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LYAPUNOV DESIGN OF STABILIZING CONTROLLERS

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Abstract: We are interested in designing a state feedback law for an affine nonlinear system to make a compact set containing the equilibrium of interest, globally attractive. Following Artstein's theorem, the problem can be solved by designing a so called Control Lyapunov function. Such a design is proposed for a nonlinear system which has alreaday been maximally linearized by feedback and diffeomorphism.

1 Introduction

We consider the following system:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where x lives in $\mathbf{R}^n$, $n \geq 2$, u is a scalar input, f and g are at least C^1 vector-fields. The state being measured, our objective is to design a state feedback to, in some sense, stabilize one equilibrium.

Stability is a topological property. Unfortunately, necessary and sufficient conditions to guarentee it are only known for linear systems. As a consequence, most of the current nonlinear control law designs meet the stability objective only indirectly, by transformation into a linear system (see for example [6]). Moreover, in some cases, it may be very constraining or even impossible to stabilize an equilibrium point. For example the origin of the following system on $\mathbf{R}$:

$$\dot{x} \;=\; x \;+\; x^2 \;(x^2 - 1)\, u \tag{2}$$

cannot be asymptotically stabilized but only practically stabilized (see a definition in [14] for example). Even more, with a bounded u, we can only guarentee that all the solutions enter the set $[-1-\varepsilon\,,\,1+\varepsilon]$, $\varepsilon > 0$ within finite time. Fortunately, for practical engineering applications, it is sometimes sufficient to have the solutions only converging to such a neighborhood of the equilibrium. In addition, this relaxed stabilization objective avoids some of the hardest mathematical problems. From these remarks, we state the following problem:
Given a compact set $\mathcal{C} \subset \mathbf{R}^n$ containing the equilibrium point of interest as an interior point, design a continuous control law such that for any initial condition, the corresponding solution of (1) enters $\mathcal{C}$ within finite time and thereafter remains in $\mathcal{C}$.
Note that by requiring the control law to be only continuous, we allow non unique solutions.

To solve this problem, inspired by Lyapunov's second method, we design the control law for the time derivative of a scalar function $h(x)$ to be strictly negative outside the desired compact set $\mathcal{C}$. For this design to meet our objective, h must be appropriately chosen and have some particular properties. Namely:

Definition 1 (P-function) *A function h from $\mathbf{R}^n$ to $\mathbf{R}_+$ (with $n \geq 2$) is said to be a P-function if the following properties are satisfied:*

1. *h is C^1.*
2. *$h(x) \to \infty$ iff $\| x \| \to +\infty$.*

From this definition, a P-function is a proper function, i.e. the preimage of a compact set is also compact.

In the forthcoming, we will use the

Notations :
Let $\Phi_t(x)$ be a solution of (1) starting from x at time $t = 0$, we denote:

$$\dot{h}(x) = \frac{d}{dt}h(\Phi_t(x))|_{t=0} = L_{f+gu}h\,(x) \tag{3}$$

In general, $K \subset \mathbf{R}^n$ is a compact set containing the origin as an interior point. ∂K is the boundary of K, $\complement_K$ is its complement set and $\overset{o}{K}$ its interior. With h be a P-function, for any compact set K, we denote K_h the compact set:

$$K_h \;=\; \{x \in \mathbf{R}^n \,/\, h(x) \leq \sup_{y \in K} h(y)\}. \tag{4}$$

$L_f h$ is the Lie derivative of h along f. Finally Z_h is the set:

$$Z_h = \{x \in \mathbf{R}^n / L_g h(x) = 0\} \tag{5}$$

The paper is organized as follows:
In section 2 we show that, thanks to Artstein 's Theorem, the solution to our problem can be reduced to designing a so called Control Lyapunov function (clf) and we observe that the complexity of this design depends mostly on the control vector field g.
In section 3, we propose such a design for a system which has already been maximally linearized by feedback and diffeomorphism and give illustrative examples.
Finally section 4 is our conclusion.

Although to simplify this paper, we deal only with globally attractive compact set, our results can also be interpreted locally. Indeed, if their assumptions are satisfied only in the open set $\{x \,|\, h(x) < M \neq 0\}$, their conclusions apply to solutions whose initial conditions are in this set.

