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# TOWARDS AN ADAPTIVE REGULATOR : LYAPUNOV DESIGN WITH A GROWTH CONDITION

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**Abstract**

Our objective in this paper is twofold :

1 – Study a growth condition,

2 – Propose a new Lyapunov design of an adaptive regulator under this condition.

The growth condition we consider has been introduced by Praly et al. in [9, Proposition (375)]. Its interest is to involve only a control Lyapunov function and not the system nonlinearities. We mention it is satisfied by strict pure feedback systems with polynomial growth non linearities and some other non-linearizable systems.

Our new Lyapunov design leads to an adaptive regulator where the adapted parameter vector is transformed before being used in the control law. Namely, the so called certainty equivalence principle is not applied. Unfortunately, the implementation of this regulator needs the explicit solution of a fixed point problem. This difficulty is rounded in a companion paper [10].

## 1 Introduction

For linear systems, it is now well established that parameterized controllers can be made adaptive (see [5] for example). In the non linear case, this is not true in general when we are concerned with global stability. As shown in the survey [9], this follows from the fact that, in general, the closed loop system depends on the parameters. Two routes have been explored to overcome this difficulty :

1 – A first route assumes that the parameters can be rejected when considered as disturbances with measured time derivatives. This is the so called matching condition, introduced by Taylor et al. [12], extended by Kanellakopoulos et al. [2] (see also [1]) and generalized by Pomet [9]. This generalized matching condition depends on the open loop system, an assignable Lyapunov function and the adaptation law. Kanellakopoulos, Kokotovic and Morse [3] have shown that, at least for systems in a pure feedback form, a simultaneous design of the control and the adaptation law allows us to satisfy systematically this generalized matching condition.

2 – The second route has been followed by Nam and Arapostathis [4], Sastry and Isidori [11] and Pomet and Praly [7, 8, 9]. Robustness is used instead of disturbance rejection as before and the matching condition is replaced by some growth condition. This latter condition – always satisfied in the linear case – is such that we can design an adaptive controller making the closed loop system Lagrange stability robust with respect to the effects of the adapted parameters. Unfortunately, on the contrary of the first route where sufficient geometric conditions on the open loop system are known for the generalized matching condition to hold [2, 9], there is no precise characterization of the systems for which the various proposed growth conditions hold.

In this paper, we follow the second route. In section 2, we present our assumptions with, in particular, the same growth condition as the one introduced in [9] for a least square estimation scheme with initialized filters. In section 3, we show that this condition is satisfied by at least some systems in a strict pure feedback form. In section 4, we design an adaptive regulator from a Lyapunov design and prove Lagrange and asymptotic Lyapunov stability.

## 2 Assumptions

Let the system to be controlled have a measured state $x$ in $\mathbb{R}^n$ and an input $u$ in $\mathbb{R}^m$. We assume :

**Assumption LP (Linear Parameterization)** (1)
*There exist two known $C^1$ functions $a$ and $A$ and an unknown vector $p^\star$ in $\mathbb{R}^l$ such that the dynamics of the system to be controlled are globally described by :*

$$\dot{x} = a(x,u) + A(x,u)\,p^\star . \tag{2}$$

We shall restrict our attention to the case where $p^\star$ is in a known convex set $\Pi^\star$. Precisely, we assume :

**Assumption ICS (Imbedded Convex Sets)** (3)
*There exists a known convex $C^2$ function $\mathcal{P} : \mathbb{R}^l \to \mathbb{R}$ such that :*

*1 – $[-1,3]$ is a subset of $\mathcal{P}(\mathbb{R}^l)$ and, by letting :*

$$\Pi_\lambda = \{p \mid \mathcal{P}(p) \le \lambda\} , \tag{4}$$

*we denote $\Pi^\star$ (respectively $\Pi_0, \Pi_1, \Pi_2, \Pi$) the set obtained for $\lambda = -1$ (respectively $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = 3$).*

*2 – there exists a strictly positive constant $N$ such that :*

$$\left\| \frac{\partial \mathcal{P}}{\partial p}(p) \right\| \ge N \quad \forall p \in \{p \mid 0 \le \mathcal{P}(p) \le 1\} , \tag{5}$$

*3 – the parameter vector $p^\star$ of the system to be controlled is in $\Pi^\star$.*

Namely, $p^\star$ is in the set $\Pi^\star$ and the sets $\Pi^\star \subsetneq \Pi_0 \subsetneq \Pi_1 \subsetneq \Pi_2 \subsetneq \Pi$ are convex and closed.

**Notation :** We denote $\delta$ the strictly positive real number defined as the minimum among the distances from $\Pi_2$ to the complement of $\Pi$ and from $\Pi^\star$ to the complement of $\Pi_0$.

