Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

ITERATIVE DESIGNS OF ADAPTIVE CONTROLLERS FOR SYSTEMS WITH NONLINEAR INTEGRATORS

Z.-P. Jiang and L. Praly
Centre Automatique et Systèmes
35 rue Saint Honoré
77305 Fontainebleau FRANCE

Abstract

We propose a slight extension of a result recently established by Kanellakopoulos, Kokotovic and Morse in [2] about the global adaptive stabilization of systems in a strict feedback form. A reinterpretation of their design algorithm in terms of the control Lyapunov function approach allows us to :

1. extend the design to a slightly broader class of systems.
2. propose a new algorithm which involves half as many parameters to update as the original Kanellakopoulos et al.'s scheme.

1 Introduction

Adaptive control of nonlinear systems has already received a great deal of attention. Full state feedback and output feedback adaptive schemes are available for local or global stabilization or tracking (see [3] and references therein). Unfortunately, these schemes apply only to linearly parametrized nonlinear systems which are constrained either by the location of their unknown parameters or by the type of nonlinearities.

Kanellakopoulos, Kokotovic and Morse [2] have recently introduced a new design which, at least for systems in a pure feedback form, allows us to obtain stabilizing adaptive controllers without requiring one of these two above mentioned constraints. This new design is based on the iterative interlacing of two techniques : the Lyapunov design of adaptive controllers [5] and the stabilization of a chain of integrators (see [4, Example 3.2] for example). Here, we shall pursue this idea of iterative interlacing.

The two basic ingredients will be studied in section 2. The stabilization of a chain of integrators will be presented via the control Lyapunov function technique [8, Theorem 3.c] instead of the change of coordinates of [4, Example 3.2] used by Kanellakopoulos et al. [2]. The interest of this technique is its ability to deal with non smooth cases – the case where the diffeomorphism would have singularities (see [7]). This will be discussed briefly in section 4. The second ingredient used by Kanellakopoulos et al. is the Lyapunov design of adaptive controller. Such a design will be recalled here. In section 3, we show how to use these ingredients to design iteratively adaptive controllers. Kanellakopoulos et al. proposed to interlace change of coordinates and Lypunov design. Here, among other things, we shall remark that the change of coordinates or more precisely the control Lyapunov function technique can be applied twice before applying an adaptive control Lyapunov design. This remark will allow us to propose an adaptive controller with half as many parameters to update as in Kanellakopoulos et al.'s scheme. Section 4 is devoted to the application of our proposed designs. We note that they apply to a slightly broader class of systems than the set of strict feedback systems.

Notations and Definition :

1. For an ordinary differential equation numbered (1), with p a constant real vector,
$$\dot{x} = f(x,p)\,, \tag{1}$$
and a function $V(x,q)$, we denote $\dot{V}\big|_{(1)}(x,p,q)$ the function of (x,p,q) given by :
$$\dot{V}\big|_{(1)}(x,p,q) = \frac{\partial V}{\partial x}(x,q)\, f(x,p) \tag{2}$$
2. $m \geq 1$ denotes an integer number.

Due to space limitations, our proofs being more or less straightforward are omitted (see [1] for a more complete version).

2 Design Ingredients

2.1 Adding one integrator

The first problem we address is :

> Knowing that a system is stabilizable, is it still stabilizable if we add one integrator ?

This problem has received a great deal of attention (see [4] and [7]). Let us reproduce these well known results but in a statement more appropriate to our problem.

We consider a single input nonlinear system :

$$\dot{x} = f(x,u,p) \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $p \in \mathbb{R}^l$ is a parameter vector, and f is a C^{m+1} function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^l$. We assume the existence of :

1 – a dynamic state feedback :
$$\begin{cases} \dot{\chi}_e = \psi_0(x,\chi_e) \\ u = u_0(x,\chi_e,p) \end{cases} \tag{4}$$
where the dynamical extension χ_e is in $\mathbb{R}^k$, ψ_0 is of class C^m and u_0 is of class C^{m+1}.

2 – a real positive C^{m+1} function $V_0(x,\chi_e,p)$ and a real positive C^0 function $W_0(x,\chi_e,p)$

such that :

(A1) for all (x,χ_e,p) in $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$, we have :

$$\dot{V}_0\big|_{(3)-(4)}(x,\chi_e,p) \leq -W_0(x,\chi_e,p)\,. \tag{5}$$

Namely, the dynamic state feedback (4) makes the function V_0 have a time derivative along the solutions of (3) less than $-W_0$.

Let us now add one integrator to the system (3) :

$$\begin{cases} \dot{x} = f(x, y, p) \\ \dot{y} = v + g(x, y, q) \end{cases} \tag{6}$$

where y is a new scalar state component, v is a new scalar input, $q \in \mathbb{R}^{l_1}$ is a new parameter vector and g is a C^m function on $\mathbb{R}^{n+1} \times \mathbb{R}^{l_1}$. We have the following Lemma [8, Theorem 3.c] :