2 The Control Lyapunov function approach

Let h be a P-function, its time derivative along the solutions of (1) is:

$$\dot{h}(x) \;=\; L_f h(x) \;+\; u\, L_g h(x) \tag{6}$$

Our stabilization problem is solved if we can assign some strictly negative value to $\dot{h}(x)$. Clearly this is possible at all points x where $L_g h(x)$ is not zero. And the whole difficulty is to deal with the points where $L_g h(x)$ is zero, i.e. the points in Z_h (see (5)). This justifies the following

Definition 2 (clf) *Let $K \subset \mathbf{R}^n$ be a compact set. We call Control Lyapunov function associated with K for system (1) and denote clf a P-function h which satisfies:*

$$Z_h \subset \{x \in \mathbf{R}^n / L_f h(x) < 0\} \cup \overset{o}{K} \tag{7}$$

It follows from this definition that a clf associated with K can be arbitrarily modified in $\overset{o}{K}$ without loosing its clf property.

We have the following extension of Artstein's Theorem [1]:

Theorem 1 *Suppose K is a compact set with a non empty interior and the vector fields f and g are C^{q+1}, $q \geq 0$. If h is a clf associated with K and if h is C^{q+1} on $\complement_{\widehat{K}}$, the complement of $\widehat{K}$, a compact set contained in $\overset{o}{K}$, then there exists $u(x)$, a C^q control law such that, for each finite initial condition, the corresponding solution of (1) enters the compact set K within finite time and thereafter remains in the compact set K_h.*

Explicit expressions outside K for the control law mentioned in this statement are known. Denoting:

$$X = L_g h(x)\,,\ Y = L_f h(x) \tag{8}$$

Sontag has proposed [11]:

$$u = \begin{cases} 0 & \text{if } Y < 0 \text{ and } X = 0 \\ -\dfrac{Y + \sqrt{Y^2 + X^4}}{X} & \text{if not} \end{cases} \tag{9}$$

We propose (closely related to the control law proposed by Tsinias [14]):

$$u = \begin{cases} -\dfrac{X\,(k\,a + Y)}{a + X^2} & \text{if } Y \leq X^2 \\ -\dfrac{Y(k\,a + Y)}{X\,(a + Y)} & \text{if } Y > k\,X^2 \\ \left\{ \begin{array}{l} -\,\beta(Y, X^2)\,\dfrac{X\,(k\,a + Y)}{a + X^2} \\ -\,(1 - \beta(Y, X^2))\,\dfrac{Y\,(k\,a + Y)}{X(a + Y)} \end{array} \right\} & \text{if not} \end{cases} \tag{10}$$

where $a(x)$ is a C^0 scalar function bounded from below by a strictly positive constant, k is a real number strictly larger than 1 and β is an at least C^0 bump function on $\mathbf{R} \times \mathbf{R}_+ - \{(0,0)\}$:

$$\beta(z_1, z_2^2) = \begin{cases} 1 & \text{if } z_1 \leq z_2^2 \\ 0 & \text{if } z_1 > k\,z_2^2\ (>\ z_2^2) \\ \in (0\,,1) & \text{if not} \end{cases} \tag{11}$$

Moreover these functions u are at least continuous at the origin if the clf h satisfies the so called "small control property", namely (see [1,11]):

Definition 3 (scp) *A P-function h is said to satisfy the small control property (scp) if:*

1. *it is a clf associated with any compact set containing the origin as an interior point i.e.*

$$Z_h \subset \{x \in \mathbf{R}^n / L_f h(x) < 0\} \cup \{0\}, \tag{12}$$

2. *for all strictly positive ε, we can find a strictly positive δ such that, for all x, $\| x \| < \delta$, $x \neq 0$, there exists u, $|u| < \varepsilon$, satisfying:*

$$L_f h(x) + u L_g h(x) < 0\,. \tag{13}$$

In this case the origin is globally continuously asymptotically stabilized.

Thanks to Artstein's theorem, the solution of the compact set stabilization problem is reduced to designing a clf h associated with a set K such that the corresponding compact set K_h (4) is contained in the desired set C.

