ADAPTIVE STABILIZATION FOR NONLINEAR SYSTEMS IN THE PLANE

B. d'Andréa-Novel, J. B. Pomet and L. Praly

Centre d'Automatique et Informatique, Ecole des Mines de Paris, 35 rue Saint Honoré, 77305 Fontainebleau Cedex, France

Abstract. We propose an adaptive controller to globally stabilize a compact set for a planar system depending linearly on unknown parameters. Global rectifiability of the control vector field is assumed.
Our approach is based on the Control Lyapunov Function and Lyapunov design technique.

Keywords. Adaptive control; linear parametrization; Lyapunov methods; planar systems; practical stability; stabilization.

1 Introduction

Important results about the stabilization of planar systems are now available. In particular, Dayawansa and Martin [4] have established a necessary and sufficient condition for stabilization by a continuous feedback. In parallel, the theory of adaptive control of linearly parameterized systems has made some progress. In this paper we combine these two fields.

We are interested in globally stabilizing a planar system depending linearly on some unknown parameters by a continuous feedback law.

In a first part, we deal with this problem in the case of a known system. Following the so called Control Lyapunov Function approach, as introduced by Artstein [2] and Sontag [12], we propose a control law which solves our problem. Moreover, we obtain an explicit expression for a Lyapunov function of the closed-loop system.

In the second part, dealing with the unknown parameter case, we apply the Lyapunov design as introduced by Parks [9]. Using a certainty equivalence modification similar to this proposed by Kanellakopoulos [6], we obtain a stabilizing adaptive controller.

2 Stabilization without parametric uncertainties

We consider the affine nonlinear system in R^2:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where the state vector x is in R^2, u is a scalar input, f and g are at least C^2 vector-fields.

Definition 1
V from R^2 to R^+ is said to be a Control Lyapunov Function, denoted clf, if the following properties are satisfied:

1. *$V(x) \to \infty$ iff $\| x \| \to +\infty$. This implies that the preimage by V of a compact set is also a compact set.*

2. *V is at least C^1 and $V(x)$ is zero iff x is zero.*

3. *Denoting $L_fV(x)$ the Lie derivative of V with respect to f,*

$$L_gV(x) = 0 \quad \textit{implies} \quad L_fV(x) < 0 \textit{ or } x = 0 \tag{2}$$

Notations :
We denote

$$\dot{V}(x) = \frac{d}{dt}V(\Phi_t(x))|_{t=0} \tag{3}$$

where $\Phi_t(x)$ is the flow of the vector-field $f + gu$ starting from x at time $t = 0$.

Our problem is to design a (as smooth as possible) control law making a (as small as possible) neighborhood of the origin globally attractive. The following important result due to Sontag [12], precising this of Artstein [2], tells us that it is sufficient to find a clf:

Theorem 1 *[12]*
If there exists a clf V for system (1), then the control law:

$$u = \begin{cases} 0 & \textit{if} \quad L_fV < 0 \\ -\dfrac{L_fV + \sqrt{(L_fV)^2 + (L_gV)^4}}{L_gV} & \textit{if not} \end{cases} \tag{4}$$

is C^0 on $\mathsf{R}^2 - \{0\}$ and makes $\dot{V}(x)$ strictly negative for all non zero x. Therefore the origin is globally stabilized.

Remark :
u can be guaranteed to be at least continuous at the origin if the clf V satisfies the so called "small control property", namely: (see [2,12])

$\forall \epsilon > 0$, $\exists \delta > 0$ such that for all x, $\| x \| < \epsilon$, $x \neq 0$, there exists u, $\| u \| < \epsilon$, satisfying:

$$L_f V(x) + u L_g V(x) < 0 . \qquad (5)$$

To obtain a *clf* for system (1), we follow the approach proposed by d'Andréa and Praly [1] and assume:

H1 :
There exist a global C^2 diffeomorphism $(\chi_1(x_1,x_2), \chi_2(x_1,x_2))$ and a function $v_0(\chi_1)$ such that:

1. (1) can be rewritten:

$$\begin{cases} \dot\chi_1 = \phi_1(\chi_1,\chi_2) \\ \dot\chi_2 = \phi_2(\chi_1,\chi_2) + u \end{cases} \qquad (6)$$

2. $v_0(0)$ is zero and v_0 and $\dfrac{\partial {v_0}^{2m-1}}{\partial \chi_1}$ are C^0 on R, for a strictly positive integer m.

