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ON OUTPUT FEEDBACK STABILIZATION FOR SYSTEMS WITH ISS INVERSE DYNAMICS AND UNCERTAINTIES

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Abstract

For systems admitting a certain global output normal form with input-to-state practically stable (ISpS) inverse dynamics, practical regulation of the output can be achieved by output feedback knowing only the relative degree, the sign of the so called "high frequency gain" and a monotone function bounding the nonlinearities for large signals. This result is only stated and discussed here. Its proof can be found in [16].

1 Problem statement

Our goal is to achieve global practical output regulation by output feedback with minimal information. Our approach is to propose a family of dynamic output feedback controllers parameterized by integer numbers, real numbers and real functions. Then, with a particular controller chosen, we characterize a class of systems such that the control objective is achieved for each plant in the class.

2 Main result

2.1 The family of controllers

For the system we want to control, let u be its single input and y be its (possibly corrupted) output measurement. The family of controllers we propose is characterized by the following $r+2$-dimensional dynamic system :

$$\left.\begin{array}{ll}\dot{\widehat{x}}_1 &= \widehat{x}_2 + \ell_1\left(\wp_a^+(y)-\widehat{x}_1\right)\\ \dot{\widehat{x}}_2 &= \widehat{x}_3 + \ell_2\left(\wp_a^+(y)-\widehat{x}_1\right)\\ &\vdots\\ \dot{\widehat{x}}_r &= \widehat{x}_{r+1} + \ell_r\left(\wp_a^+(y)-\widehat{x}_1\right)+u\\ \dot{\widehat{x}}_{r+1} &= \quad + \ell_{r+1}\left(\wp_a^+(y)-\widehat{x}_1\right)\\ \dot{\widehat{k}} &= \sigma\dfrac{\exp\left(-\frac{1}{\max\left\{V_r\left(y,\widehat{x}_2,\ldots,\widehat{x}_r,\widehat{k}\right)-\varpi,0\right\}}\right)}{1+\left|\frac{\partial V_r}{\partial k}\left(y,\widehat{x}_2,\ldots,\widehat{x}_r,\widehat{k}\right)\right|}\\ u &= u_{r,2}\left(y,\widehat{x}_2,\ldots,\widehat{x}_r,\widehat{k}\right)-\widehat{x}_{r+1}\end{array}\right\}\qquad(1)$$

with $\widehat{k}$ initialized at any strictly positive value. This controller is parameterized by the following items :

1 – a positive integer number r which will correspond to the relative degree of the system to be controlled,

2 – real numbers ℓ_i's, chosen as Hurwitz gains, which are coefficients of the characteristic polynomial of what will be interpreted as an observer,

3 – a smooth real function $\wp_a^+$, introduced to "invert" the "sensor mapping",

4 – a real number σ chosen in $[0, \exp(1)]$,

5 – a real number ϖ whose square root is the threshold we assign for the output to remain below, i.e. it is a dead-zone,

6 – a C^1 function $u_{r,2}$ and a positive C^2 function V_r which are derived from choosing three C^1 functions of class K : $\gamma_{\wp_a^+}$, Γ and $\Gamma_\wp$.

In the following we consider the controller to be fixed, implying that all these parameters are given.

2.2 The class of plants

By means of four assumptions, we now characterize a class of nonlinear systems such that global practical output regulation will be achieved using a given controller of the form (1). The first two assumptions address the feasibility of using dynamic output feedback while the third assumption insures the compatibility of the system with the given controller. The fourth assumption is a technical condition needed only if the control objective is convergence to a desired set point. These assumptions are discussed in more detail in sections 3 and 4.

Assumption ST (Structure) : *The system to be controlled can be globally described by :*

$$\begin{aligned}\dot{z} &= h(z,x_1,t)\\ \dot{x}_i &= x_{i+1} + f_i(z,x_1,t)\,,\ i\in\{1,\ldots,m-1\}\\ \dot{x}_m &= x_{m+1} + f_m(z,x_1,t) + u\\ \dot{x}_{m+1} &= 0\\ y &= \wp(z,x_1,t)\end{aligned}\qquad(2)$$

with a single input u, a single measurement y, coordinates $(z, x_1, \ldots, x_{m+1})$ in $\mathbb{R}^n \times \mathbb{R}^{m+1}$, and functions f_i's, h and $\wp$ sufficiently smooth.