A first question to be addressed for this design is the possibility of finding a P-function h for which the associated set Z_h is contained in C. Noting that $L_g h$ is nothing but the time derivative of h along the solutions of:

$$\dot{x} = g(x) \tag{14}$$

the compactness property of Z_h only depends on g. We then have the following result:

Proposition 1 (Appendix A.1) *1- If for each compact set in $\mathbf{R}^n$ there exists a solution of (14) whose closure does not intersect this set, then there is no P-function h such that the corresponding Z_h is compact.*
2- Suppose the desired compact set C is a manifold with boundary and g is transversal to the boundary ∂C. If, may be after changing g in $-g$, all the solutions of (14) enter C within finite time and thereafter remain in C, then there exists a P-function h such that $L_g h(x)$ is strictly negative for all x not in C. Consequently h is a clf associated with C and $C_h = C$.

Example 1 : Let:

$$g(x) = Ax. \tag{15}$$

If $A + A'$ is positive (resp. negative) definite, the origin is a global exponential repellor (resp. attractor) for (14) and therefore point 2 of Proposition 1 applies.

The simplest example about point 1 is the case where g is a constant vector-field. Proposition 1 tells us that, in such a case, for each P-function h, the corresponding set Z_h is not compact. In particular this is the case of a single input linear system on $\mathbf{R}^n$:

$$\dot{x} = Fx + Gu \tag{16}$$

However, if this system is stabilizable, there exists a matrix C such that the eigenvalues of $F - GC$ are all in the open left half complex plane. Consequently, for any symmetric positive definite matrix Q, there exists a symmetric positive definite matrix P such that for all non zero x:

$$x'PFx - x'PGCx = -x'Qx < 0 \tag{17}$$

Hence, by choosing the P-function h as:

$$h(x) = x'Px \tag{18}$$

we get:

$$Z_h = \{x \in \mathbf{R}^n \,/\, x'PG = 0\} \,,\; L_f h(x) = x'PFx. \tag{19}$$

If x belongs to Z_h and is not the origin we have:

$$L_f h(x) \;=\; x'PFx \;=\; -x'Qx \;<\; 0 \tag{20}$$

It follows that the stabilizability assumption is sufficient for linear systems to find a *clf* satisfying *scp*.

For a nonlinear system, if its tangent linearization at the origin is stabilizable, we may choose (18) for h. In such a case, h is a *clf* satisfying *scp*, at least on the neighborhood of the origin. More generally, if there exists a state feedback making the origin asymptotically stable, by applying converse Lyapunov theorems (see [5, Theorem 4.2.]), we know the existence of a P-function h for which $\dot{h}$ can be made locally strictly negative for all non zero x. One idea is to try this P-function to solve the global compact set stabilization problem.

3 Design of a *clf*

In system (1), if $g(0)$ is not zero, there exists an integer m, $m \geq 1$, such that this system is locally equivalent by static state feedback and diffeomorphism to a system of the form:

$$\begin{cases} \dot{z} &= k(y_1, z) \\ \dot{y}_1 &= y_2 \\ & \vdots \\ \dot{y}_m &= u \end{cases} \tag{21}$$

$z \in \mathbf{R}^{n-m}$ and $y \in \mathbf{R}^m$. In the following we assume that (21) makes sense globally on $\mathbf{R}^n$. In this form, g is a constant vector field. From point 1 in Proposition 1, we know that for any P-function h, the associated set Z_h is not compact. To design a *clf* in this more complex situation, we assume the knowledge of a P-function h_0 and a control law u_0 such that:

$$L_{k(u_0(z),z)} h_0(z) \;<\; 0 \tag{22}$$

For this latter assumption to be less restrictive we choose m as large as possible. An algorithm maximizing this m has been proposed by Marino (see [9]).