To design our adaptative controller, we consider the system to be controlled as a particular element of $\{S_p\}_{p\in\Pi}$, the following family of systems indexed by $p \in \Pi$ :

$$\dot{x} \;=\; a(x,u) \;+\; A(x,u)\,p \,. \tag{6}$$

We assume that each element in this family is Lagrange stabilizable in the following sense :

**Assumption LS (Lagrange Stabilizability) (7)**
*There exist two known functions :*

$$\begin{array}{ll} u_n : \mathbb{R}^n \times \Pi \to \mathbb{R}^m & \text{which is } C^1 \\ V : \mathbb{R}^n \times \Pi \to \mathbb{R}_+ & \text{which is } C^2 \end{array}$$

*such that :*

1 - *for all positive real number $K_v$ and all compact subset $\Pi_c$ of $\Pi$, $\{x \mid \exists p \in \Pi_c : V(x,p) \le K_v\}$ is a compact subset of $\mathbb{R}^n$,*

2 - *for all $(x,p)$ in $\mathbb{R}^n \times \Pi$, we have :*

$$\frac{\partial V}{\partial x}(x,p)\left[a(x,u_n(x,p)) + A(x,u_n(x,p))\,p\right] \le 0 \tag{8}$$

*In the following, we denote $-W(x,p)$ the left hand side of this inequality.*

Namely, the following closed loop system :

$$\dot{x} \;=\; a(x,u_n(x,p)) \;+\; A(x,u_n(x,p))\,p \tag{9}$$

is Lagrange stable with Lyapunov function $V$ whose time derivative is $-W$. For the more stronger asymptotic Lyapunov stability of a desired set point $\mathcal{E}^\star$, we will invoke :

**Assumption AS (Asymptotic Stabilizability) (10)**
*For all $C^1$ time function $\widehat{p} : \mathbb{R}_+ \to \Pi$ with bounded derivative, the only bounded solution of :*

$$\dot{x} \;=\; a(x,u_n(x,\widehat{p}(t))) \;+\; A(x,u_n(x,\widehat{p}(t)))\,p^\star \,, \tag{11}$$

*satisfying for all $t \in \mathbb{R}_+$ :*

$$W(x(t),\widehat{p}(t)) \;=\; 0 \tag{12}$$

*is the trivial solution $x(t) = \mathcal{E}^\star$.*

Indeed, with this assumption, asymptotic Lyapunov stability will follow from LaSalle's Theorem.

The closed loop system (9) depends on the parameter vector $p$. More precisely, the Lyapunov function $V$ depends on $p$. As mentioned in Introduction and discussed in [9], this is the origin of most of the difficulties in adaptive non linear control. To specify this dependence in our approach, we assume :

**Assumption GC (Growth Condition) (13)**
*There exists a known positive real number $\gamma$ such that, for all $(x,p)$ in $\mathbb{R}^n \times \Pi$, we have :*

$$\begin{aligned} \left\|\tfrac{\partial V}{\partial p}(x,p)\right\| &\le \gamma\,\left(1 + V(x,p)\right) \,, \\ \left\|\tfrac{\partial^2 V}{\partial p^2}(x,p)\right\| &\le \gamma^2\,\left(1 + V(x,p)\right) \,. \end{aligned} \tag{14}$$

This assumption GC (13) is the same as the one considered in [9, Proposition (375)] with an estimation design. The interest of the growth condition (14) is that the system nonlinearities are not involved explicitly. It concerns only the parameter dependence of the control Lyapunov function $V$.

## 3 Examples

Let us illustrate our assumptions by means of examples :

### 3.1 Strict pure feedback systems

We consider a system which may be after parameter dependent diffeomorphism and feedback can be written in the following form called strict pure feedback form in [3] :

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \quad + f_1(x_1)\,p_1^\star \\ \quad\vdots \\ \dot{x}_i = x_{i+1} + f_i(x_1,\dots,x_i)\,p_i^\star \\ \quad\vdots \\ \dot{x}_n = u \end{array}\right. \tag{15}$$

where the $x_i$'s in $\mathbb{R}$ are measured, the $f_i$'s are known $C^\infty$ function row vectors and the $p_i$'s are unknown parameter vectors in known convex compact sets. For this system, we are interested in asymptotically stabilizing the set point $\mathcal{E}^\star = (0, e_2^\star, \dots, e_n^\star)$ which is uniquely defined by :

$$e_{i+1}^\star \;=\; -f_i\left(0,\dots,e_i^\star\right)p_i^\star \,. \tag{16}$$

Clearly assumption LP (1) is satisfied. To show that functions $u_n$ and $V$ can be found to satisfy assumptions LS (7), AS (10) and GC (13), we apply the iterative Lyapunov design procedure suggested by [13, Theorem 3.c]. For this, we introduce the following more compact notations :

$$\begin{array}{rclcrcl} X_i &=& \left(X_{i-1}^\top, x_i\right)^\top & , & X_1 &=& x_1 \\ P_i &=& \left(P_{i-1}^\top, p_i^\top\right)^\top & , & P_1 &=& p_1 \end{array}$$
$$F_i\left(X_i\,,\,x_{i+1}\,,\,P_i\right) \;=\; \begin{pmatrix} x_2 + f_1(x_1)\,p_1 \\ \vdots \\ x_{i+1} + f_i(x_1,\dots,x_i)\,p_i \end{pmatrix} \tag{17}$$