Lemma 1 (Adding one integrator)
With assumption (A1), consider the following two real functions V_1 of class C^m and W_1 of class C^0, with r in $\mathbb{R}^l$:

$$V_1(x, y, \mathcal{X}_e, r, p) = V_0(x, \mathcal{X}_e, p) + \tfrac{1}{2}\left(y - u_0(x, \mathcal{X}_e, r)\right)^2 \tag{7}$$

$$W_1(x, y, \mathcal{X}_e, r, p) = W_0(x, \mathcal{X}_e, p) + \left(y - u_0(x, \mathcal{X}_e, r)\right)^2 \tag{8}$$

and the following C^m function :

$$\begin{aligned} u_1(x, y, \mathcal{X}_e, p, q, r) = &-g(x, y, q) - (y - u_0(x, \mathcal{X}_e, r)) \\ &+ \tfrac{\partial u_0}{\partial x}(x, \mathcal{X}_e, r)\, f(x, y, p) + \tfrac{\partial u_0}{\partial \mathcal{X}_e}(x, \mathcal{X}_e, r)\, \psi_0(x, \mathcal{X}_e) \\ &- \tfrac{\partial V_0}{\partial x}(x, \mathcal{X}_e, p) \int_0^1 \varphi(x, y, \mathcal{X}_e, r, p, s)\, ds \end{aligned} \tag{9}$$

with :

$$\varphi = \frac{\partial f}{\partial y}\left(x\,,\, u_0(x, \mathcal{X}_e, r) + [y - u_0(x, \mathcal{X}_e, r)]s\,,\, p\right)\ . \tag{10}$$

By applying, to the system (6), the controller :

$$\begin{cases} \dot{\mathcal{X}}_e = \psi_0(x, \mathcal{X}_e) \\ v = u_1(x, y, \mathcal{X}_e, p, q, p) \end{cases} \tag{11}$$

we obtain, for all $(x, y, \mathcal{X}_e, p, q)$, by evaluating at $r = p$,

$$\dot{V}_1 \big|_{(6)-(11)} (x, y, \mathcal{X}_e, p) \ \leq\ - W_1(x, y, \mathcal{X}_e, p, p)\ . \tag{12}$$

Remark 1 :
1 – From (12), we see that the functions u_1, V_1 and W_1 satisfy assumption (A1) for the augmented system (6). It follows that Lemma 1 applies again to this system (6) augmented with another integrator. Therefore, we can recursively add integrators. Note however that one degree of smoothness is lost at each addition. Namely, starting with C^m functions, only m integrators can be added this way. Nevertheless, using the so called desingularizing function technique introduced in [7], it may be possible to add more integrators. This possibility will be discussed in section 4.
2 – For the meaning of Lemma 1, the presence of the parameter vectors p and q is not important. However, in view of the iterative design proposed in the next section, it is important to know how these parameter vectors appear in V_1, W_1 or u_1. It is to allow us to describe more precisely this appearance that we have introduced the dummy variable r. In particular, we can make the following key observations :
2.1 – V_1 does not depend on r if u_0 does not.
2.2 – u_1 always depends not only on the new parameter vector q but also on the old one p.

2.2 Removing the parameter dependence in the control law

In Remark 1 above, we observed that when we add one integrator, the new control law depends on the parameter vector. We are now facing a second problem :

Knowing a parametrized stabilizing control law for a linearly parametrized system, is it possible to design a stabilizing controller not depending on the system parameters ?

Precisely, consider a single input nonlinear system as one element of the following family of systems, parametrized in θ,

$$\dot{x}_e \;=\; a(x_e) \;+\; A(x_e)\theta \;+\; b(x_e)\,\overline{u} \tag{13}$$

where $x_e \in \mathbb{R}^n$ is the state, $\overline{u} \in \mathbb{R}$ is the input, $\theta \in \mathbb{R}^l$ is a vector of unknown constant parameters, and a, A and b are C^m functions on $\mathbb{R}^n$. For this family, we assume the existence of :
1 – a dynamic state feedback

$$\begin{cases} \dot{\mathcal{X}} = \psi(x_e, \mathcal{X}) \\ \overline{u} = u(x_e, \mathcal{X}, \theta) \end{cases} \tag{14}$$

where the dynamical extension $\mathcal{X}$ is in $\mathbb{R}^k$, ψ is of class C^m and u is of class C^m,
2 – a real positive C^{m+1} function $V(x_e, \mathcal{X}, \widehat{\theta}, \theta)$ and a real positive C^0 function $W(x_e, \mathcal{X}, \widehat{\theta}, \theta)$ satisfying :

(A2.1) $\frac{\partial V}{\partial x_e}(x_e, \mathcal{X}, \widehat{\theta}, \theta)$ is independent of θ.

(A2.2) $\frac{\partial V}{\partial \mathcal{X}}(x_e, \mathcal{X}, \widehat{\theta}, \theta)$ is linear in θ.

(A3) there exists a known C^m function h: $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^l$ satisfying :

$$\frac{\partial V}{\partial \widehat{\theta}}(x_e, \mathcal{X}, \widehat{\theta}, \theta) = \left(\frac{\partial V}{\partial x_e}(x_e, \mathcal{X}, \widehat{\theta}, \theta)\, b(x_e)\right) h(x_e, \mathcal{X}, \widehat{\theta}) \tag{15}$$

such that :
(A4) for all $(x_e, \mathcal{X}, \theta)$ and by evaluating at $\widehat{\theta} = \theta$,

$$\dot{V}\,\big|_{(13)-(14)}\,(x_e, \mathcal{X}, \theta) \leq -\, W(x_e, \mathcal{X}, \theta, \theta)\ . \tag{16}$$

Again, we have introduced the dummy variable $\widehat{\theta}$ to allow us to precise the role played by some components of the parameter vector. Assumption (A4) tells us that the parametrized dynamic state feedback (14) makes the function V have a time derivative along the solutions of the parametrized system (13) less than $-W$. Assumption (A3) is a sufficient condition for the extended matching condition as generalized in [3, p. 371] to hold. Assumptions (A2.1) and (A2.2) allow us to introduce the following function – not depending on θ –,

$$\begin{aligned} Z(x_e, \mathcal{X}, \widehat{\theta}) \;=\; &\tfrac{\partial V}{\partial x_e}(x_e, \mathcal{X}, \widehat{\theta}, \theta)\, A(x_e) \\ &+ \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \mathcal{X}}(x_e, \mathcal{X}, \widehat{\theta}, \theta)\, \psi(x_e, \mathcal{X})\right)\ . \end{aligned} \tag{17}$$

With these assumptions we can solve our second problem by using a Lyapunov design [5]. More precisely we follow here a variant (compare with [3, p. 377]) of the Lyapunov design proposed by Slotine and Li.