For the case $m = 1$, (21) is simply:

$$\begin{cases} \dot{z} &= k(y_1, z) \\ \dot{y}_1 &= u \end{cases} \tag{23}$$

Thanks to our assumption (22), to find a *clf* for this system, it is sufficient to find a P-function h such that the following two implications are satisfied:

$$y = u_0(z) \implies \frac{\partial h}{\partial y_1}(y_1, z) = 0 \tag{24}$$

$$y = u_0(z) \implies \frac{\partial h}{\partial z}(y_1, z)\, k(y_1, z) = L_{k(u_0(z),z)} h_0(z) \tag{25}$$

We could take:

$$h(y_1, z) \;=\; \int_{u_0(z)}^{y_1} (y - u_0(z))\; \mathrm{d}y \;+\; h_0(z) \tag{26}$$

If u_0 is smooth enough, this function h is an appropriate *clf* associated with the compact set $\{(y_1, z) \mid z \in K_0\,,\, |y_1 - u_0(z)| \leq \varepsilon \neq 0\}$. For example, it can be used to reestablish the Property [8, Corollary 3.2]:
If $\dot{z} = k(v, z)$ is smoothly stabilizable, the cascaded system (21) is smoothly stabilizable as well.

However, in general, the given control law u_0 is not smooth enough for h in (26) to be a P-function. A first solution to this problem would be to replace u_0 by a smoother global stabilizer. This is always possible from Artstein 's Theorem [12, Corollary sect 7]. Unfortunately, for engineering applications, this regularization is usually unpractical. Another solution is to replace, in (26), $y - u_0(z)$ by a so called "desingularizing" function $\varphi(y_1, z)$. We have:

Lemma 1 (Appendix A.2)
Suppose $K_0 \subset \mathbf{R}^{n-1}$ is a compact set containing the origin as an interior point of $\mathbf{R}^{n-1}$.
If there exist a P-function h_0 and a C^0 scalar function $u_0(z)$ such that:

1. *Maybe after a C^1 change of the z-coordinates in $\mathbf{R}^{n-1}$, for all i in $\{1, \ldots, n-1\}$ and all $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n-1})$ in $\mathbf{R}^{n-1}$, the real numbers z_i where $\frac{\partial u_0}{\partial z_i}(z_1, \ldots, z_i, \ldots, z_{n-1})$ is not defined are isolated in $\mathbf{R}$,*
2. *There exists a scalar C^0 function $\varphi(y_1, z)$ such that:*

$$\varphi(y_1, z) \;=\; 0 \quad\iff\quad y_1 \;=\; u_0(z) \tag{27}$$

$\Phi(y_1, z)$ is C^1 in $\mathbf{R}^n$ and, for all z in $\mathbf{R}^{n-1}$, $\Phi(y_1, z) \to +\infty$ if $|y_1| \to +\infty$, where Φ denotes the primitive:

$$\Phi(y_1, z) \;=\; \int_0^{y_1} \varphi(y, z)\, dy \,, \tag{28}$$

3. *For all z not in the interior of K_0 we have:*

$$L_{k(u_0(z),z)} h_0(z) \;<\; 0 \tag{29}$$

then, with α a real number larger or equal to 1,

$$h_1(y_1, z) \;=\; \Phi(y_1, z) \;-\; \Phi(u_0(z), z) \;+\; h_0(z)^\alpha \tag{30}$$

is a clf for system (23) associated with the compact set:

$$K_1(\varepsilon_1) \;=\; \{(y_1, z) \mid z \in K_0\,,\, |y_1 - u_0(z)| \;\leq\; \varepsilon_1\} \tag{31}$$

with ε_1 any strictly positive constant.

This Lemma provides a *clf* associated with $K_1(\varepsilon_1)$. Hence we have answered the problem only if there exists $\varepsilon_1 > 0$ such that $K_1(\varepsilon_1)_h$ is contained in the desired compact set C. This is an extra constraint on h_0, u_0 and K_0. If this constraint is satisfied, we have reduced the design of h_1 to looking for a desingularizing function φ. If there exists a C^1 scalar function s and a strictly positive integer p such that $(u_0(z) - s(z))^{2p-1}$ is C^1, we may choose:

$$\varphi(y_1, z) \;=\; (y_1 - s(z))^{2p-1} \;-\; (u_0(z) - s(z))^{2p-1} \tag{32}$$