Then, it can be proved (see [10] for more details) by induction that assumptions LS (7), AS (10) and GC (13) are satisfied for the system :

$$\left\{\begin{array}{l} \dot{X}_{i-1} = F_{i-1}(X_{i-1}, x_i, P_{i-1}) \\ \dot{x}_i \;\;= x_{i+1} + f_i(X_{i-1}, x_i)\,p_i \\ \dot{x}_{i+1} = u_{i+1} \end{array}\right. \tag{18}$$

by the following $C^\infty$ functions :

$$\begin{aligned} u_{i+1} \;=\; &-\left(\tfrac{\partial f_i}{\partial X_i}p_i - \tfrac{\partial u_i}{\partial X_i}\right)F_i - V_i^{m_i-1}\tfrac{\partial V_i}{\partial x_i} \\ &- \left(x_{i+1} + f_i p_i - u_i\right) \,, \end{aligned} \tag{19}$$

and :

$$\begin{aligned} V_{i+1}(X_i, x_{i+1}, P_i) \;=\; &\tfrac{1}{m_i}\,V_i(X_i, P_{i-1})^{m_i} \\ &+ \tfrac{1}{2}\left(x_{i+1} + f_i(X_i)p_i - u_i(X_i, P_{i-1})\right)^2 \,, \end{aligned} \tag{20}$$

where $m_i$ is an integer number. However, for the induction argument to apply, we have to be able to find at each step the integer $m_i$ such that, for all $(X_{i-1}, x_i, P_{i-i})$, we have :

$$\begin{aligned} \|f_i\| + \left\|\tfrac{\partial u_i}{\partial P_{i-1}}\right\| &\le \mu_i(P_{i-1})\,(1 + V_i^{\frac{m_i}{2}}) \,, \\ \left\|\tfrac{\partial^2 u_i}{\partial P_{i-1}{}^2}\right\| &\le \lambda_i(P_{i-1})\,(1 + V_i^{\frac{m_i}{2}}) \,, \end{aligned} \tag{21}$$

where $\mu_i$ and $\lambda_i$ are some positive continuous functions. Fortunately, this extra assumption can always be satisfied if the $f_i$'s and their successive derivatives have a polynomial growth.

We conclude that our assumptions LP (1), LS (7), AS (10) and GC (13) are satisfied at least for linearly parameterized systems in a strict pure feedback form with polynomial growth non linearities and a parameter vector in a known convex compact set – a subclass of the family of systems considered by Kanellakopoulos, Kokotovic and Morse in [3] –.

### 3.2 A three dimensional system

Our assumptions may apply also to systems which cannot be written in pure feedback form and are not even feedback linearizable. To illustrate this point, we consider the following system:

$$\begin{cases} \dot{x} = p_1^\star z + p_2^\star z^2 \\ \dot{y} = z + p_3^\star y^3 \\ \dot{z} = u \end{cases} \tag{22}$$

where the parameters $p_1^\star$, $p_2^\star$, $p_3^\star$ are unknown. We are interested in asymptotically stabilizing the set point $\mathcal{E}^\star = (0,0,0)$.

Equations (22) being linear in the $p_i^\star$'s, assumption LP (1) is satisfied.

For assumption ICS to hold, it is sufficient to know that the vector $(p_1^\star, p_2^\star, p_3^\star)^\top$ satisfies:

$$(p_1^\star - p_{01})^2 + (p_2^\star - p_{02})^2 + (p_3^\star - p_{03})^2 \leq R^2 - \delta^2 \tag{23}$$

where the $p_{0i}$'s and $R > \delta > 0$ are arbitrary but known. Indeed, in this case we may define the function $\mathcal{P}$ by:

$$\mathcal{P}(p_1, p_2, p_3) = \tag{24}$$
$$\frac{1}{\delta^2}\left[(p_1 - p_{01})^2 + (p_2 - p_{02})^2 + (p_3 - p_{03})^2 - R^2\right] .$$