Lemma 2 (Removing parameter dependence)
With assumptions (A2.1), (A2.2), (A3) and (A4), consider the following two real functions $\overline{V}$ of class C^{m+1} and $\overline{W}$ of class C^0 :

$$\begin{aligned} \overline{V}(x_e, \mathcal{X}, \widehat{\theta}, \theta) &= V(x_e, \mathcal{X}, \widehat{\theta}, \theta) \;+\; \tfrac{1}{2}\|\widehat{\theta} - \theta\|^2 \\ \overline{W}(x_e, \mathcal{X}, \widehat{\theta}) &= W(x_e, \mathcal{X}, \widehat{\theta}, \widehat{\theta}) \end{aligned} \tag{18}$$

By applying, to the system (13), the following C^m dynamic state feedback :

$$\begin{cases} \dot{\mathcal{X}} = \psi(x_e, \mathcal{X}) \\ \dot{\widehat{\theta}} = Z(x_e, \mathcal{X}, \widehat{\theta})^\top \\ \overline{u} = u(x_e, \mathcal{X}, \widehat{\theta}) - h(x_e, \mathcal{X}, \widehat{\theta})\, Z(x_e, \mathcal{X}, \widehat{\theta})^\top \end{cases} \tag{19}$$

where Z is defined in (17), we obtain, for all $(x_e, \mathcal{X}, \widehat{\theta}, \theta)$,

$$\dot{\overline{V}}\big|_{(13)-(19)}(x_e, \mathcal{X}, \widehat{\theta}) \le -\overline{W}(x_e, \mathcal{X}, \widehat{\theta}) . \tag{20}$$

Remark 2 :
1 – Without assumptions (A2.1) and (A2.2), Z in (17) would depend on θ and consequently the controller (19) would still depend on the system parameters. Note also that $\overline{V}$ defined in (18) statisfies these assumptions.
2 – The dynamic controller (19) and $\overline{V}$ and $\overline{W}$ defined in (18) satisfy assumption (A1) of Lemma 1 with $\mathcal{X}_e = (\mathcal{X}^\top, \widehat{\theta}^\top)^\top$. This implies that we can add one integrator to the system (13).

3 Adding integrators with unknown parameters

As a step towards iterative designs of adaptive controllers for linearly parametrized nonlinear systems, let us address a third problem :

> Knowing that a system is stabilizable, is it still stabilizable if we add several integrators with unknown parameters ?

Precisely, our starting point is the same single input nonlinear system as in section 2.1 but with f linear in p, i.e. :

$$\dot{x} = a_0(x, u) + A_0(x, u)\, p , \tag{21}$$

where a_0 and A_0 are C^{m+2} functions. We assume assumption (A1) holds, namely, there exist functions ψ_0, of class C^{m+1}, u_0, of class C^{m+2}, V_0, positive of class C^{m+2}, and W_0, positive of class C^0, such that :

(A1) for all $(x, \mathcal{X}_e, p)$, we have :

$$\dot{V_0}\big|_{(21)-(4)}(x, \mathcal{X}_e, p) \le -W_0(x, \mathcal{X}_e, p) . \tag{22}$$

We will need also the following extra assumption :

(A1.1) u_0 does not depend on the parameter vector p and is rewritten $u_0(x, \mathcal{X}_e)$.

As mentioned in Remark 2, these two assumptions would be satisfied if V_0, W_0, ψ_0 and u_0 were given by Lemma 2.

3.1 Adding one integrator with unknown parameters

Let us now add one integrator to the system (21) :

$$\begin{cases} \dot{x} = a_0(x, y) + A_0(x, y)\, p \\ \dot{y} = \overline{u}_1 + a_1(x, y) + A_1(x, y)\, \theta_1 \end{cases} \tag{23}$$

where y is a new scalar state component, $\overline{u}_1$ is a new scalar input, $\theta_1 \in \mathbb{R}^{l_1}$ is a new parameter vector, and a_1 and A_1 are C^{m+1} functions.

Applying Lemma 1, without r thanks to (A1.1) (see point 2.1 of Remark 1), we get two real functions $V_1(x, y, \mathcal{X}_e, p)$ of class C^{m+2} and $W_1(x, y, \mathcal{X}_e, p)$ of class C^0 and a new C^{m+1} control law $u_1(x, y, \mathcal{X}_e, p, \theta_1)$:

$$\begin{aligned} u_1(x, y, \mathcal{X}_e, p, \theta_1) = {} & \tfrac{\partial u_0}{\partial x}(x, \mathcal{X}_e)\left(a_0(x, y) + A_0(x, y) p\right) \\ & - a_1(x, y) - A_1(x, y)\,\theta_1 - (y - u_0(x, \mathcal{X}_e)) \\ & - \tfrac{\partial V_0}{\partial x}(x, \mathcal{X}_e, p) \int_0^1 \overline{\varphi}(x, y, \mathcal{X}_e, p, s)\, ds \\ & + \tfrac{\partial u_0}{\partial \mathcal{X}_e}(x, \mathcal{X}_e)\, \psi_0(x, \mathcal{X}_e) , \end{aligned} \tag{24}$$

with :

$$\overline{\varphi} = \frac{\partial(a_0 + A_0 p)}{\partial y}\left(x\, ,\, u_0(x, \mathcal{X}_e) + [y - u_0(x, \mathcal{X}_e)] s\right) , \tag{25}$$

such that, for all $(x, y, \mathcal{X}_e, p, \theta_1)$:

$$\dot{V_1}\big|_{(23)-(24)}(x, y, \mathcal{X}_e, p) \le -W_1(x, y, \mathcal{X}_e, p) . \tag{26}$$