Taking h_1 in place of h in (9) or (10), we obtain appropriate control laws for stabilizing the compact set $K_1(\varepsilon_1)_h$. However, when it has a continuous extension at the zeros of $\frac{\partial\Phi}{\partial y}$, the more adapted state feedback:

$$\begin{aligned} u_1(y_1,z) = \{ & \left(\frac{\partial\Phi}{\partial z}(y_1,z) - \frac{\partial\Phi}{\partial z}(u_0(z),z)\right) k(y_1,z) \\ & - \left(\frac{\partial\Phi}{\partial y}(y_1,z) - \frac{\partial\Phi}{\partial y}(u_0(z),z)\right) k(y_1,z) \\ & - h_0(z)^{\alpha-1}\frac{\partial h_0}{\partial z}(y_1,z)\,(k(y_1,z) - k(u_0(z),z)) \\ & + \left(\frac{\partial\Phi}{\partial y}(y_1,z)\right)^2 \} \times \left(\frac{\partial\Phi}{\partial y}(y_1,z)\right)^{-1} \end{aligned} \tag{33}$$

leads usually to simpler expressions.

Example 2 : Consider the following planar system [7]:

$$\left\{ \begin{array}{l} \dot z = z - y^3 \\ \dot y = u \end{array} \right. \tag{34}$$

The tangent linearization at the origin is not stabilizable and therefore there is no C_1 control law asymptotically stabilizing this point. However, Kawski has proposed a general method for small-time locally controllable systems in the plane. In this case, it gives a locally Hölder control law guarenteeing asymptotic stabilization.
To apply our method, we check that assumptions of Lemma 1 are satisfied when we choose in (32):

$$\left. \begin{array}{c} h_0(z) = \beta\, z^2 \ , \ u_0(z) = [(1+c)\, z]^{\frac{1}{3}} \ , \ s(z) = 0 \\ p \geq 2 \ , \ \alpha \geq 1 \ , \ \beta > 0 \end{array} \right\} \tag{35}$$

with c a strictly positive real number. A *clf* is:

$$h(y,z) = \frac{y^{2p}}{2p} - y\,[(1+c)\,z]^{\frac{2p-1}{3}} + \frac{2p-1}{2p}\,[(1+c)\,z]^{\frac{2p}{3}} + \beta\,\frac{z^{2\alpha}}{2\alpha} \tag{36}$$

We note that to satisfy *scp* we must have:

$$2\alpha > \frac{2p-1}{3} \tag{37}$$

Moreover, choosing $\alpha = p/3$ and $p \geq 3$, by homogeneity, *scp* is satisfied and the larger p is, the smoother on $\mathbf{R}^2 - \{(0,0)\}$ a stabilizing control law can be designed. With this choice the origin is made continuously asymptotically stable.

For the case m larger than 1, we proceed by induction. Starting from $i = 1$, we obtain recursively a compact set K_i and a P-function h_i which is a *clf* associated with K_i for the system:

$$\left\{ \begin{array}{l} \dot z = k(y_1,z) \\ \dot y_1 = y_2 \\ \quad \vdots \\ \dot y_i = u_i \end{array} \right. \tag{38}$$

Artstein's Theorem 1 gives us a control law $u_i(z,y_1,\ldots,y_{i-1},y_i)$ to continue the recursion. At each step i, the difficulty is to find a desingularizing function to go around the roughness of u_{i-1}.

The above arguments show that, to solve the problem of controlling system (21), it is sufficient to find appropriate *clf* h_0 and function u_0 satisfying (29) for all z outside a compact set $K_0 \subset \mathbf{R}^{n-m}$. Such a result has been pointed out many times in the litterature (see [9, Theorem 5] or [8, Corollary 3.2] for example). For system (23) in $\mathbf{R}^2$, existence of h_0 and u_0 satisfying (29) (point 3 in Lemma 1) is necessary:

Lemma 2 *[[13, Lemma 3.1], Appendix A.3]*
A necessary condition for the existence of a continuous control law u making all the solutions of:

$$\left\{ \begin{array}{l} \dot z = k(y,z) \\ \dot y = u \end{array} \right. \tag{39}$$

enter a connected compact set $K \subset \mathbf{R}^2$, *containing* $(0,0)$, *within finite time is:*
for every P-function h_0 *with no stationary point outside the set* $\{z \mid \exists y : (z,y) \in K\}$, *and for every* z *outside this set, there exists* y *such that* $\frac{dh_0}{dz}(z)\,k(y,z)$ *is strictly negative.*

What may not be necessary in Lemma 1 are the smoothness assumptions in points 1 and 2. In particular, Dayawansa and Martin [4] have established that, if k is a real analytic function, then point 3 (more precisely, an even weaker condition than point 3) is necessary and sufficient for the existence of a locally asymptotically stabilizing C^0 control law.