Let $\Phi(z, x_1, t)$ be the vector in $\mathbb{R}^{m+1}$ whose components are the f_i's with $f_{m+1} = 0$, except for the first one, obtained from the equation satisfied by $\dot{y}$:

$$\begin{aligned}\Phi_1(z,x_1,t) \;=\; &\frac{\partial\wp}{\partial z}(z,x_1,t)h(z,x_1,t)\\ &+\frac{\partial\wp}{\partial x_1}(z,x_1,t)f_1(z,x_1,t) + \frac{\partial\wp}{\partial t}(z,x_1,t)\,.\end{aligned}\qquad(3)$$

Assumption QL (Qualitative) :
__QL1__ : *The z-subsystem is ISpS. That is : there exist*

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functions β, of class $\mathcal{KL}$, and γ, of class $\mathcal{K}$[1]*, and a positive real number d such that, for any real numbers t_0 and T, with $t_0 < T$, for any initial condition z_0 and any C^0 function $x_1 : [t_0, T) \to \mathbb{R}$, there exists a unique solution $z(t)$ of :*

$$\dot{z} \;=\; h(z, x_1(t), t) \quad , \qquad z(t_0) \;=\; z_0 \; . \qquad (4)$$

It is defined on $[t_0, T)$ and, for all s and t satisfying $t_0 \leq s \leq t < T$, we have :

$$|z(t)| \leq \beta\left(|z(s)|, (t-s)\right) + \gamma\left(\sup_{s \leq \tau \leq t}\{|x_1(\tau)|\}\right) + d \qquad (5)$$

If d is equal to 0, the system is said to be ISS.

QL2 : *There exist a C^1 nondecreasing positive function $\gamma_\wp$ and two positive real numbers $\zeta \in (0,1]$ and $d_\wp \in \mathbb{R}_+$, satisfying, for all (z,t) in $\mathbb{R}^n \times \mathbb{R}$,*

$$\textbf{QL2.1} \;:\; \wp(z, x_1, t) = 0 \quad \Longrightarrow \quad |x_1| \leq d_\wp \; , \quad (6)$$

$$\textbf{QL2.2} \;: \zeta \;\leq\; \frac{\partial \wp}{\partial x_1}(z, x_1, t) \;\leq\; \gamma_\wp(|x_1|) \quad \forall x_1 \in \mathbb{R} \, . \quad (7)$$

QL3 : *There exist two functions γ_{x_1} and γ_z of class $\mathcal{K}$ and three positive real numbers C, s_1 and d_Φ such that :*

QL3.1 *: for all (z, x_1, t) in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$,*

$$|\Phi(z, x_1, t)| \;\leq\; \gamma_{x_1}(|x_1|) \;+\; \gamma_z(|z|) \;+\; d_\Phi \qquad (8)$$

$$\textbf{QL3.2} \;: \qquad \gamma_z(s) \;\leq\; C s \qquad \forall s \in [0, s_1] \; , \qquad (9)$$

$$\textbf{QL3.3} \;: (\gamma_{x_1} + \gamma_z \circ 2\gamma)(s) \leq C s \quad \forall s \in [0, s_1] \; . (10)$$

Assumption QT (Quantitative) :
Let r, $\wp_a^+$, $\gamma_{\wp_a^+}$, Γ and $\Gamma_\wp$ be given by the controller.

QT1 : *There exists a positive real number $d_{\wp_a^+}$ such that, for all (z, x_1, t) in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$,*

$$\left|x_1 - \wp_a^+(\wp(z, x_1, t))\right| \;\leq\; \gamma_{\wp_a^+}(|x_1|) \;+\; d_{\wp_a^+} \; . \quad (11)$$

QT2 : *There exists a positive real number s_2 such that we have :*

$$\textbf{QT2.1} : (\gamma_{x_1} + \gamma_z \circ 2\gamma)(s) \leq \Gamma(s) \;\; \forall s \in [s_2, \infty) \;\; (12)$$

$$\textbf{QT2.2} : \;\; \gamma_\wp(s) - \gamma_\wp(0) \leq \Gamma_\wp(s) \;\; \forall s \in [s_2, \infty) \; , \;\; (13)$$

QT3 : *The relative degree m is equal to r.*

Assumption T (Technical) : *There exist two strictly positive real numbers $s^\star$ and $q \geq 2$ such that :*

$$\beta(s^\star, \cdot) \;\in\; L^q([0, +\infty)) \; . \qquad (14)$$

Our main result is the following:

Theorem 1 *By applying the controller (1) to any dynamic system satisfying assumptions ST, QL and QT, we obtain existence, uniqueness and boundedness of all the solutions of the closed loop system. Moreover, if σ is chosen strictly positive, the output y of each of the solutions satisfies :*

$$\limsup_{t \to \infty} |y(t)|^2 \;\leq\; \varpi \; . \qquad (15)$$

Furthermore, if d_Φ, d, $d_\wp$, $d_{\wp_a^+}$ are zero and assumption T is satisfied, then the "dead-zone" ϖ can be set equal to 0, and, in this case, all the solutions converge to a 2-dimensional manifold where we have in particular $z = 0$ and $x_1 = \ldots = x_r = 0$.

3 Feasibility of output feedback

3.1 Assumption ST: the normal form

Byrnes and Isidori, in [1], have given necessary and sufficient conditions under which a system can be written globally in the form :

$$\left.\begin{array}{ll} \dot{z} = \mathcal{H}(z, \chi_1) & \\ \dot{\chi}_i = \chi_{i+1} & i \in \{1, \ldots, r-1\} \\ \dot{\chi}_r = \mathcal{F}(z, \chi_1, \ldots, \chi_r) + \mathcal{G}(z, \chi_1, \ldots, \chi_r) u & \\ y = \chi_1 & \end{array}\right\} \qquad (16)$$

Compared to this form, assumption ST imposes two restrictions : the functions f_i's must depend only on (z, x_1) and the function g must be identically equal to 1; and it allows one relaxation : the output measurements may be corrupted, i.e. $y \neq \chi_1$.

3.1.1 f_i depends only on (z, x_1)

Rewriting (2) with the time derivatives of x_1 as coordinates, we get, when (2) is time invariant, $\wp$ is the identity function and x_{r+1} is zero,

$$\left.\begin{array}{ll} \dot{z} = \mathcal{H}(z, \chi_1) & \\ \dot{\chi}_i = \chi_{i+1} & i \in \{1, \ldots, r-1\} \\ \dot{\chi}_r = \displaystyle\sum_{i=1}^{r} f_i(z, \chi_1)^{(r-i)} \;+\; u & \\ y = \chi_1 & \end{array}\right\} \qquad (17)$$

where $(\cdot)^{(r-i)}$ denotes the $(r-i)$th Lie derivative along the vector field given by (2). This exhibits a very particular structure for the function $\mathcal{F}$ in (16). The motivation for this restricted structure is the several counterexamples, given in [13], to global output feedback stabilization for systems in the form (16).

Instead of the constraint (17) imposed here, one may restrict the behavior at infinity of $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{G}$. For example, Khalil and Saberi have proved :

Theorem 2 ([8]) *If, for the system (16),*

a) *the zero solution of the following system is globally exponentially stable :*

$$\dot{z} \;=\; \mathcal{H}(z, 0) \; , \qquad (18)$$

b) $$\mathcal{H}(0,0) \;=\; 0 \quad , \qquad \mathcal{F}(0, 0, \ldots, 0) \;=\; 0 \; , \quad (19)$$

c) *the sign of $\mathcal{G}(z, \chi_1, \ldots, \chi_r)$ is definite and known,*

d) *$\mathcal{H}$ and $\mathcal{F}$ are globally Lipschitz continuous and $\mathcal{G}$ is bounded,*

then there exists a dynamic output feedback controller which guarantees global exponential stability of the origin of the closed loop system.

3.1.2 The input vector field, $\mathcal{G} = 1$

Again, by comparing (17) and (16), we see that $\mathcal{G}$ in (17) must be known, depend only on χ_1 and have a constant sign (then change $\mathcal{G}(\chi_1)\, u$ into u). Khalil