To meet assumption LS (7), we choose the following control Lyapunov function:

$$\begin{aligned} V(x,y,z,p_1,p_2,p_3) &= \tfrac{1}{2}\left(x - h(y,p_1,p_2,p_3)\right)^2 \quad (25) \\ &\quad + \tfrac{1}{8} y^8 + \tfrac{1}{2}\left(z + (p_3+1)y^3\right)^2 , \end{aligned}$$

where, to simplify the notations, we have let:

$$h(y,p_1,p_2,p_3) = p_1(p_3+1)\,y - \frac{p_2(p_3+1)^2}{4} y^4 . \tag{26}$$

Then a Lyapunov design gives the following control law:

$$\begin{aligned} u_n(x,y,z,p_1,p_2,p_3) &= -\left(z + (p_3+1)y^3\right) - y^7 \quad (27) \\ &\quad -(x-h)(p_2 z - p_3(p_1 - p_2(p_3+1)y^3)) \\ &\quad -3(p_3+1)y^2(z + p_3 y^3) . \end{aligned}$$

It follows that (8) in assumption LS (7) is satisfied with:

$$W(x,y,z,p_1,p_2,p_3) = y^{10} + \left(z + (p_3+1)y^3\right)^2 \tag{28}$$

Assumption AS (10) holds also if the set $\Pi$ defined by (24) is such that:

$$(p_1,p_2,p_3) \in \Pi \implies p_1 p_3 \neq 0 . \tag{29}$$

Indeed for any $C^1$ time function $(\widehat{p}_1(t), \widehat{p}_2(t), \widehat{p}_3(t)) \in \Pi$, any solution $(x(t), y(t), z(t))$ of (22), with:

$$u = u_n(x,y,z,\widehat{p}_1(t),\widehat{p}_2(t),\widehat{p}_3(t)) , \tag{30}$$

which satisfies:

$$W(x,y,z,\widehat{p}_1(t),\widehat{p}_2(t),\widehat{p}_3(t)) = 0 \quad \forall t , \tag{31}$$

is necessarily such that:

$$y(t) = z(t) = 0 \quad \forall t . \tag{32}$$

But, from (26)-(27), this implies:

$$\widehat{p}_1(t)\,\widehat{p}_3(t)\,x(t) = 0 \quad \forall t . \tag{33}$$

The conclusion follows from (29).

Thanks to a straightforward computation and the fact that the parameter vector is in a compact set, it is easy to check that assumption GC (13) holds.

Finally, we remark that, as far as we know, up to now, there is no adaptive controller guaranteeing a global stabilization for this system without invoking assumption GC (13).

## 4 A theoretical adaptive controller

To design a controller guaranteeing at least Lagrange stability for the system (2) with $p^\star$ unknown, we follow the standard adaptive control procedure and propose the following dynamic state feedback:

$$\begin{cases} \dot{\widehat{p}} = v(x,\widehat{p}) \\ u = u_n(x,\widehat{p}) \end{cases} \tag{34}$$

where the new control $v$ is to be designed.

Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be some positive proper $C^1$ function with a strictly positive derivative denoted $L'$. This function $L$ will be precised later. For the time being, we use it to define the function:

$$S(x,\widehat{p}) \stackrel{\text{def}}{=} L(V(x,\widehat{p})) . \tag{35}$$

The time derivative of this function along the solutions of (2)-(34) is:

$$\dot{S} = L'(V)\left[\frac{\partial V}{\partial x}\left[a(\cdot,u_n) + A(\cdot,u_n)\,p^\star\right] + \frac{\partial V}{\partial p} v\right] \tag{36}$$

Then, using (8) in assumption LS (7) and our dynamic controller (34), we get the following differential inequalities system:

$$\begin{cases} \dot{S} \leq -L'W - L'\frac{\partial V}{\partial x} A\left(\widehat{p} - p^\star\right) + L'\frac{\partial V}{\partial p} v \\ \dot{\overline{\left(\widehat{p} - p^\star\right)}} = v \end{cases} \tag{37}$$

In view of point 1 of assumption LS (7), Lagrange stability would follow if we were able to find functions $L$ and $v$ so that $S$ and $\widehat{p}$ remain bounded. To simplify the notations, let:

$$\begin{aligned} \psi^\top &= L'\tfrac{\partial V}{\partial x} A \\ \varphi^\top &= L'\tfrac{\partial V}{\partial p} \\ \widetilde{p} &= \left(\widehat{p} - p^\star\right) \\ y &= S - L'\tfrac{\partial V}{\partial p}\left(\widehat{p} - p^\star\right) = S - \varphi^\top \widetilde{p} . \end{aligned} \tag{38}$$

By using the positivity of $L'$ and $W$, we get from (37):

$$\begin{cases} \dot{y} \leq -\left(\dot{\varphi} + \psi\right)^\top \widetilde{p} \\ \dot{\widetilde{p}} = v \end{cases} \tag{39}$$