3. We have for all non zero χ_1 :

$$\chi_1 \phi_1(\chi_1, v_0(\chi_1)) < 0 \qquad (7)$$

Remarks :
1. Property (7) is known to be a necessary condition for the asymptotic stabilizability of the origin for system (6), see [3,4,1].

2. To illustrate assumption **H1** we consider the non C^1-stabilizable system studied by Kawski [7]:

$$\begin{cases} \dot x_1 = x_1 - x_2^3 \\ \dot x_2 = u \end{cases} \qquad (8)$$

The global diffeomorphism is trivially the Identity and we may choose:

$$v_0(\chi_1) = 2\chi_1^{\frac{1}{3}} \qquad (9)$$

to meet (7). Then point 2. in **H1** is satisfied with m larger or equal to 2.

With **H1**, we can propose the following *clf*:

Lemma 1
Under assumption **H1**, *the function V defined by:*

$$V = \frac{\chi_2^{2m}}{2m} - \chi_2 v_0(\chi_1)^{2m-1} + \frac{2m-1}{2m} v_0(\chi_1)^{2m} + \frac{\chi_1^2}{2} \qquad (10)$$

is a clf for system (1). Moreover if:

$$v_0(\chi_1) = 0 \implies \chi_1 = 0 \qquad (11)$$

the following control law is C^0 on $\mathbb{R}^2 - \{0\}$ and makes V strictly decreasing:

$$\begin{aligned} u(\chi_1,\chi_2) = & - c\,(\chi_2^{2m-1} - v_0(\chi_1)^{2m-1}) \\ & + \frac{\partial {v_0}^{2m-1}}{\partial \chi_1}(\chi_1)\, \frac{\phi_1(\chi_1,\chi_2)}{R(v_0(\chi_1),\chi_2)} \\ & - \chi_1 \frac{\phi_1(\chi_1,\chi_2) - \phi_1(\chi_1, v_0(\chi_1))}{\chi_2^{2m-1} - v_0(\chi_1)^{2m-1}} \end{aligned} \qquad (12)$$

where c is a strictly positive constant and $R(v_0,\chi_2)$ is the polynomial defined by:

$$\chi_2^{2m-1} - v_0^{2m-1} = (\chi_2 - v_0)\, R(v_0,\chi_2) \qquad (13)$$

Proof :
First, by writing:

$$v_0^{2m} = \left| v_0^{2m-1} \right|^{1+\frac{1}{2m-1}} \qquad (14)$$

we notice that, v_0 being C^0 and v_0^{2m-1} being C^1, v_0^{2m} is C^1 and we have:

$$\frac{2m-1}{2m} \frac{\partial v_0^{2m}}{\partial \chi_1} = v_0 \frac{\partial v_0^{2m-1}}{\partial \chi_1} \qquad (15)$$

It follows that V is C^1.

Second, defining

$$X = \frac{v_0(\chi_1)}{\chi_2} \qquad (16)$$

we rewrite V in:

$$V(\chi_1,\chi_2) = \frac{{\chi_2}^{2m}}{2m} F(X) + \frac{\chi_1^2}{2} \qquad (17)$$

where

$$F(X) = 1 - 2mX^{2m-1} + (2m-1)X^{2m} \qquad (18)$$

F is continuous and positive since the derivative of F changes sign only at $X = 1$ where F is zero. Since both χ_1^2 and ${\chi_2}^{2m}F(X)$ are positive, if $V(\chi_1,\chi_2)$ is bounded, the same holds for χ_1^2 and ${\chi_2}^{2m}F(X)$. On the other hand, since v_0 is continuous, $v_0(\chi_1)$ is bounded. F being continuous and $F(0)$ equal to 1, this implies that χ_2 is bounded. Hence $V(x) \to \infty$ iff $\| x \| \to +\infty$ and $V(x)$ is zero iff x is zero.