Note that V_1 does not depend on the new parameter vector θ_1. Also, thanks to assumption (A1.1), if $\frac{\partial V_0}{\partial x}$ does not depend on p, the same holds for $\frac{\partial V_1}{\partial x}$ and $\frac{\partial V_1}{\partial y}$. Therefore, if the system (21) satisfies assumptions (A1), (A1.1) and

(A1.2.1) $\frac{\partial V_0}{\partial x}(x, \mathcal{X}_e, p)$ is independent of p,

(A1.2.2) $\frac{\partial V_0}{\partial \mathcal{X}_e}(x, \mathcal{X}_e, p)$ is linear in p,

then, by identifying :

$$\begin{aligned} & x_e = (x^\top, y)^\top, \mathcal{X} = \mathcal{X}_e , \theta = (p^\top, \theta_1^\top)^\top, \widehat{\theta} = (\widehat{p}_1^\top, \widehat{\theta}_1^\top)^\top \\ & W = W_1 , u = u_1 , \psi = \psi_0 , \\ & V(x_e, \mathcal{X}, \widehat{\theta}, \theta) = V_1((x, y), \mathcal{X}_e, p) , \end{aligned} \tag{27}$$

assumptions (A2.1), (A2.2), (A3) and (A4) are satisfied for the system (23). In particular $h = 0$ in assumption (A3) and Z defined in (17) has two parts :

$$\begin{aligned} Z_{\theta_1}(x, y, \mathcal{X}_e) &= (y - u_0(x, , \mathcal{X}_e))\, A_1(x, y) . \\ Z_p(x, y, \mathcal{X}_e) &= \frac{\partial}{\partial p}\left(\frac{\partial V_0}{\partial \mathcal{X}_e}(x, \mathcal{X}_e, p)\, \psi(x, \mathcal{X}_e)\right) \\ & + \left[\tfrac{\partial V_0}{\partial x}(x, \mathcal{X}_e, p) - (y - u_0(x, \mathcal{X}_e))\, \tfrac{\partial u_0}{\partial x}(x, \mathcal{X}_e)\right] A_0(x, y) \end{aligned} \tag{28}$$

A straightforward consequence of Lemma 2 is :

Proposition 1 (Adding one integrator with unknown parameters)
With assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2), consider the following two real functions $\overline{V}_1$ of class C^{m+2} and $\overline{W}_1$ of class C^0 :

$$\begin{aligned} \overline{V}_1(x, y, \mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1, p, \theta_1) = {} & V_0(x, \mathcal{X}_e, p) + \tfrac{1}{2}(y - u_0(x, \mathcal{X}_e))^2 \\ & + \tfrac{1}{2}\|\widehat{p}_1 - p\|^2 + \tfrac{1}{2}\left\|\widehat{\theta}_1 - \theta_1\right\|^2 \end{aligned} \tag{29}$$

$$\overline{W}_1(x, y, \mathcal{X}_e, \widehat{p}_1) = W_0(x, \mathcal{X}_e, \widehat{p}_1) + (y - u_0(x, \mathcal{X}_e))^2 . \tag{30}$$

By applying, to the system (23), the following C^{m+1} dynamic state feedback :

$$\begin{cases} \dot{\mathcal{X}}_e = \psi_0(x, \mathcal{X}_e) \\ \dot{\widehat{p}}_1 = Z_p(x, y, \mathcal{X}_e)^\top \\ \dot{\widehat{\theta}}_1 = Z_{\theta_1}(x, y, \mathcal{X}_e)^\top \\ \overline{u}_1 = u_1(x, y, \mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1) \end{cases} \tag{31}$$

with u_1 given in (24) and Z_p and Z_{θ_1} given in (28), we obtain, for all $(x, y, \mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1, p, \theta)$,

$$\dot{\overline{V}}_1\big|_{(23)-(31)}(x, y, \mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1, p, \theta) \le -\overline{W}_1(x, y, \mathcal{X}_e, \widehat{p}_1) \tag{32}$$

Remark 3 :
1 – If, in the system (23), we know $\theta_1 = p$, then, in (31),

there is no need to add two estimators $\widehat{p}_1$ and $\widehat{\theta}_1$. We let simply :

$$\dot{\widehat{p}}_1 = \dot{\widehat{\theta}}_1 = Z_p(x,y,\mathcal{X}_e)^\top + Z_{\theta_1}(x,y,\mathcal{X}_e)^\top \qquad (33)$$

2 – With assumption (A1.2.1), $\overline{u}_1$ in (31), $\frac{\partial \overline{V}_1}{\partial x}$ and $\frac{\partial \overline{V}_1}{\partial y}$ do not depend on (p,θ_1). With assumption (A1.2.2), $\frac{\partial \overline{V}_1}{\partial \mathcal{X}_e}$, $\frac{\partial \overline{V}_1}{\partial \widehat{p}_1}$ and $\frac{\partial \overline{V}_1}{\partial \widehat{\theta}_1}$ are linear in (p,θ_1). It follows that assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) are satisfied by the system (23), with this time :

$$\mathcal{X} = (\mathcal{X}_e^\top, \widehat{p}_1^\top, \widehat{\theta}_1^\top)^\top . \qquad (34)$$

Therefore, by using Proposition 1 recurrently, other integrators with unknown parameters could again be added. This fundamental observation is due to Kanellakopoulos, Kokotovic and Morse [2]. Each time, Proposition 1 is applied, we extend the dynamics of the controller by a new estimate $\widehat{p}_1$ of all the old parameter vector p, and one estimate $\widehat{\theta}_1$ the new parameter vector θ. Hence for k integrators, with one new parameter vector for each, we will get $\frac{k(k+1)}{2}$ parameter vector estimates.