This is the standard form to apply Parks' Lyapunov design [6]. Namely, with $\alpha$ a strictly positive real number to be chosen, we define the function :

$$U = y + \frac{1}{2\alpha}\|\widetilde{p}\|^2 \tag{40}$$

and we evaluate its time derivative along the solutions of (39). We get :

$$\dot{U} \leq \left(\tfrac{1}{\alpha}v - \dot{\varphi} - \psi\right)^\top \widetilde{p}\,. \tag{41}$$

Therefore, by choosing :

$$v = \alpha\left(\psi + \dot{\varphi}\right)\,, \tag{42}$$

we are guarenteed that $\dot{U}$ is negative. This leads us to propose the following control $v$ and a corresponding control Lyapunov function $U$ :

$$\dot{\widehat{p}} = v = \alpha\left(L'\,A^\top\frac{\partial V}{\partial x}^\top + \overline{L'\,\frac{\partial V}{\partial p}}^{\,\dot{}\,\top}\right) \tag{43}$$

and :

$$U = L(V) - L'(V)\frac{\partial V}{\partial p}(\widehat{p} - p^\star) + \frac{1}{2\alpha}\|\widehat{p} - p^\star\|^2 \tag{44}$$

$$= L(V) - \frac{\alpha}{2}\left\|L'(V)\frac{\partial V}{\partial p}\right\|^2 \tag{45}$$

$$+\frac{1}{2\alpha}\left\|\widehat{p} - p^\star - \alpha\,L'(V)\frac{\partial V}{\partial p}^\top\right\|^2 .$$

Two questions about this proposition :

1- Is the equation for $\widehat{p}$ realizable — a time derivative is involved in the right hand side of (43) ?

2- Is $U$ a positive proper function of the state vector $(x, \widehat{p})$ of the closed loop system ?

The answer to the first question is simple. A state space realization of a system giving $\widehat{p}$ is :

$$\begin{cases} \dot{\widehat{q}} = \alpha\,L'(V(x,\widehat{p}))\,A(x, u_n(x,\widehat{p}))^\top\,\frac{\partial V}{\partial x}(x,\widehat{p})^\top \\ \widehat{p} = \widehat{q} + \alpha\,L'(V(x,\widehat{p}))\,\frac{\partial V}{\partial p}(x,\widehat{p})^\top \end{cases} \tag{46}$$

Unfortunately, a difficulty remains since the last equation is implicit in $\widehat{p}$. We shall address this point after the following answer to the second question : Since the function $L$ is to be designed as a positive proper function and $V$ is positive and satisfies point 1 of assumption LS (7) , $U$ is a positive proper function of $(x, \widehat{p})$ if there exists a strictly positive real number $\varepsilon < 1$ such that :

$$\frac{\alpha}{2}\left\|L'(V)\frac{\partial V}{\partial p}\right\|^2 \leq (1-\varepsilon)\,L(V)\,. \tag{47}$$

But, from assumption GC (13) , this inequality is satisfied if $\widehat{p}$ is in the set $\Pi$ and :

$$\frac{\alpha}{2}L'(V)^2\gamma^2\left(1+V\right)^2 \leq (1-\varepsilon)\,L(V)\,. \tag{48}$$

Therefore it is sufficient to choose :

$$L(V) = 1 + \log(1+V)\quad,\qquad \alpha < \frac{2}{\gamma^2}\,. \tag{49}$$

Now, coming back to the problem of implicit definition of $\widehat{p}$, we note that, not only a solution should exist, but also, for (47) to hold with this function $L$ in (49), this solution $\widehat{p}$ should be in $\Pi$. The second equation of (46) is :

$$\widehat{p} = \widehat{q} + \alpha\frac{\frac{\partial V}{\partial p}(x,\widehat{p})^\top}{1+V(x,\widehat{p})}\,. \tag{50}$$

It follows from a standard fixed point argument (see Lemma A.1) that assumption GC (13) and :

$$\alpha < \min\left\{\frac{\delta}{\gamma}\,,\,\frac{1}{2\gamma^2}\right\} \tag{51}$$

imply the existence of a $C^1$ function $\rho : \mathbb{R}^n \times \Pi_2 \rightarrow \Pi$ such that, for all $(x, \widehat{q})$ in $\mathbb{R}^n \times \Pi_2$,

$$\rho(x,\widehat{q}) = \widehat{q} + \alpha\frac{\frac{\partial V}{\partial p}(x,\rho(x,\widehat{q}))^\top}{1+V(x,\rho(x,\widehat{q}))}\,. \tag{52}$$

To use this function $\rho$ in (46), it remains to guarantee that $\widehat{q}$ is in $\Pi_2$. This is achieved by using the standard projection trick.