Finally, we compute:

$$L_g V(\chi_1,\chi_2) = \frac{\partial V}{\partial \chi_2} = {\chi_2}^{2m-1} - v_0(\chi_2)^{2m-1} \qquad (19)$$

$2m-1$ being odd, $L_g V(\chi_1,\chi_2)$ is zero iff

$$\chi_2 = v_0(\chi_1). \qquad (20)$$

In this case we have with (15):

$$L_f V(\chi_1, v_0(\chi_1)) = \chi_1\, \phi_1(\chi_1, v_0(\chi_1)) \qquad (21)$$

Consequently, assumption (7) implies that (2) is satisfied for system (1).
Hence V is a *clf*.

Applying u given by (12) to system (1) with V given by (10), we obtain:

$$\dot V = -c\,(\chi_2^{2m-1} - v_0(\chi_1)^{2m-1})^2 + \chi_1\, \phi_1(\chi_1, v_0(\chi_1)) \qquad (22)$$

which is strictly negative except at the origin. Moreover, ϕ_1 being C^1, u is C^0, except possibly when $R(v_0,\chi_2)$ is zero. With assumption (11), this is possible only at the origin. □

Remarks :
1. Applying this Lemma to system (8), assumption (11) is satisfied and therefore (12) is a well defined stabilizing control law. It is continuous at the origin only for $m = 2$.

2. The fact that u is only C^0 and not locally Lipschitz may lead to uniqueness problems. As shown by Kawski [7], this difficulty can be rounded in some cases.

3. Assumption (11) is not needed when $m = 1$ in **H1**. The case when 0 is an isolated zero of v_0 and v_0 is C^1 except may be at 0 can also be handled.

Under assumption **H1**, with Lemma 1 and Theorem 1, we can obtain a control law which is C^0 except may be at the origin. It follows that at least any prespecified compact set containing the origin as an interior point can be made globally attractive. Namely, assumption **H1** is sufficient for practical stability of the origin. To precisely state this result, we notice, with the definition of a *clf*, that:

$$K_\epsilon = \{x \,/\, V(x) \le \epsilon\} \tag{23}$$

defines, with ϵ in $]0, +\infty[$, an imbedded family of compact sets containing the origin as an interior point. Moreover these compact sets are connected for all sufficiently small ϵ.

Theorem 2
Let us choose a strictly positive ϵ. Under assumption **H1**, *the control law given by (4) (or (12) if (11) is met) is C^0 on $\mathbb{R}^2 - \{0\}$ and makes $Max(V(x) - \epsilon, 0)$ a strictly decreasing function of time, as long as it is not zero and therefore globally stabilizes K_ϵ. Moreover, if the control law is continuous at 0, ϵ can be chosen equal to zero.*

Proof :
With V given by Lemma 1, let $\bar{V}$ be:

$$\bar{V}(x) = \mathrm{Max}\{V(x) - \epsilon, 0\} \tag{24}$$

Let ψ be a C^∞ function equal to 1 outside the compact set K_ϵ and belonging to $[0,1]$ inside K_ϵ. Let u_1 be the control law given by (4) (or (12) if (11) is met) and u_2 be any C^0 function on K_ϵ. We consider the control law:

$$u = u_1\psi + (1-\psi)u_2 \tag{25}$$

u is C^0 and along the solution of (1) in closed-loop with (25) we have:

$$\bar{V} \neq 0 \implies \dot{\bar{V^2}} = \bar{V}\dot{V} < 0 \qquad \square \tag{26}$$

3 Stabilization with parametric uncertainties

We consider now a family of systems:

$$\dot{x} = f(x,p) + ug(x,p) \tag{27}$$

with

$$\begin{cases} f(x,p) = f_0(x) + F(x)p \\ g(x,p) = g_0(x) + G(x)p \end{cases} \tag{28}$$

indexed by the parameter vector p in $\mathbb{R}^n$. The following properties are assumed:

H2 :
For any x, $g_0(x)$ is not zero and each column of the $G(x)$ matrix is colinear to $g_0(x)$.