3.2 Adding two integrators with unknown parameters

Let us now consider the case where two integrators are added to the system (21) :

$$\begin{cases} \dot{x} = a_0(x,y) + A_0(x,y)\,p \\ \dot{y} = z + a_1(x,y) + A_1(x,y)\,\theta_1 \\ \dot{z} = \overline{u}_2 + a_2(x,y,z) + A_2(x,y,z)\,\theta_2 \end{cases} \qquad (35)$$

where y and z are two new scalar state components, $\overline{u}_2$ is a new scalar input, $\theta_1 \in \mathbb{R}^{l_1}$ and $\theta_2 \in \mathbb{R}^{l_2}$ are new parameter vectors, and a_i and A_i, $i = 1,2$ are C^{m+1} functions.

From our Remark 3, under assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2), an adaptive controller for this system (35) can be obtained by applying Proposition 1 twice. However, proceeding this way, by applying this Proposition once, the state vector of the dynamic controller obtained will be $(\mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1)$. And therefore, the dynamic controller we will obtain for the system (35) after applying this Proposition twice will have the state : $(\mathcal{X}_e, \widehat{p}_1, \widehat{\theta}_1, \widehat{p}_2, \widehat{\theta}_{12}, \widehat{\theta}_2)$. Namely, this state contains the original state $\mathcal{X}_e$, two estimates $\widehat{p}_1$ and $\widehat{p}_2$ of p, two estimates $\widehat{\theta}_1$ and $\widehat{\theta}_{12}$ of θ_1 and one estimate $\widehat{\theta}_2$ of θ_2. This explosion of the number of estimates is the main drawback of this iterative procedure proposed in [2]. Our intent, now, is to show the existence of a dynamic controller with a state $(\mathcal{X}_e, \widehat{p}, \widehat{\theta}_1, \widehat{\theta}_2)$ of smaller dimension.

Applying Lemma 1 twice (see point 1 of Remark 1), with $r = (\widehat{p}, \widehat{\theta}_1)$ the second time, we get two real functions V_2 of class C^{m+1} and W_2 of class C^0 and a new C^m control law u_2 :

$$V_2(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1,p) = V_0(x,\mathcal{X}_e,p) + \tfrac{1}{2}(y - u_0(x,\mathcal{X}_e))^2 + \tfrac{1}{2}(z - u_1(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))^2 \qquad (36)$$

$$W_2(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1,p) = W_0(x,\mathcal{X}_e,p) + (y - u_0(x,\mathcal{X}_e))^2 + (z - u_1(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))^2 \qquad (37)$$

$$\begin{aligned} u_2(x,y,z,\mathcal{X}_e,p,\theta_1,\theta_2) = &- (y - u_0(x,\mathcal{X}_e)) \qquad (38) \\ &- a_2(x,y,z) - A_2(x,y,z)\,\theta_2 - (z - u_1(x,y,\mathcal{X}_e,p,\theta_1)) \\ &+ \tfrac{\partial u_1}{\partial y}(x,y,\mathcal{X}_e,p,\theta_1)\,(z + a_1(x,y) + A_1(x,y)\theta_1) \\ &+ \tfrac{\partial u_1}{\partial x}(x,y,\mathcal{X}_e,p,\theta_1)\,(a_0(x,y) + A_0(x,y)p) \\ &+ \tfrac{\partial u_1}{\partial \mathcal{X}_e}(x,y,\mathcal{X}_e,p,\theta_1)\,\psi_0(x,\mathcal{X}_e) , \end{aligned}$$

with u_1 given in (24). These functions are such that, for all $(x,y,z,\mathcal{X}_e,p,\theta_1,\theta_2)$ and by evaluating at $(\widehat{p},\widehat{\theta}_1) = (p,\theta_1)$:

$$\dot{V}_2\big|_{(35)-(38)}(x,y,z,\mathcal{X}_e,p,\theta_1) \leq -W_2(x,y,z,\mathcal{X}_e,p,\theta_1,p)$$

Now, to apply Lemma 2, we identify :

$$\begin{aligned} &x_e = (x^\top, y, z)^\top, \ \mathcal{X} = \mathcal{X}_e, \ \theta = (p^\top, \theta_1^\top, \theta_2^\top)^\top, \\ &\widehat{\theta} = (\widehat{p}^\top, \widehat{\theta}_1^\top, \widehat{\theta}_2^\top)^\top, \ u = u_2, \ \psi = \psi_0, \qquad (40) \\ &V(x_e,\mathcal{X},\widehat{\theta},\theta) = V_2((x,y,z),\mathcal{X}_e,(\widehat{p},\widehat{\theta}_1),p), \\ &W(x_e,\mathcal{X},\widehat{\theta},\theta) = W_2((x,y,z),\mathcal{X}_e,(\widehat{p},\widehat{\theta}_1),p) . \end{aligned}$$