4 Conclusion

For an affine nonlinear system, we have studied the problem of making a given compact set globally attractive. Our solution consists in assigning the dynamical behavior of a Lyapunov function. The resulting control law has singularities. If all the integral curves of the control field intersect the given compact set, the control law can be chosen such that its singularities are in this set. In the case of non compact singularities, Artstein's Theorem tells us that it is necessary and sufficient to find a Lyapunov function such that the open loop dynamic makes this function decrease at the singular points. In particular, this allows us to design a control law likely to enlarge the attractivity domain of a C^1 stabilizable equlibrium point.

For systems which are already maximally linearized by feedback and diffeomorphism, we design a state feedback assuming the knowledge of a control law stabilizing the equilibrium of the remaining nonlinear subsystem. In particular, for planar systems, this gives sufficient conditions and necessary conditions for a compact set containing the equilibrium as an interior point to be stabilized.

A Appendix

A.1 Proof of Proposition 1

Point 1: Let h be a P-function and g a C^1 vector-field on $\mathbf{R}^n$. Let us first prove that the closure of each solution of (14) intersects Z_h. We denote $\Phi_t(x_0)$ a solution of (14) starting from x_0 at time $t = 0$. Let I be its maximal interval of definition. To obtain a contradiction we suppose that $L_g h$ has no zero in the closure of $\{\Phi_t(x_0) \,|\, t \in I\}$, say $L_g h$ is positive. Since:

$$\dot{h}(x) = L_g h(x) \qquad (40)$$

$h(\Phi_t(x_0))$ is increasing in t, i.e.:

$$0 \le h(\Phi_t(x_0)) \le h(x_0) \quad , \quad \forall t \in I, t \le 0 \qquad (41)$$

Because h is proper, $\Phi_t(x_0)$ belongs to a compact set. We can find a positive constant ϵ such that:

$$\dot{h}(\Phi_t(x_0)) \ge \epsilon \quad , \quad \forall t \in I, t \le 0 \qquad (42)$$

It follows:

$$0 \le h(\Phi_t(x_0)) \le \epsilon t + h(x_0) \,,\ \forall t \in I, t \le 0 \qquad (43)$$

which implies

$$t \ge -\frac{h(x_0)}{\epsilon} \,,\ \forall t \in I \qquad (44)$$

Consequently, I has a lower bound T. Since I is maximal and h is a P-function, we get:

$$\lim_{t \to T_+} h(\Phi_t(x_0)) = +\infty \qquad (45)$$

which contradicts (41).
Knowing now that, for any P-function h, the closures of all the solutions of (14) intersect the corresponding set Z_h, the assumption implies that Z_h cannot be compact.

Point 2: We may apply a converse Lyapunov Theorem to get a P-function h. For example, we can choose $h(x)$ as the time when $\bar{\Phi}_t(x)$ enters $\mathcal{C}$, with $\bar{\Phi}_t(x)$ the solutions of:

$$\dot{x} = \frac{g(x)}{\|g(x)\|} \qquad (46)$$

This system is well defined outside $\mathcal{C}$ since all the solutions of (14) enter $\mathcal{C}$ within a finite time imply that g is not zero outside $\mathcal{C}$. In [10], we have established that the proposed function h is a P-function and we have:

$$L_g h(x) = \|g(x)\| \frac{dh}{dt}(\bar{\Phi}_t(x))_{|t=0} = -\|g(x)\| < 0. \quad \square \qquad (47)$$