H3 :
There exist a convex subset π of $\mathbb{R}^n$ and a family of global C^2 diffeomorphisms $(\chi_1(x), \chi_2(x,p))$ and functions $v_0(\chi_1, p)$ such that:

1. (27) can be rewritten:

$$\dot{\chi} = \phi(\chi, p, u) \tag{29}$$

with

$$\phi = (\phi_1(\chi_1, \chi_2, p)\,,\, \phi_2(\chi_1, \chi_2, p) + u) \tag{30}$$

2. $v_0(0,p)$ is zero and v_0, $\frac{\partial v_0}{\partial p}$ and $\frac{\partial v_0^{2m-1}}{\partial \chi_1}$ are C^0 on $\mathbb{R} \times \pi$, for some strictly postive integer m.

3. We have for all p in π and all non zero χ_1:

$$\chi_1\phi_1(\chi_1, v_0(\chi_1,p), p) < 0 \tag{31}$$

4. There exists a scalar C^1 function Π on $\mathbb{R}^n$, such that:
1.1 $\Pi(\pi)$ contains the set $[0,1]$.
1.2 For each α in $[0,1]$, the set:

$$\pi_\alpha = \{p \mid \Pi(p) \le \alpha\} \tag{32}$$

is convex and

$$\Pi(p) = \alpha \implies \frac{\partial \Pi}{\partial p}(p) \neq 0 \tag{33}$$

1.3 The set π is exactly π_1.

5. Finally either m above is 1 or else:

$$v_0(\chi_1, p) = 0 \implies \frac{\partial v_0}{\partial p}(\chi_1, p) = 0 \tag{34}$$

Remarks :
1. (30) implies for all p in π :

$$L_{g(x,p)}\chi_1(x) = 0 \tag{35}$$

Since from **H2** the span of g does not depend on p, the assumption:

$$\frac{\partial \chi_1}{\partial p} = 0 \tag{36}$$

is not too restrictive.

2. For all p, q in π and x in $\mathbb{R}^2$, we have:

$$L_{g(x,q)}\chi_2(x,p) \neq 0 \tag{37}$$

This follows from **H2** which implies the existence of a scalar function $\lambda(x,p,q)$ to satisfy:

$$g(x,q) = \lambda(x,p,q)\, g(x,p) \tag{38}$$

and from (30) which implies:

$$L_{g(x,p)}\chi_2(x,p) = 1 \tag{39}$$

Hence, $g(x,p)$ is not zero and therefore the same is for λ and

$$L_{g(x,q)}\chi_2(x,p) = \lambda(x,p,q) \tag{40}$$

3. The Π function will be used only to construct a smooth projection of vector fields on the set π. Other construction and associated assumption are possible.

4. If m is not 1, assumption (34) can be relaxed if 0 is an isolated zero of v_0 and v_0 is C^1 except may be at 0.

5. Applying Theorem 2, we know a control law, denoted hereafter $\bar{u}(\chi, p)$ (adding the p dependance), making $\bar{V}^2$ strictly decreasing along the solutions of (29) with:

$$\bar{V}(\chi, p) = \mathrm{Max}\{V(\chi,p) - \epsilon, 0\} \tag{41}$$

and V given in (10). Moreover ϵ can be chosen 0 if $\bar{u}$ has a continuous extension at $\chi = 0$.

The system to be controlled is supposed to be one of the family (27), namely the one obtained by taking $p = p^*$.

p^* is unknown and our objective is to design a control law guaranteeing the same properties as the ones given by the control law of Section 2 for p^* known.

To design this control law, with the diffeomorphism given in **H3**, we define a new set of coordinates

$$\hat{\chi} = (\hat{\chi}_1, \hat{\chi}_2) = (\chi_1(x_1, x_2)\,,\, \chi_2(x_1, x_2, \hat{p})) \qquad (42)$$

where $\hat{p}$ is a time function interpreted as an estimate of p^*. In these new coordinates, (27) becomes:

$$\dot{\hat{\chi}} = \phi(\hat{\chi}, \hat{p}, u) + Z_p(x, \hat{p}, u)(p^* - \hat{p}) + \frac{\partial \hat{\chi}}{\partial \hat{p}}(x, \hat{p})\dot{\hat{p}} \qquad (43)$$

with $Z_p(x, \hat{p}, u)$ a $2 \times n$ matrix satisfying:

$$Z_p(p^* - \hat{p}) = \begin{pmatrix} L_{F(p^*-\hat{p})}\hat{\chi}_1 \\ L_{F(p^*-\hat{p})+uG(p^*-\hat{p})}\hat{\chi}_2 \end{pmatrix} \qquad (44)$$

(43) appears as system (29) (for which we have the control law $\bar{u}$) perturbed by $(p^* - \hat{p})$ and $\dot{\hat{p}}$ terms.