Then, we observe from the definitions (36) and (37) of V_2 and W_2 that assumptions (A2.1), (A2.2), (A3) and (A4) are satisfied for the system (35) if, again assumptions (A1.2.1) and (A1.2.2) hold for the system (21). In particular the function h satisfying (A3) is :

$$h(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) = -\left(\tfrac{\partial u_1}{\partial p}(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1),\ \tfrac{\partial u_1}{\partial \theta_1}(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1),\ 0\right) \qquad (41)$$

and Z defined in (17) has three parts :

$$\begin{aligned} Z_p(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) = &\tfrac{\partial}{\partial p}\left(\tfrac{\partial V_0}{\partial \mathcal{X}_e}(x,\mathcal{X}_e,p)\,\psi(x,\mathcal{X}_e)\right) \\ &- \Big[(z - u_1(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))\tfrac{\partial u_1}{\partial x}(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) \\ &+ (y - u_0(x,\mathcal{X}_e))\ \tfrac{\partial u_0}{\partial x}(x,\mathcal{X}_e) - \tfrac{\partial V_0}{\partial x}(x,\mathcal{X}_e,p)\Big]\,A_0(x,y) \end{aligned}$$

$$\begin{aligned} Z_{\theta_1}(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) = &\Big[(y - u_0(x,\mathcal{X}_e)) \\ &-(z - u_1(x,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))\,\tfrac{\partial u_1}{\partial y}(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1)\Big]\,A_1(x,y) \end{aligned} \qquad (42)$$

$$Z_{\theta_2}(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) = (z - u_1(x,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))\,A_2(x,y,z) .$$

A straightforward consequence of Lemma 2 is :

Proposition 2 (Adding two integrators with unknown parameters)
With assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2), consider the following two real functions $\overline{V}_2$ of class C^{m+1} and $\overline{W}_2$ of class C^0 :

$$\begin{aligned} \overline{V}_2(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1,\widehat{\theta}_2,p,\theta_1,\theta_2) = &V_0(x,\mathcal{X}_e,p) \qquad (43) \\ &+ \tfrac{1}{2}(y - u_0(x,\mathcal{X}_e))^2 + \tfrac{1}{2}(z - u_1(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))^2 \\ &+ \tfrac{1}{2}\|\widehat{p} - p\|^2 + \tfrac{1}{2}\left\|\widehat{\theta}_1 - \theta_1\right\|^2 + \tfrac{1}{2}\left\|\widehat{\theta}_2 - \theta_2\right\|^2 \end{aligned}$$

$$\begin{aligned} \overline{W}_2(x,y,z,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1) = &W_0(x,\mathcal{X}_e,\widehat{p}) \qquad (44) \\ &+ (y - u_0(x,\mathcal{X}_e))^2 + (z - u_1(x,y,\mathcal{X}_e,\widehat{p},\widehat{\theta}_1))^2 . \end{aligned}$$

By applying, to the system (35), the following C^m dy-

namic state feedback :

$$\begin{cases} \dot{\chi}_e = \psi_0(x,\chi_e) \\ \dot{\widehat{p}} = Z_p(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1)^\top \\ \dot{\widehat{\theta}}_1 = Z_{\theta_1}(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1)^\top \\ \dot{\widehat{\theta}}_2 = Z_{\theta_2}(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1)^\top \\ \overline{u}_2 = u_2(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1,\widehat{\theta}_2) \\ \qquad + \frac{\partial u_1}{\partial p}(x,y,\chi_e,\widehat{p},\widehat{\theta}_1)\, Z_p(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1) \\ \qquad + \frac{\partial u_1}{\partial \theta_1}(x,y,\chi_e,\widehat{p},\widehat{\theta}_1) Z_{\theta_1}(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1) \end{cases} \tag{45}$$

with u_1 given in (24), u_2 in (38) and $(Z_p, Z_{\theta_1}, Z_{\theta_2})$ in (42) we obtain, for all $(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1,\widehat{\theta}_2,p,\theta_1,\theta_2)$,

$$\dot{\overline{V}}_2 \big|_{(35)-(45)} (x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1,\widehat{\theta}_2) \le -\overline{W}_2(x,y,z,\chi_e,\widehat{p},\widehat{\theta}_1)$$

Remark 4 :
1 – If, in the system (35), we know $\theta_2 = \theta_1 = p$, then, in (45), there is no need to add three estimators.
2 – As for Proposition 1, the Lyapunov design for two integrators of this Proposition 2 implies that assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) are satisfied by the system (35) if they were satisfied by the system (21). It follows that Proposition 2 can be used recurrently. Each time, this Proposition is applied, we extend the dynamics of the controller by a new estimate of all the old parameter vectors, and one estimate for each of the new two parameter vectors. Hence for $2k$ integrators, with one parameter vector, we will get $k(k+1)$ parameter vector estimates. This is to be compared with the $k(2k+1)$ parameter vector estimates given by applying Proposition 1, i.e. the scheme proposed by Kanellakopoulos Kokotovic and Morse in [2].

4 Applications

4.1 Systems in strict feedback form

The idea of iterative design has been proposed by Kanellakopoulos, Kokotovic and Morse [2] originally to solve the problem of globally stabilizing the following single input strict feedback system :

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1)\, p \\ \dot{x}_2 = x_3 + f_2(x_1,x_2)\, p \\ \quad \vdots \\ \dot{x}_{n-1} = x_n + f_{n-1}(x_1,\cdots,x_{n-1})\, p \\ \dot{x}_n = u + f_n(x_1,\cdots,x_n)\, p \end{cases} \tag{47}$$

where $x = (x_1,x_2,\cdots,x_n)^\top$ is the state, $p \in \mathbb{R}^l$ is the vector of constant unknown parameters and $f_i, i = 1,2,\cdots,n$ are known smooth functions. For such systems, Kanellakopoulos, Kokotovic and Morse [2] have proposed to apply recurrently Proposition 1.

Following Remark 4, Proposition 2 applies also recurrently. Indeed, starting with $u_0 \equiv 0$, $V_0 \equiv 0$, $W_0 \equiv 0$, we can add integrators two by two. Getting at the end a controller with $n/2$ estimates of the parameter vector p, instead of the n estimates involved in the Kanellakopoulos et al.'s scheme. Let us illustrate this point with the following famous example.