[1] See [17] for a definition

and Saberi need only that $\mathcal{G}$ be bounded and bounded away from 0. Further, they allow it to depend on all the variables. In [11, 10, 6], Marino and Tomei and Kanellakopoulos et al., respectively, impose that $\mathcal{G}$ depends only on $\mathcal{X}_1$ but this function is known only up to a multiplicative strictly positive real number (see Theorems 4 and 5 below). In our framework, this latter case can be considered as sensor corruption. Indeed consider the following system :

$$\begin{aligned} \dot{z} &= \overline{h}(z, \overline{x}_1, t) \\ \dot{\overline{x}}_i &= \overline{x}_{i+1} + \overline{f}_i(z, \overline{x}_1, t) \quad i \in \{1, \dots, r-1\} \\ \dot{\overline{x}}_r &= \theta\, u + \overline{f}_r(z, \overline{x}_1, t) \\ y &= \overline{x}_1 \end{aligned} \tag{20}$$

with θ an unknown strictly positive real number. Then, by letting :

$$x_i = \frac{1}{\theta} \overline{x}_i \quad i \in \{1, \dots, r\}, \tag{21}$$

we can rewrite this system as :

$$\begin{aligned} \dot{z} &= h(z, x_1, t) \\ \dot{x}_i &= x_{i+1} + f_i(z, x_1, t) \quad i \in \{1, \dots, r-1\} \\ \dot{x}_r &= u + f_r(z, x_1, t) \\ y &= \theta\, x_1 \end{aligned} \tag{22}$$

with :

$$\left.\begin{aligned} h(z, x_1, t) &= \overline{h}(z, \theta x_1, t) \\ f_i(\overline{z}, x_1, t) &= \frac{\overline{f}_i(z, \theta x_1, t)}{\theta}\ . \end{aligned}\right\} \tag{23}$$

This system is in the form (2), with :

$$\wp(z, x_1, t) = \theta\, x_1\ . \tag{24}$$

Then it is easy to show that, with θ any positive real number, if assumption QL holds for (20) then it also holds for (22). Note that QT1 and QT2.2 hold with :

$$\wp_d^+(y) = 0\ , \quad \gamma_{\wp_d^+}(s) = s\ , \quad \Gamma_\wp(s) = s\ . \tag{25}$$

3.1.3 The role of x_{m+1}

In (16) and (17), $x_{m+1} \equiv 0$. In our context, a nonzero value for $x_{m+1}(0)$ may be useful for handling nonzero set points or nonvanishing nonlinearities. For example, consider the following system with input u and output $\overline{y}$,

$$\left.\begin{aligned} \dot{\overline{z}} &= \overline{h}(\overline{z}, \overline{x}_1, t) \\ \dot{\overline{x}}_i &= \overline{x}_{i+1} + \overline{f}_i(\overline{z}, \overline{x}_1, t)\ i \in \{1, \dots, r-1\} \\ \dot{\overline{x}}_r &= u + \overline{f}_r(\overline{z}, \overline{x}_1, t) \\ \overline{y} &= \overline{x}_1 \end{aligned}\right\} \tag{26}$$

Let us denote by $\overline{y}_d$ the desired rest point for the output $\overline{y}$. We assume that this particular value $\overline{y}_d$ is achievable, ie. there exists a real number u_d such that the system (26) above with the constant control u_d has an equilibrium point, whose components are denoted $(\overline{z}_d, \overline{x}_{1d}, \dots, \overline{x}_{rd})$, satisfying :

$$\overline{y}_d = \overline{x}_{1d}\ . \tag{27}$$

We remark that this implies :

$$\left.\begin{aligned} \overline{x}_{id} &= -\overline{f}_{i-1}(\overline{z}_d, \overline{x}_{1d}, t) \quad i \in \{2, \dots, r\} \\ u_d &= -\overline{f}_r(\overline{z}_d, \overline{x}_{1d}, t) \qquad \forall t \in \mathbb{R}\ . \end{aligned}\right\} \tag{28}$$

Under this condition the system (26) can be rewritten in the form (2) whose desired rest point for y is 0. Indeed, this is obtained by letting, with i in $\{1, \dots, r\}$,

$$\left.\begin{aligned} z &= \overline{z} - \overline{z}_d \\ h(z, x_1, t) &= \overline{h}(\overline{z}, \overline{x}_1, t) \\ x_i &= \overline{x}_i - \overline{x}_{id} \\ f_i(z, x_1, t) &= \overline{f}_i(\overline{z}, \overline{x}_1, t) - \overline{f}_i(\overline{z}_d, \overline{x}_{1d}, t) \\ x_{r+1} &= u_d \\ y &= \overline{y} - \overline{y}_d \end{aligned}\right\} \tag{29}$$

3.2 The qualitative assumptions

3.2.1 Assumption QL1 : the z-subsystem is ISpS

Detectability with no input information implies at least that the origin is a globally asymptotically stable solution of the zero dynamics :

$$\dot{z} = \mathcal{H}(z, 0)\ . \tag{30}$$

The interest of such a stability property is well known for linear systems : it is sufficient to know the relative degree r and the sign of the so called high frequency gain to be able to design a dynamic output feedback providing global asymptotic stabilization.