As a result, we have designed the following adaptive controller :

$$\begin{cases} \dot{\widehat{q}} = \alpha\mathrm{Proj}\left(\widehat{q}, \frac{A(x,u_n(x,\rho(x,\widehat{q})))^\top\,\frac{\partial V}{\partial x}(x,\rho(x,\widehat{q}))^\top}{1+V(x,\rho(x,\widehat{q}))}\right) \\ u = u_n(x,\rho(x,\widehat{q})) \end{cases} \tag{53}$$

where the locally Lipschitz continuous function Proj : $\Pi \times \mathbb{R}^l \rightarrow \mathbb{R}^l$ is defined by – see [9, Lemma (103)] –

$$\mathrm{Proj}(q,y) = \tag{54}$$

$$\begin{cases} y & \text{if } \mathcal{P}(q) \leq 0 \text{ or } \frac{\partial \mathcal{P}}{\partial p}(q)y \leq 0 \\ y - \frac{\mathcal{P}(q)\frac{\partial \mathcal{P}}{\partial p}(q)y}{\left\|\frac{\partial \mathcal{P}}{\partial p}(q)\right\|^2}\frac{\partial \mathcal{P}}{\partial p}(q)^\top & \text{if } \mathcal{P}(q) > 0 \text{ and } \frac{\partial \mathcal{P}}{\partial p}(q)y > 0 \end{cases}$$

and satisfies for all $(q, p, y) \in \mathbb{R}^l \times \Pi_0 \times \mathbb{R}^l$ :

$$(q-p)^\top\,\mathrm{Proj}(q,y) \leq (q-p)^\top y\,. \tag{55}$$

Moreover, an appropriate Lyapunov function to study the dynamics of the closed loop system should be :

$$U(x,\widehat{q}) = 1 + \log(1 + V(x,\rho(x,\widehat{q}))) \tag{56}$$

$$-\frac{\alpha}{2}\left\|\frac{\frac{\partial V}{\partial p}(x,\rho(x,\widehat{q}))}{1+V(x,\rho(x,\widehat{q}))}\right\|^2 + \frac{1}{2\alpha}\|\widehat{q} - p^\star\|^2 .$$

We have :

**Proposition 1** *Let assumptions LP (1) , ICS (3) , LS (7) and GC (13) hold and $\alpha$ be chosen such that :*

$$0 < \alpha < \min\left\{\frac{\delta}{\gamma}\,,\,\frac{1}{2\gamma^2}\right\}\,. \tag{57}$$

*Under these conditions, for any initial condition $(x(0), \widehat{q}(0))$ in $\mathbb{R}^n \times \Pi_0$, there exists a unique solution $(x(t), \widehat{q}(t))$ of (2)-(53) which is bounded on $[0, +\infty)$. Moreover, if assumption AS (10) holds, we have also :*

$$\lim_{t\rightarrow+\infty} x(t) = \mathcal{E}^\star\,. \tag{58}$$

*Proof of Proposition 1 :*

The closed loop system we consider is :

(59)

$$
\begin{cases}
\dot{x} = a(x, u_n(x, \rho(x, \widehat{q}))) + A(x, u_n(x, \rho(x, \widehat{q})))\, p^\star \\
\dot{\widehat{q}} = \alpha \mathrm{Proj}\left(\widehat{q}, \dfrac{A(x, u_n(x, \rho(x, \widehat{q})))^\top \frac{\partial V}{\partial x}(x, \rho(x, \widehat{q}))^\top}{1 + V(x, \rho(x, \widehat{q}))}\right)
\end{cases}
$$

From our smoothness assumptions on the functions $a$, $A$, $u_n$ and $V$, Lemma A.1 and [9, Lemma (103)] this system has a locally Lipschitz continuous right hand side in the open set $\mathbb{R}^n \times \mathring{\Pi}_2$. It follows that, for any initial condition $(x(0), \widehat{q}(0))$ in this open set and therefore in particular in $\mathbb{R}^n \times \Pi_0$, there exists a unique solution $(x(t), \widehat{q}(t))$ defined on a right maximal interval $[0, T)$, with $T$ may be infinite. Moreover, from [9, Lemma (103) point 5], we know that $\widehat{q}(t) \in \Pi_1 \subset \mathring{\Pi}_2$ for all $t$ in $[0, T)$.