Example 1 (A third order strict-feedback system) : Consider the following strict feedback system :

$$\begin{cases} \dot{x}_1 = x_2 + p x_1^2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases} \tag{48}$$

where $x = (x_1,x_2,x_3)^\top \in \mathbb{R}^3$ is the state, u is a scalar input and p is an unknown constant parameter.
Step 0 : We start with $V_0 \equiv 0$, $W_0 \equiv 0$ and $u_0 \equiv 0$. In this way, assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) are trivially satisfied.
Step 1 : We add two integrators, i.e. we consider the following system :

$$\begin{cases} \dot{x}_1 = x_2 + p x_1^2 \\ \dot{x}_2 = \overline{u}_2 \end{cases} \tag{49}$$

For this system, with Step 0, Proposition 2 applies. We get two functions $\overline{V}_2$ and $\overline{W}_2$:

$$\overline{V}_2(x_1,x_2,\widehat{p}_1,p) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_2 + x_1 + \widehat{p}_1 {x_1}^2)^2 + \tfrac{1}{2}|\widehat{p}_1 - p|^2 \tag{50}$$

$$\overline{W}_2(x_1,x_2,\widehat{p}_1) = x_1^2 + (x_2 + x_1 + \widehat{p}_1\, x_1^2)^2 .$$

It corresponds :

$$\begin{aligned} Z_2 &= \left[x_1 + (x_2 + x_1 + \widehat{p}_1 {x_1}^2)(1 + 2x_1\widehat{p}_1)\right] x_1^2 \\ h_2 &= x_1^2 \end{aligned} \tag{51}$$

It follows that, by applying, to the system (49), the following dynamic state feedback :

$$\begin{cases} \dot{\widehat{p}}_1 = \left[x_1 + (x_2 + x_1 + \widehat{p}_1 x_1^2)(1 + 2x_1\widehat{p}_1)\right] x_1^2 \qquad (52) \\ \overline{u}_2 = -(1 + 2x_1\widehat{p}_1)(x_2 + \widehat{p}_1 x_1^2) - (x_2 + x_1 + \widehat{p}_1 {x_1}^2) \\ \qquad - x_1 - x_1^2\left[x_1 + (x_2 + x_1 + \widehat{p}_1 x_1^2)(1 + 2x_1\widehat{p}_1)\right] x_1^2 \\ \quad \overset{\text{def}}{=} \overline{u}_2(x_1,x_2,\widehat{p}_1) \end{cases}$$

we obtain :

$$\dot{\overline{V}}_2 \big|_{(49)-(52)} (x_1,x_2,\widehat{p}_1) = -\overline{W}_2(x_1,x_2,\widehat{p}_1) . \tag{53}$$

Therefore $\overline{V}_2$, $\overline{W}_2$ and $\overline{u}_2$ satisfy assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) for the system (49).
Step 2 : We add another integrator to (49) to obtain our actual system (48). For this system, with Step 1, Proposition 1 applies. We get two functions $\overline{V}_3$ and $\overline{W}_3$:

$$\overline{V}_3(x_1,x_2,x_3,\widehat{p}_1,\widehat{p}_2,p) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_2 + x_1 + \widehat{p}_1 {x_1}^2)^2 + \tfrac{1}{2}(x_3 - \overline{u}_2(x_1,x_2,\widehat{p}_1))^2 + \tfrac{1}{2}|\widehat{p}_1 - p|^2 + \tfrac{1}{2}|\widehat{p}_2 - p|^2 \tag{54}$$

$$\overline{W}_3(x_1,x_2,x_3,\widehat{p}_1) = x_1^2 + (x_2 + x_1 + \widehat{p}_1 {x_1}^2)^2 + (x_3 - \overline{u}_2(x_1,x_2,\widehat{p}_1))^2 . \tag{55}$$

It corresponds :

$$\begin{aligned} Z_3 &= -\left(x_3 - \overline{u}_2(x_1,x_2,\widehat{p}_1)\right) \tfrac{\partial \overline{u}_2}{\partial x_1}(x_1,x_2,\widehat{p}_1)\, x_1^2 \\ h_3 &= 0 \end{aligned} \tag{56}$$

Therefore, since $\overline{V}_3$ is of class C^1 and proper in $(x_1,x_2,x_3,\widehat{p}_1,\widehat{p}_2)$ and $\overline{W}_3$ is of class C^0 and positive, the following dynamic state feedback :

$$\begin{cases} \dot{\widehat{p}}_1 = \left[x_1 + (x_2 + x_1 + \widehat{p}_1 x_1^2)(1 + 2x_1\widehat{p}_1)\right] x_1^2 \qquad (57) \\ \dot{\widehat{p}}_2 = -\left(x_3 - \overline{u}_2(x_1,x_2,\widehat{p}_1)\right) \tfrac{\partial \overline{u}_2}{\partial x_1}(x_1,x_2,\widehat{p}_1)\, x_1^2 \\ u = \tfrac{\partial \overline{u}_2}{\partial x_1}(x_1,x_2,\widehat{p}_1)(x_2 + \widehat{p}_2 {x_1}^2) + \tfrac{\partial \overline{u}_2}{\partial x_2}(x_1,x_2,\widehat{p}_1) x_3 \\ \quad + \tfrac{\partial \overline{u}_2}{\partial \widehat{p}_1}(x_1,x_2,\widehat{p}_1)\dot{\widehat{p}}_1 \\ \quad - (x_2 + x_1 + \widehat{p}_1 {x_1}^2) - (x_3 - \overline{u}_2(x_1,x_2,\widehat{p}_1)) \end{cases}$$

guarantees that any $(x_1(t), x_2(t), x_3(t), \widehat{p}_1(t), \widehat{p}_2(t))$, solution of (49)-(57), is well defined, bounded on $[0, +\infty)$ and satisfies :

$$\lim_{t \to +\infty} (|x_1(t)| + |x_2(t)| + |x_3(t)|) = 0 . \quad (58)$$

As announced, the dynamic state feedback (57), has dimension two instead of three as in [2]. This dimension is the same as the one of the controller proposed by Pomet and Praly [6] who have followed a different route. ∎