Unfortunately, such a property does not extend to the nonlinear case as shown by the counterexamples given in [13]. Imposing the ISpS property is one possible way to go around the difficulty. The ISS, and therefore ISpS, properties hold for the special cases considered in [8], [9, 10, 11, 12] and [6]. The ISS property has been introduced by Sontag in [17].

3.2.2 Corrupted output measurements

The actual system output x_1 is not directly measured. Instead we have access to y as :

$$y = \wp(z, x_1, t) \tag{31}$$

where $\wp$ is supposed to represent the effects of a sensor. However, the constraint QL2 implies that, for each (z, t), the x_1 to y relation is strictly increasing and the function $\wp(\cdot, x_1, \cdot)$ is bounded for each x_1. This means that, for each x_1, the measurement y differs from the actual output x_1 by only a finite amount, uniformly in (z, t).

3.2.3 Assumptions QL3

The function Φ being smooth, there exists a smooth function $\overline{\Phi}$ such that, for all (z, x_1, t) in $\mathbb{R}^r \times \mathbb{R} \times \mathbb{R}$,

$$\Phi(z, x_1, t) = \overline{\Phi}(z, x_1, t) \begin{pmatrix} z \\ x_1 \end{pmatrix} + \Phi(0, 0, t)\ . \tag{32}$$

Then, if the functions $\overline{\Phi}(z, x_1, \cdot)$ and $\Phi(0, 0, \cdot)$ are bounded for each (z, x_1) in $\mathbb{R}^n \times \mathbb{R}$, it follows that assumptions QL3.1 and QL3.2 are satisfied with the following well defined quantities :

$$\left.\begin{aligned} \gamma_{x_1}(s) &= 2\, s \sup_{\substack{|z| \leq |x_1| \leq s \\ t \in \mathbb{R}}} \left\{ \left| \overline{\Phi}(z, x_1, t) \right| \right\} \\ \gamma_z(s) &= 2\, s \sup_{\substack{|x_1| \leq |z| \leq s \\ t \in \mathbb{R}}} \left\{ \left| \overline{\Phi}(z, x_1, t) \right| \right\} \\ d_\Phi &= \sup_{t \in \mathbb{R}} \{ |\Phi(0, 0, t)| \} \end{aligned}\right\} \tag{33}$$

In particular γ_{x_1} is linearly bounded on a neighborhood of 0. It follows that QL3.3 is an assumption only on the composition $\gamma_z \circ 2\gamma$ which captures, with an L^∞-norm, the behavior of the input-output operator given by the z-subsystem with x_1 as input and Φ as output. Precisely, with γ the function of the ISpS property of the z-subsystem, QL3.3 is satisfied if :

$$\limsup_{s\to 0} \frac{\gamma(s)\left[\sup_{\substack{|z|\leq 2\gamma(s)\\ t\in\mathbb{R}}} \left\{|\overline{\Phi}(z,0,t)|\right\}\right]}{s} < +\infty . \tag{34}$$

This holds in particular if γ is linearly bounded on a neighborhood of 0. This can always be satisfied by increasing d in (5). However, with d not zero, only practical regulation can be achieved. Nevertheless if, besides assumption QL1, the system :

$$\dot{z} = h(z,0,t) \tag{35}$$

has a zero solution which is locally exponentially stable, then γ can be chosen linearly bounded on a neighborhood of 0 together with d being zero.

To be more specific, let us consider the case where (see [9, 10, 11, 12, 6]) :

– the time dependence is introduced by time varying parameters $\theta : \mathbb{R} \to \mathcal{C} \subset \mathbb{R}^p$, with $\mathcal{C}$ compact :

$$\left.\begin{array}{l}\overline{\Phi}(z,x_1,t) = \overline{\Phi}_\theta(z,x_1,\theta(t)) \ ,\\ h(z,x_1,t) = h_\theta(z,x_1,\theta(t)) \ ,\end{array}\right\} \tag{36}$$

where, maybe thanks to a reparameterization, the functions $\overline{\Phi}_\theta$ and h_θ are smooth on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$

