smooth functions w_n and τ_n are constructed at Step n such that, with :

$$\dot{\widehat{\vartheta}} = \tau_n(x_1, \ldots, x_n, r, \widehat{\vartheta}) \quad (43)$$

$$u = w_n(x_1, \ldots, x_n, r, \widehat{\vartheta}) , \quad (44)$$

the time derivative of the positive function

$$V_n = \sum_{j=1}^{n}\frac{1}{2a_j}\overline{x}_j^2 + r + \frac{1}{2\lambda}(\widehat{\vartheta} - \vartheta)^2 \quad (45)$$

along the solutions of (1), (12), (43) and (44) satisfies :

$$\dot{V}_n \le -\sum_{j=1}^{n}(k_j - n + j)\overline{x}_j^2 - \frac{\bar{c}}{2^n}r + d_n(t) \quad (46)$$

where $d_n(t) \ge 0$ and $= 0$ for all $t \ge T^o$.

4. MAIN RESULT

Theorem 4.1 *Under Assumptions (H1)-(H4), for any initial condition* $(x^o, z^o, r^o, \widehat{\vartheta}^o)$ *in* $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}_{>0} \times \mathbb{R}$, *the associate solution* $(x(t), z(t), r(t), \widehat{\vartheta}(t))$ *of the closed loop system (1), (12), (43) and (44) is well defined on* $[0, +\infty)$, *unique and bounded. Furthermore, there exists a positive real number* ϑ_∞ *such that*

$$\lim_{t\to+\infty}(|x(t)| + |z(t)| + r(t)) = 0 \quad (47)$$

$$\lim_{t\to+\infty}\widehat{\vartheta}(t) = \vartheta_\infty . \quad (48)$$

Proof (Outline): Since $d_n(t)$ is defined for each $t \ge 0$ and equal to zero for all $t \ge T^o$, from (46), it follows :

$$V_n(t) \le V_n(x(0), r(0), \widehat{\vartheta}(0)) + \int_0^{T^o} d_n(s)ds \quad (49)$$

This, in conjunction with (45), (25) and (13), implies that the corresponding solution $(x(t), z(t), r(t), \widehat{\vartheta}(t))$ of the closed-loop system is well defined on $[0, +\infty)$ and bounded.

Since $d_n(t) = 0$ for all $t \ge T^o$, (46) implies :

$$\dot{V}_n \le -\sum_{j=1}^{n}(k_j - n + j)\overline{x}_j^2 - \frac{\bar{c}}{2^n}r , \quad t \ge T^o$$

The proof is completed with the help of LaSalle's invariance principle and equations (22), (25) and (37).

Remark 4.1 Theorem 4.1 applies to systems (1) having the origin as a fixed equilibrium point (see (H2)). If this is not the case, for example, ϕ_{i1} and/or ϕ_{i2} are not zero at zero, it is useful to employ the idea of σ-modification proposed in (Ioannou and Kokotović, 1984) in the design of adaptive law to prevent the possible parameter drift instability. However, in this case, only the global boundedness property can be established for the closed-loop solutions. With this in mind, extension to the tracking case is direct.

Example 4.1 Consider the nonlinear system :

$$\begin{aligned} \dot{z} &= -z + x_1^2 \\ \dot{x}_1 &= a_1 x_2 + \theta_1 x_1 e^{\theta_2 x_1} + \theta_3 z \\ \dot{x}_2 &= a_2 u + \theta_4 x_2^2 + \theta_5 z^2 x_1 \end{aligned} \tag{50}$$

where $a_1 > 0$, $a_2 > 0$, $\theta_i \, (1 \leq i \leq 5)$ are unknown constant parameters and z is unmeasured.

Clearly, Assumption (H1) is checked. Assumption (H2) holds with :

$$\begin{aligned} \phi_{11}(s) &= s\, e^{0.5s^2} , \quad \phi_{12}(s) = s , \\ \phi_{21}(s) &= 0.5s^2 , \quad \phi_{22}(s) = 0.5s^4 \\ \vartheta^* &= \max\{|\theta_1| e^{0.5\theta_2^2} , |\theta_3| , 2|\theta_4| , |\theta_5|\} \end{aligned}$$

It is direct to verify that Assumption (H3) holds for the z-subsystem with :

$$V_z(z) = z^2 , \quad c = 1.2 , \quad \gamma(s) = 1.25s^4 \tag{51}$$

Finally, Assumption (H4) is satisfied.

Therefore, applying the control design procedure in Section 3 yields a self-tuning globally regulating partial-state feedback controller for system (50).

5. CONCLUSION

A class of uncertain nonlinear systems with nonlinearly appeared unknown parameters and stable dynamic uncertainties was considered in this paper. Inspired by some early work in adaptive linear control (Praly, 1990), an available dynamic signal is introduced to bound the dynamic uncertainty. The philosophy underlying the proposed constructive control scheme is an iterative use of the now standard "adding one integrator" techniques (see, e.g., (Byrnes and Isidori, 1989; Tsinias, 1989)). The main advantages of earlier adaptive control algorithms for feedback linearizable systems are also kept: for example, one does not demand neither the matching condition nor the usual growth conditions as required in previous work (Pomet and Praly, 1992; Praly *et al.*, 1991; Jiang and Praly, 1992; Jiang, 1995). However, the control design procedure proposed in this paper brings new advantages: nonlinear parametrization is allowed and the common feature of overparameterization in earlier adaptive schemes is removed here.

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