4.2 Extension of strict-feedback systems

Our starting point for applying Proposition 1 or 2 is a system in the form (3) and satisfying assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) with some functions (V_0, W_0, u_0, ψ_0). For the strict feedback systems considered in [2] and section 4.1, this system is reduced to nothing with $V_0 = W_0 = u_0 = \psi_0 = 0$. Introducing a less trivial system allows us to enlarge easily the family of systems which can be adaptively stabilized. Let us illustrate this point with the following example :

Example 2 (System satisfying a matching condition augmented by a strict feedback system) : Let us consider the following forth order system :

$$\begin{cases} \dot{x}_1 = x_2 + p\,(x_1 + x_2)(2x_1 + 2x_2 + x_3) \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \overline{u}_1 + q\,x_4^2 \end{cases} \quad (59)$$

where (p, q) are two constant unknown parameters. Though this system cannot be written in a strict feedback form, Proposition 1 applies. Indeed, we start with the first three equations and :

$$V_0(x_1, x_2, x_3, \widehat{p}_1, p) = \frac{x_1^2}{2} + \frac{(x_1+x_2)^2}{2} + \frac{(2x_1+2x_2+x_3)^2}{2} + \tfrac{1}{2}(\widehat{p}_1 - p)^2 \quad (60)$$

$$W_0(x_1, x_2, x_3) = x_1^2 + (x_1 + x_2)^2 + (2x_1 + 2x_2 + x_3)^2 \quad (61)$$

and the following dynamic state feedback :

$$\begin{cases} \dot{\widehat{p}}_1 = (2x_1 + 2x_2 + x_3)(x_1 + x_2)(6x_1 + 5x_2 + 2x_3) \\ u = -3x_1 - 5x_2 - 3x_3 \\ \qquad - \widehat{p}_1(x_1 + x_2)(6x_1 + 5x_2 + 2x_3) \\ \quad \stackrel{\text{def}}{=} u_0(x_1, x_2, x_3, \widehat{p}_1) \end{cases} \quad (62)$$

Therefore, assumptions (A1), (A1.1), (A1.2.1) and (A1.2.2) are satisfied. It follows that Proposition 1 can be used to get a dynamical state feedback for the actual four equations system. ∎

Another possible extension we have mentioned concerns the non smooth case, i.e. the case where the diffeomorphim allowing us to write the system in a strict feedback form has singularities. A systematic way to handle this case is for example to replace V_1 in (7) by :

$$V_1(x, y, \chi_e, r, p) = V_0(x, \chi_e, p) + \Phi(y, u_0(x, \chi_e, r)) \quad (63)$$

where $\Phi(y, u_0)$ is a positive function, proper in y with :

$$\frac{\partial \Phi}{\partial y}(y, u_0) = 0 \quad \Longrightarrow \quad y = u_0 . \quad (64)$$

This possibility is illustrated in [1, Example3].

5 Conclusion

In this paper, we have deepened the idea of iterative design of adaptive controller introduced by Kanellakopoulos, Kokotovic and Morse [2]. First, by giving an interpretation of the scheme in [2] in terms of the control Lyapunov function technique, we have shown that it can be applied to a slightly broader family of systems than the strict feedback systems. Second, by noticing that this design involves implicitely at each step a matching condition which is known to hold not only for one integrator but also for two integrators, we have been able to propose a new adaptive controller with in general half as many parameters to update. An noticeable aspect of these designs is that together with an adaptive controller, we get a corresponding control Lypunov function. This is interesting when in a next stage, we will study robustness of stability to extraneous disturbances, unmodeled dynamics or time variations.

References

[1] Jiang Z.P., Praly L. : *Iterative designs of adaptive controllers for systems with nonlinear integrators.* Internal Report CAS. March 1991.

[2] Kanellakopoulos I., Kokotovic P.V., Morse A.S. : *Systematic design of adaptive controllers for feedback linearizable systems* Technical Report UILU-ENG-90-2249, DC-124, University of Illinois at Urbana-Champaign, October 1990. Submitted for publication in IEEE Transactions on Automatic Control.

[3] Kokotovic P.V., Ed. : *Foundations of Adaptive Control.* Lecture Notes in Control and Informations Sciences. Springer-Verlag, Berlin, 1991.

[4] Kokotovic P.V., Sussmann H.J. : *A positive real condition for global stabilization of nonlinear systems.* Systems & Control Letters 13 (1989) 125-133.

[5] Parks P. : *Lyapunov Redesign of Model Reference Adaptive Control Systems.* I.E.E.E. Transactions on Automatic Control 11 No 3, pp 362-367, 1966.

[6] Pomet J.-B., Praly L. : *Adaptive non-linear regulation: equation error from the Lyapunov equation.* 28th IEEE CDC, Tampa, December 1989. To appear in I.E.E.E. Transactions on Automatic Control.

[7] Praly L., d'Andréa-Novel, J.-M. Coron : *Lyapunov design of stabilizing controllers for cascaded systems.* To appear in IEEE Transations on Automatic Control. April 1991. See also Proc. 28th IEEE CDC, Tampa, 1989.

[8] Tsinias J. : *Sufficient Lyapunov-like conditions for stabilization.* Math. Control Signals Systems 2 (1989) 343-357.