– the function h_θ is linear in z, i.e. :

$$h_\theta(z,x_1,\theta) = A_z(\theta)\, z + H(x_1,\theta) \tag{37}$$

where the matrix $A_z(\theta(t))$ is such that its transition matrix ϕ satisfies, with some strictly positive real numbers c and α,

$$|\phi(t,t_0)| \leq c\, \exp(-\alpha(t-t_0))\ \forall (t,t_0)\in\mathbb{R}^2 \tag{38}$$

– the nonlinearities are zero at the origin :

$$H(0,\theta(t)) = 0\ ,\ \Phi_\theta(0,0,\theta(t)) = 0\ \forall t\in\mathbb{R} \tag{39}$$

In this case, we can take :

$$\gamma_{x_1}(s) = 2\,s\left[\sup_{S_1}\left\{|\overline{\Phi}_\theta|\right\} + \sup_{S_2}\left\{|\overline{\Phi}_\theta|\right\}\right] \tag{40}$$

$$\gamma_z(s) = 2\,s\left[\sup_{S_3}\left\{|\overline{\Phi}_\theta|\right\} + \sup_{S_4}\left\{|\overline{\Phi}_\theta|\right\}\right] \tag{41}$$

$$\gamma(s) = s\,\frac{c}{\alpha}\left[\sup_{S_5}\left\{\left|\frac{H}{x_1}\right|\right\} + \sup_{S_6}\left\{\left|\frac{H}{x_1}\right|\right\}\right] \tag{42}$$

where the suprema are taken with respect to (z,x_1,θ) in the sets :

$$\begin{array}{c} S_1 = \left\{\begin{array}{c}|z| \leq |x_1| \leq s\\ |\theta| \leq s\end{array}\right\}\ ,\ S_2 = \left\{\begin{array}{c}|z| \leq |x_1| \leq |\theta|\\ \theta\in\mathcal{C}\end{array}\right\}\\ S_3 = \left\{\begin{array}{c}|x_1| \leq |z| \leq s\\ |\theta| \leq s\end{array}\right\}\ ,\ S_4 = \left\{\begin{array}{c}|x_1| \leq |z| \leq |\theta|\\ \theta\in\mathcal{C}\end{array}\right\}\\ S_5 = \left\{\begin{array}{c}|x_1| \leq s\\ |\theta| \leq s\end{array}\right\}\ ,\ S_6 = \left\{\begin{array}{c}|x_1| \leq |\theta|\\ \theta\in\mathcal{C}\end{array}\right\}\end{array} \tag{43}$$

So assumption QL3 is satisfied.

4 Relaxing the required information

Given that we only consider plants for which assumptions ST and QL are satisfied, Theorem 1 states that assumption QT makes precise what information about the plant is required to guarantee that a particular controller will achieve the control objective. The inequalities involved in assumption QT attempt to encompass a wide class of systems including uncertainties (see section 5 for an example). When more information about the system to be controlled is available, many results are available in the literature. For example consider the following particular case of (2) :

$$\left.\begin{array}{ll}\dot{z} = A_z\, z\ +\ H(x_1) & \\ \dot{x}_i = x_{i+1}\ +\ f_i(x_1) & i\in\{1,\ldots,r-1\}\\ \dot{x}_r = F_z\, z\ +\ f_r(x_1)\ +\ u & \\ y = x_1 & \end{array}\right\} \tag{44}$$

where A_z and F_z are matrices of appropriate dimensions and the functions f_i's and H satisfy :

$$H(0) = 0\ ,\quad f_i(0) = 0\quad \forall i\in\{1,\ldots,r\}\ . \tag{45}$$

For this system, when all the matrices and functions are known, Marino and Tomei in [9, 10] and Kanellakopoulos et al. in [7] have proved, via three different techniques, the following result :

Theorem 3 ([9, 10, 7]) *If, for the system (44), the matrix A_z is strictly Hurwitz and (45) holds, then there exists a dynamic output feedback controller which guarantees global asymptotic stability of the origin of the closed loop system.*

Various attempts have been pursued to relax the required information about the system (44). As far as we are aware and now with Theorem 1, results for the following cases are available :

1 – unknown parameters entering linearly,
2 – unknown parameters entering nonlinearly,
3 – nonlinear zero dynamics,
4 – a bounding function dominating only "at infinity",
5 – nonvanishing nonlinearities,
6 – corrupted output measurement as in section 3.2.2.

Theorem 1 covers these six points simultaneously.