Then, we compute the time derivative of the function $U(x(t), \widehat{q}(t))$ defined in (56). With assumption LS (7) and (50), we get, with $\widehat{p} = \rho(x, \widehat{q})$ :

$$
\begin{aligned}
\dot{U} = & \frac{1}{1+V}\left[\frac{\partial V}{\partial x}\left(a + A\,p^\star\right) + \frac{\partial V}{\partial p}\dot{\widehat{p}}\right] \\
& - \alpha \frac{\frac{\partial V}{\partial p}}{1+V} \overbrace{\frac{\frac{\partial V}{\partial p}^\top}{1+V}}^{\cdot} + \frac{1}{\alpha}\left(\widehat{q} - p^\star\right)^\top \dot{\widehat{q}} \qquad (60) \\
\leq & -\frac{W}{1+V} + \left(\widehat{q} + \alpha \frac{\frac{\partial V}{\partial p}^\top}{1+V} - p^\star\right)^\top \left[\frac{\dot{\widehat{q}}}{\alpha} - \frac{A^\top \frac{\partial V}{\partial x}^\top}{1+V}\right]
\end{aligned}
$$

But, since $\widehat{q}(t)$ is in $\Pi_1 \subset \Pi_2$, $\widehat{p}$ is in $\Pi$. And $\widehat{p}$ in $\Pi$, assumptions GC (13) and (57) imply :

$$
\left\| \alpha \frac{\frac{\partial V}{\partial p}(x, \widehat{p})^\top}{1 + V(x, \widehat{p})} \right\| \leq \alpha\,\gamma < \delta\,. \qquad (62)
$$

Since, $p^\star$ is in $\Pi^\star$, the definition of $\delta$ and this inequality implies that $p^\star - \alpha \frac{\frac{\partial V}{\partial p}^\top}{1+V}$ is in $\Pi_0$. Therefore, with the expression of $\dot{\widehat{q}}$ and (55), we get finally :

$$
\dot{U} \leq -\frac{W(x(t), \rho(x(t), \widehat{q}(t)))}{1 + V(x(t), \rho(x(t), \widehat{q}(t)))}\,. \qquad (63)
$$

This implies that $U(x(t), \widehat{q}(t))$ is a non increasing time function. Moreover, we have with assumption GC (13) :

$$
\begin{aligned}
U \geq & \left(1 - \frac{\alpha\gamma^2}{2}\right)\left(1 + \log(1+V)\right) \\
& + \frac{\alpha}{2}\left(\gamma^2 - \left\|\frac{\frac{\partial V}{\partial p}}{1+V}\right\|^2\right) + \frac{1}{2\alpha}\|\widehat{q} - p^\star\|^2 \qquad (64) \\
\geq & \left(1 - \frac{\alpha\gamma^2}{2}\right)\left(1 + \log(1+V)\right) + \frac{1}{2\alpha}\|\widehat{q} - p^\star\|^2 \geq 0
\end{aligned}
$$

It follows from (57), Lemma A.1 and point 1 of assumption LS (7) that, for $\widehat{q}$ in $\Pi_2$, $U$ is a positive function proper in $x$ and $\widehat{q}$. Therefore, by contradiction $T$ is infinite and the solution $(x(t), \widehat{q}(t))$ is bounded on $[0, +\infty)$.

Finally, it follows from [14, 5.2.81] and (63) that any solution of the autonomous system (59) converges to the largest invariant set of points $(x, \widehat{q})$ satisfying $W(x, \rho(x, \widehat{q})) = 0$. But any bounded solution in this invariant set is also a solution of (11)-(12) in assumption AS (10) with :

$$
\widehat{p}(t) = \rho(x(t), \widehat{q}(t))\,. \qquad (66)
$$

The conclusion follows readily. □

Unfortunately, if, from a theoretical point of view, the adaptive controller (53) may be satisfactory, it is not yet a practical solution. Its implementation requires an explicit expression for the function $\rho$. It is clear from our examples of section 3 that an analytical expression for this function is unaccessible in general. In a companion paper [10], we propose a more practical adaptive regulator where the static fixed point equation is replaced by a dynamical system with this fixed point as equilibrium.

## 5 Conclusion

Our corner stone in this paper is the growth condition GC (13) . It does not involve explicitly the system nonlinearities and is satisfied by a class of systems encompassing at least the strict pure feedback systems with polynomial growth non linearities and parameter vector in a known compact set, a subclass of the family of systems considered by Kanellakopoulos, Kokotovic and Morse in [3]. This condition has been introduced by Praly et al. in [9, Proposition (375)]. They have shown that it is sufficient to obtain an adaptive controller by an estimation design. Their controller contains a least squares algorithm and therefore has a vanishing adaptation gain. But worse, for the regulation result to hold, the filters feeding this algorithm must be initialized to a particualr value depending on the state initial condition. Here, we have applied a Lyapunov design and obtained an adaptive regulator with a not necessarily vanishing adaptation gain and without requiring a specific initialization.

Our adaptive regulator is of a new type since the adapted parameter vector is transformed before being used in the control law. This means that it does not rely on the so called certainty equivalence principle. Unfortunately this transformation is given as the solution of a fixed point problem which cannot be computed explicitly in general. Nevertheless, we have shown in [10] that a more practical adaptive regulator can be obtained where the static fixed point equation is replaced by a dynamical system with this fixed point as equilibrium.