To counteract these perturbations, and particularly the $\dot{\hat{p}}$ terms, we use an auxiliary control v, namely we take:

$$u = \bar{u} + v \qquad (45)$$

Then, as will be clear later, it is appropriate to rewrite (43) in:

$$\begin{aligned} \dot{\hat{\chi}} = & + \phi_B(\chi, \hat{p}) + \frac{\partial \hat{\chi}}{\partial \hat{p}}\dot{\hat{p}} + \frac{\partial \hat{\chi}}{\partial x} g(\hat{q}, x)v \qquad (46) \\ & + Z_p(x, \hat{p}, \bar{u})(p^* - \hat{p}) + Z_q(x, \hat{p}, v)(p^* - \hat{q}) \end{aligned}$$

where we have introduced another p^* estimate denoted $\hat{q}$ and used (45). ϕ_B denotes:

$$\phi_B = (\phi_1\,,\, \phi_2 + \bar{u}) \qquad (47)$$

and $Z_q(x, \hat{p}, v)$ is a $2 \times n$ matrix satisfying:

$$Z_q(p^* - \hat{q}) = \begin{pmatrix} 0 \\ vL_{G(p^*-\hat{q})}\hat{\chi}_2 \end{pmatrix} \qquad (48)$$

To determine v and how $\hat{p}$ and $\hat{q}$ are updated, we consider the scalar function $W(x, \hat{p}, \hat{q})$:

$$W = \frac{1}{2}\left(\bar{V}^2 + \| \hat{p} - p^* \|^2 + \| \hat{q} - p^* \|^2\right) \qquad (49)$$

It is defined on $\mathbb{R}^2 \times \pi \times \pi$ where it satisfies:

$$W(x, \hat{p}, \hat{q}) \to \infty \quad \text{iff} \quad \| (x, \hat{p}, \hat{q}) \| \to +\infty \qquad (50)$$

Its time variation along (46) is:

$$\begin{aligned} \dot{W} = & + \bar{V}\frac{\partial V}{\partial \chi}\phi_B \qquad (51) \\ & + \left[-\dot{\hat{p}}^T + \bar{V}\frac{\partial V}{\partial \chi}Z_p\right](p^* - \hat{p}) \\ & + \left[-\dot{\hat{q}}^T + \bar{V}\frac{\partial V}{\partial \chi}Z_q\right](p^* - \hat{q}) \\ & + \bar{V}\left[\left(\frac{\partial V}{\partial \chi}\frac{\partial \hat{\chi}}{\partial \hat{p}} + \frac{\partial V}{\partial \hat{p}}\right)\dot{\hat{p}} + \frac{\partial V}{\partial \chi}\frac{\partial \hat{\chi}}{\partial x}g(\hat{q}, x)v\right] \end{aligned}$$

We notice:

1. with (35) and (40) that, for all $\hat{p}$ and $\hat{q}$ in π, we have:

$$\frac{\partial \hat{\chi}}{\partial x}g(\hat{q}, x) = (0\,,\, \lambda(x, \hat{p}, \hat{q})) \neq 0 \qquad (52)$$

2. With (36) we get:

$$\frac{\partial V}{\partial \chi}\frac{\partial \hat{\chi}}{\partial \hat{p}} = \frac{\partial V}{\partial \chi_2}\frac{\partial \hat{\chi}_2}{\partial \hat{p}} \qquad (53)$$

3. Finally, from (10) and (13) we obtain:

$$\frac{\partial V}{\partial \hat{p}} = -\frac{\partial V}{\partial \chi_2}\frac{(2m-1)\,v_0^{2m-2}}{R(v_0(\hat{\chi}_1, \hat{p}), \hat{\chi}_2)}\frac{\partial v_0}{\partial \hat{p}} \qquad (54)$$

where, thanks to (34), the right hand side term is continuous.