4.1 Unknown parameters entering linearly

Marino and Tomei in [11] and Kanellakopoulos et al. in [6] have considered the case where, may be after reparameterization, the system can be written as :

$$\left.\begin{array}{l}\dot{z}\ = A_z(\theta)\, z + H(x_1,\theta)\\ \dot{x}_i\ = x_{i+1} + f_i(x_1,\theta)\ i\in\{1,\ldots,r-1\}\\ \dot{x}_r = F_z(\theta)\, z + f_r(x_1,\theta) + g(\theta)u\\ y\ = x_1\end{array}\right\} \tag{46}$$

where $A_z(\theta)$, $F_z(\theta)$, $H(x_1,\theta)$, $f_i(x_1,\theta)$ and $g(\theta)$ depend linearly on some parameter vector θ contained in a closed subset $\mathcal{C}$ of $\mathbb{R}^p$ and satisfy :

$$H(0,\theta) = 0\ ,\quad f_i(0,0,\theta) = 0\qquad \forall\theta\in\mathcal{C}\ . \tag{47}$$

For the case where the vector θ is unknown, Marino and Tomei and Kanellakopoulos et al. have proved :

Theorem 4 ([11, 6]) *If, for the system (46),*

a) *for each vector θ in $\mathcal{C}$, the matrix $A_z(\theta)$ is strictly Hurwitz,*
b) *(47) holds,*
c) *for each vector θ in $\mathcal{C}$, the real number $g(\theta)$ is say positive,*

then there exists a dynamic output feedback controller such that, for each vector θ in $\mathcal{C}$, all the solutions of the closed loop system are bounded and their components z, x_1, ..., x_r tend to 0.

4.2 Unknown parameters entering nonlinearly

Marino and Tomei in [10] have considered the case where (46) holds but with an arbitrary dependence on the parameter vector θ. They have proved :

Theorem 5 ([10]) *If, for the system (46),*

a) *the set $\mathcal{C}$ is compact and known,*
b) *for each vector θ in $\mathcal{C}$, the matrix $A_z(\theta)$ is strictly Hurwitz,*
c) *(47) holds,*
d) *for each vector θ in $\mathcal{C}$, the real number $g(\theta)$ is say positive,*

then there exists a parameterized dynamic output feedback controller such that, with appropriately tuned parameters and whatever the vector θ in $\mathcal{C}$ is, the origin of the closed loop system is globally asymptotically stable.

Marino and Tomei have further relaxed the assumptions of this Theorem in [12]. Adaptation of the controller parameters is introduced in order to allow self-tuning and assumption (47) has been relaxed.

4.3 Nonlinear zero dynamics

In [15], Praly and Jiang have proved, for the system (2) with :

$$x_{r+1} = 0 \,, \quad h(0,0) = 0 \,, \quad f_i(0,0) = 0 \,, \qquad (48)$$

Theorem 6 ([15]) *If, for the system (2),*

a) *assumption QL1 holds with $d = 0$,*
b) *(48) holds,*
c) *(12) holds for $s_2 = 0$,*
d) *(10) holds,*

then there exists a dynamic output feedback controller which guarantees global asymptotic stability of the equilibrium of the closed loop system.

4.4 A bounding function dominating only "at infinity"

According to QT2.1, a controller, parameterized by a function Γ, is appropriate for a particular system with perfect output measurement, i.e. $y = x_1$, if the behavior of the input-output operator given by the z-subsystem with x_1 as input and Φ as output is captured, in an L^∞-sense, by the function Γ when $\sup_{0\leq s\leq t}\{|x_1(t)|\}$ is large enough. In the specific case where (36)-(39) hold, QT2.1 can be rewritten as : for all s large enough, we have :

$$\Gamma(s) \;\geq\; 2\,s \sup_{S_1}\left\{\left|\overline{\Phi}_\theta(z,x_1,\theta)\right|\right\} \;+\; 2\,s^2 \qquad (49)$$
$$+\; 2\,\rho(s) \sup_{S_7}\left\{\left|\overline{\Phi}_\theta(z,x_1,\theta)\right|\right\} \;+\; 2\,s\,\rho(s)$$

where S_1 is given in (43) and S_7 and $\rho(s)$ denote :

$$S_7 \;=\; \left\{ \begin{array}{c} |x_1| \leq |z| \leq 2\rho(s) \\ |\theta| \leq 2\rho(s) \end{array} \right\} \qquad (50)$$

$$\rho(s) \;=\; s^2 \sup_{\substack{|x_1|\leq s \\ |\theta|\leq s}} \left\{\left|\frac{H(x_1,\theta)}{x_1}\right|\right\} \;+\; s^2 \,. \quad (51)$$