Finally, we observe that our adaptive regulator or the one considered in [9, Proposition (375)], when applied to strict pure feedback systems with polynomial growth non linearities, do not have the drawback of having to estimate more than the necessary number of parameters on the contrary of [3].

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## A The fixed point $\rho(x,\widehat{q})$

**Lemma A.1** *Let $\Pi_2 \subsetneq \Pi$ be two closed convex subsets of $\mathbb{R}^l$ such that :*

$$\delta \stackrel{\text{def}}{=} \inf_{p_1\in\Pi_2,\, p_2\notin\Pi} \|p_1 - p_2\| \;>\; 0\,. \tag{67}$$

*Let also $V : \mathbb{R}^n \times \Pi \to \mathbb{R}_+$ be a $C^2$ function such that, for all $(x,p)$ in $\mathbb{R}^n \times \Pi$, we have :*

$$\begin{aligned} \left\|\frac{\partial V}{\partial p}(x,p)\right\| &\leq \gamma\,(1+V(x,p))\,. \\ \left\|\frac{\partial^2 V}{\partial p^2}(x,p)\right\| &\leq \gamma^2\,(1+V(x,p))\,. \end{aligned} \tag{68}$$

*If $\alpha$ is a postive real number satisfying :*

$$\alpha \;<\; \min\left\{\frac{\delta}{\gamma}\,,\,\frac{1}{2\gamma^2}\right\}\,, \tag{69}$$

*then, there exists a $C^1$ function $\rho : \mathbb{R}^n \times \Pi_2 \to \Pi$ such that :*

$$\rho(x,q) \;=\; q \;+\; \alpha\,\frac{\frac{\partial V}{\partial p}(x,\rho(x,q))^{\top}}{1+V(x,\rho(x,q))}\,. \tag{70}$$

*Proof :*
Let $f$ be defined by :

$$f(x,p,q) \stackrel{\text{def}}{=} q \;+\; \alpha\,\frac{\frac{\partial V}{\partial p}(x,p)^{\top}}{1+V(x,p)}\,. \tag{71}$$

For all $(x,p)$ in $\mathbb{R}^n \times \Pi$, we have :

$$\|f(x,p,q) - q\| \;\leq\; \alpha\gamma\,. \tag{72}$$

Therefore, from (67) and (69), $f(x,q,p)$ is an interior point of $\Pi$ for all $(x,q,p)$ in $\mathbb{R}^n \times \Pi_2 \times \Pi$. Now, with assumption (68), we have :

$$\begin{aligned} &\sup_{(x,p)\in\mathbb{R}^n\times\Pi}\left\|\frac{\partial}{\partial p}\left(\frac{\frac{\partial V}{\partial p}^{\top}}{1+V}\right)\right\| = \\ &\qquad \sup_{(x,p)\in\mathbb{R}^n\times\Pi}\left\|\frac{\frac{\partial^2 V}{\partial p^2}}{1+V} - \left(\frac{\frac{\partial V}{\partial p}}{1+V}\right)^2\right\| \;\leq\; 2\gamma^2\,. \end{aligned} \tag{73}$$

Then, from this inequality, the Mean Value Theorem and the convexity of $\Pi$, we have, for all $(x,q)$ in $\mathbb{R}^n \times \Pi_2$ and $p_1$ and $p_2$ in $\Pi$,

$$\begin{aligned} &\|f(x,p_1,q) - f(x,p_2,q)\| \;= \\ &\qquad \alpha\left\|\frac{\frac{\partial V}{\partial p}(x,p_1)}{1+V(x,p_1)} - \frac{\frac{\partial V}{\partial p}(x,p_2)}{1+V(x,p_2)}\right\| \;\leq\; 2\,\alpha\,\gamma^2\,\|p_1 - p_2\|\,. \end{aligned} \tag{74}$$

Since $\Pi$ is a complete metric space, it follows from the Contraction Mapping Theorem that (69) implies the existence of a unique function $\rho : \mathbb{R}^n \times \Pi_2 \to \Pi$ such that :

$$\rho(x,q) \;=\; f(x,\rho(x,q),q) \tag{75}$$

Moreover, from (73) and (69), the matrix :

$$I - \frac{\partial f}{\partial p}(x,p,q) \;=\; I - \alpha\frac{\partial}{\partial p}\left(\frac{\frac{\partial V}{\partial p}(x,p)}{1+V(x,p)}\right) \tag{76}$$

is non singular for all $(x,p)$ in $\mathbb{R}^n \times \Pi$. It follows from the Implicit Function Theorem and the uniqueness of $\rho$ that this function $\rho$ is $C^1$ as is $V$ and $\frac{\partial V}{\partial p}$. □