Knowing from Theorem 2 that $\bar{V}\frac{\partial V}{\partial \chi}\phi_B$ is negative, $\dot{W}$ will be negative if we choose $\dot{\hat{p}}$, $\dot{\hat{q}}$ and v as follows:

$$\begin{aligned} \dot{\hat{p}} &= Z_p^T\frac{\partial V}{\partial \chi}^T\bar{V} + \Delta_p \qquad (55) \\ \dot{\hat{q}} &= Z_q^T\frac{\partial V}{\partial \chi}^T\bar{V} + \Delta_q \qquad (56) \\ v &= \frac{1}{\lambda(x, \hat{p}, \hat{q})}\left[\frac{(2m-1)v_0^{2m-2}}{R(v_0(\hat{\chi}_1, \hat{p}), \hat{\chi}_2)}\frac{\partial v_0}{\partial \hat{p}} - \frac{\partial \hat{\chi}_2}{\partial \hat{p}}\right]\dot{\hat{p}} \qquad (57) \end{aligned}$$

where Δ_p and Δ_q are introduced to constrain $\hat{p}$ and $\hat{q}$ to remain in π and must satisfy:

$$\Delta_p^T(p^* - \hat{p}) \geq 0\;;\quad \Delta_q^T(p^* - \hat{q}) \geq 0 \qquad (58)$$

To define this Δ vector field, we use the Π function of assumption **H3**. Let φ be a non decreasing C^∞ function satisfying:

$$\varphi(y) = \begin{cases} = 0 & \text{if } y \leq 0 \\ > 0 & \text{if } 0 < y < 1 \\ = 1 & \text{if } 1 \leq y \end{cases} \qquad (59)$$

We define (and similarly for Δ_q):

$$\Delta_p = -\varphi(\Pi(\hat{p}))\,\text{Max}\left\{0\,,\, \frac{\frac{\partial \Pi}{\partial \hat{p}}Z_p^T\frac{\partial V}{\partial \chi}^T\bar{V}}{\|\frac{\partial \Pi}{\partial \hat{p}}\|^2}\right\}\frac{\partial \Pi}{\partial \hat{p}}^T \qquad (60)$$

Then $\Delta_p(x, \hat{p}, \hat{q})$ is at least C^0 on $\mathbb{R}^2 \times \pi \times \pi$ and satisfies (58) if:

$$\Pi(p^*) \leq 0 \qquad (61)$$

We have established:

Theorem 3
*Under assumption **H3**, if the system to be controlled satisfies (61), the adaptive controller defined by (45), (55), (56) and (57) with $\bar{u}$ and V given by Theorem 2 is well defined and:*
1. any solution with initial condition in $\mathbb{R}^2 \times \pi \times \pi$ exists and remains in this set.
2. For all positive constant c, a solution whose initial condition satisfies:

$$\begin{aligned} &\hat{p}(0) \in \pi \quad \hat{q}(0) \in \pi \\ &\bar{V}(x(0), \hat{p}(0))^2 + \| \hat{p}(0) - p^* \|^2 + \| \hat{q}(0) - p^* \|^2 \leq c \end{aligned} \qquad (62)$$

is bounded and converges to the compact set:

$$\left\{(x, \hat{p}, \hat{q})\,\middle|\, V(x, \hat{p}) \leq \epsilon,\, \| \hat{p} - p^* \|^2 + \| \hat{q} - p^* \|^2 \leq c\right\}$$

Moreover if $\bar{u}(x, \hat{p})$ given by Theorem 2 is continuous at $(0, \hat{p})$ for all $\hat{p}$ in π, then, by choosing $\epsilon = 0$, the x-component of the solution goes to the origin.

Proof :
Existence of solutions with initial condition in $\mathbb{R}^2 \times \pi \times \pi$ follows from:
1. the continuity on this set of the right hand side of the ordinary differential equation which defines them.
2. The property

$$\Pi(\hat{p}) = 1 \quad \Longrightarrow \quad \frac{\partial \Pi}{\partial \hat{p}} \dot{\hat{p}} \leq 0 \tag{63}$$

which yields:

$$\hat{p}(0) \in \pi \quad \Longrightarrow \quad \hat{p}(t) \in \pi \,,\, \forall t \geq 0 \tag{64}$$

and similarly for $\hat{q}$.