This expression is obtained by using the fact that, $\mathcal{C}$ being a compact subset and the functions being smooth, expressions like $\sup_{S_2}\left\{\left|\overline{\Phi}_\theta(z,x_1,\theta)\right|\right\}$ or positive real numbers like c/α can be bounded by s or $\rho(s)$, for s large enough. Therefore we see that the lower bound (49) for $\Gamma(s)$ does not depend on the compact set $\mathcal{C}$ nor on the ratio $\frac{c}{\alpha}$. In particular, in the case where the assumptions (36)-(39) are satisfied, the extra assumption [12, (5.58)] is not needed to deal with the case where the compact set $\mathcal{C}$ is unknown.

Also, if, in (2), the functions h, f and $\wp$ were linear, we could simply take :

$$\Gamma(s) \;=\; s^2 \,. \qquad (52)$$

Before concluding, we remark that, in the case of linearly parameterized bounds on the nonlinearities, the existence, for large enough signals, of a bounding function independent of the parameters has been mentioned previously by Kanellakopoulos [5]. In fact the approach in [5] differs from what we follow here. Nevertheless this idea of Kanellakopoulos was one of the key ingredients which led us to the statement of Theorem 1. Its main consequence is that the evolution of the controller parameter $\widehat{k}$ can be frozen, i.e. σ can be set equal to 0, without destroying the boundedness of all the solutions.

4.5 About assumption T

The function β in the ISS or ISpS property captures the behavior of the system when the effects of the initial condition dominate those of the forcing input. Our assumption T implies, in nonrigorous terms, that the solutions of the zero dynamics (30) should be in some L^q space, if their initial condition is small enough. This is a very weak assumption (see [16]).

5 An example : Robust output regulation

To illustrate how very little information about the system to be controlled is required in Theorem 1, we consider the following disturbed "linear" system :

$$\mathcal{A}\left(\frac{d}{dt}\right)\,(y+d_y) \;=\; \mathcal{B}\left(\frac{d}{dt}\right)\,(u+d_u) \qquad (53)$$

where $\mathcal{A}$ and $\mathcal{B}$ are polynomials in the time derivation and d_u and d_y are outputs of a nonlinear system with y as inputs, i.e. :

$$\left.\begin{array}{l} \dot{z}_1 = h_1(z_1,y,t) \\ d_y = C_1(z_1,t) \\ d_u = C_3(z_1,y,t) \end{array}\right\} . \qquad (54)$$

Theorem 1 applies, if :

a) We know the difference of the degrees of the polynomials $\mathcal{A}$ and $\mathcal{B}$ and the sign of the highest degree coefficient of $\mathcal{B}$

b) The system (54) is ISpS and the polynomial B is strictly Hurwitz.

c) The origin is a locally exponentially stable solution of :

$$\dot{z}_1 = h_1(z_1, 0, t) \tag{55}$$

d) We know the bounding function Γ associated with the system (54).

6 A further comment

With Theorem 1 and the ensuing discussion, we have stated that practical output regulation can be achieved in a fashion robust to unmodeled effects which can be captured by the ISpS z-subsystem and the measurement corruption $\wp$. However, the controller we propose to obtain this result is of a high gain type. In fact two classes of high gains are involved :

1. a nonlinear one for large signals embedded in the fact that the functions Γ and $\Gamma_\wp$ should grow fast enough,

2. a linear one $\widehat{k}$ which is tuned on line by increasing it as long as the output is not within a prescribed distance of the desired rest point.

In these regards, one can see our result as an extension to some nonlinear systems of the series of publications devoted to high gain adaptive stabilization (see [2] and references therein). However, as in the linear case, one may wonder about the robustness of the achieved stability. The high gain structure may be incompatible with neglected dynamics which would reduce the relative degree. Also, the dead-zone threshold ϖ should not be set to 0. Otherwise the adaptive structure with $\widehat{k}$ positive is likely to exhibit a drift phenomenon as described by Ioannou and Kokotovic in [3].

In [14], Pomet gave a set of sufficient information for feedback regulation. Both qualitative and quantitative assumptions were also needed. It would be a very interesting issue to compare these two sets of sufficient information which have some definite similarities.

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