A.2 Proof of Lemma 1

First step: h_1 is a P-function

1.1: h_1 is C^1: To show that h_1 in (30) is C^1 it is sufficient to prove that $\Phi(u_0(z), z)$ is C^1. Let us denote:

$$\Psi(z) = \Phi(u_0(z), z) \qquad (48)$$

For all $z = (z_1, \dots, z_{n-1})$ where $\frac{\partial u_0}{\partial z_i}(z)$ exists, we have:

$$\frac{\partial \Psi}{\partial z_i}(z) = \frac{\partial \Phi}{\partial y}(u_0(z), z)\frac{\partial u_0}{\partial z_i}(z) + \frac{\partial \Phi}{\partial z_i}(u_0(z), z) \qquad (49)$$

Since the definition of Φ implies:

$$\frac{\partial \Phi}{\partial y}(u_0(z), z) = \varphi(u_0(z), z) = 0 \qquad (50)$$

for all $z = (z_1, \dots, z_{n-1})$ where $\frac{\partial u_0}{\partial z_i}(z)$ exists, we obtain:

$$\frac{\partial \Psi}{\partial z_i}(z) = \frac{\partial \Phi}{\partial z_i}(u_0(z), z) \qquad (51)$$

Now, for $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{n-1})$ fixed at any arbitrary value in $\mathbf{R}^{n-2}$ and for any i, $\Psi(z)$ and $\frac{\partial \Phi}{\partial z_i}(u_0(z), z)$ are C^0 functions of z_i and (51) is satisfied maybe except at isolated points of $\mathbf{R}$. It follows from [3, Proposition I.2.6] that $\frac{\partial \Psi}{\partial z_i}(z)$ is defined and continuous on whole $\mathbf{R}^{n-1}$. Since this holds for all i, Ψ is C^1 (see [2, Statement II.1.3]).

1.2: h_1 is positive and proper: It is sufficient to show that for every y_1 and z we have:

$$\Phi(y_1, z) - \Phi(u_0(z), z) \ge 0 \qquad (52)$$

Indeed if this is the case, when $h_1(y_1, z)$ is bounded, $h_0(z)$ is bounded. Moreover h_0 being proper, the same holds for z and consequently for $\Phi(u_0(z), z)$, $\Phi(y_1, z)$ and y.
To prove (52) let us study:

$$\Phi(y_1, z) - \Phi(u_0(z), z) \qquad (53)$$

as a function of y_1 with z fixed. Its derivative can be written:

$$\frac{\partial \Phi}{\partial y}(y_1, z) = \varphi(y_1, z) \qquad (54)$$

which is zero iff y_1 is equal to $u_0(z)$. Since (53) is continuous, positive at infinity and zero if $y_1 = u_0(z)$, (52) is satisfied.

Second step: h_1 is a *clf* associated with K_1

By construction, we have:

$$L_g h_1(y_1, z) = \varphi(y_1, z) = 0 \quad \text{iff} \quad y_1 = u_0(z) \qquad (55)$$

Hence, if z belongs to K_0 but $|y_1 - u_0(z)|$ is larger than $\varepsilon_1 \ne 0$, $L_g h_1(y_1, z)$ is not zero. And, when $L_g h_1(y_1, z)$ is zero, we obtain with (51):

$$L_f h_1(y_1, z) = \alpha h_0^{\alpha-1}(z) L_{k(u_0(z), z)} h_0(z) \qquad (56)$$

which is strictly negative for all z not in the interior of K_0.

A.3 Proof of Lemma 2

Since u is continuous, the solutions $(z(t), y(t))$ of (39) exist and are C^1 for any initial conditions. Hence, by assumption, for any initial condition, there exists a C^1 time function y such that the corresponding solution of:

$$\dot{z} = k(y(t), z) \qquad (57)$$

enters the compact set $\{z \,|\, \exists y : (z, y) \in K\}$, within finite time. Following a trivial extension of [13, Lemma 3.1], this implies:
for every z outside the set $\{z \,|\, \exists y : (z, y) \in K\}$, there exists y such that $z\, k(y, z)$ is strictly negative. To conclude, we note that, h_0 being a P-function with no stationary point outside the connected set $\{z \,|\, \exists y : (z, y) \in K\}$, which contains 0, z and $\frac{dh_0}{dz}(z)$ have same sign outside this set. $\square$

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