Moreover, by construction, we have:

$$\dot{W} \leq \bar{V} \frac{\partial V}{\partial \chi} \phi_B \tag{65}$$

With Theorem 2, it follows that W is strictly decreasing as long as $\bar{V}$ is not zero. Hence any solution satisfying (62) is bounded. The conclusion follows from the same argument as those invoked in Lasalle Theorem [5, Theorem X.1.3]. □

Comments :
1. The method we have used to derive the adaptive controller has been introduced by Parks [9] and is known as Lyapunov design.

2. The idea of counteracting the effects of $\dot{\hat{p}}$ by introducing an auxiliary control v is not new. It has been used by Middleton and Goodwin [8], Pomet and Praly [11] and more recently by Kanellakopoulos et al. [6]. This leads to an implicit equation in v, which is not always solvable. According to Pomet [10], in [6] the diffeomorphism in **H3** (with may be $\frac{\partial \chi_1}{\partial \hat{p}} \neq 0$) is chosen for ϕ_B in (46) to be independent of $\hat{p}$. Then v is computed to cancel the $\dot{\hat{p}}$ term in this equation. A necessary and sufficient condition for this to be possible is: **H2** and for all x, p

$$F(x) \in \text{Span}\left\{g(x,p)\,,\,[f,g](x,p)\right\} \tag{66}$$

In our result, this latter condition is not necessary. This follows from the fact that v is computed to make $\dot{W}$ (instead of $\dot{\hat{\chi}}$) independent of $\dot{\hat{p}}$ as proposed in [8] and [11].

3. To understand why we have introduced $\hat{q}$, let us rederive the computation without it. (46) becomes:

$$\begin{aligned} \dot{\hat{\chi}} = & + \ \phi_B(\chi, \hat{p}) + \frac{\partial \hat{\chi}}{\partial \hat{p}} \dot{\hat{p}} + v \\ & + \ Z_p(x, \hat{p}, \bar{u} + v)(p^* - \hat{p}) \end{aligned} \tag{67}$$

and (55) remains unchanged. But now, Z_p depends on v and consequently with (55), (57) is an implicit equation in v. Its solvability requires an extra-assumption [6].

References

[1] B. d'Andréa-Novel, L. Praly. *Compact set stabilization for nonlinear systems. Proc. of 28th IEEE CDC, 1989.*

[2] Z. Artstein. *Stabilization with relaxed controls. Nonlinear Analysis, TMA7, 1983, 1163-1173.*

[3] R.W. Brockett. *Asymptotic stability and feedback stabilization, R.W. Brockett, R.S. Millman, H.J. Sussmann Eds., Differential Geometric Control Theory, Birkhauser, Boston, MA 1983, 181-191.*

[4] W.P. Dayawansa, C.F. Martin. *Asymptotic stabilization of two dimensional real-analytic systems, Systems and Control Letters 12, April 1989.*

[5] J.K. Hale. *Ordinary differential equations, Krieger publishing company, 1980.*

[6] I. Kanellakopoulos, P.V. Kokotovic, R. Marino, *Robustness of adaptive nonlinear control under an extended matching condition, IFAC Symposium on Nonlinear System Design, Capri, Italy, 1989.*

[7] M. Kawski. *Stabilization of nonlinear systems in the plane, Systems and Control Letters 12, March 1989.*

[8] R.H. Middleton, G.C. Goodwin. *Adaptive computed torque control for rigid links manipulators, Systems and Control Letters, Vol. 10, January 1988.*

[9] P.C. Parks. *Lyapunov redesign of Model Reference Adaptive Control Systems, IEEE AC 11 No. 3, 1966.*

[10] J.B. Pomet. *Sur la commande adaptative des systèmes non linéaires, Thèse Ecole des Mines de Paris, 1989.*

[11] J.B. Pomet, L. Praly. *Nonlinear Adaptive Control: an estimation-based algorithm, Coll. Int. d'Automatique non linéaire, Nantes, 1988.*

[12] E.D. Sontag. *A "universal" construction of Artstein's theorem on nonlinear stabilization. Report Sycon 89-03.*