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PERIODIC SOLUTIONS IN ADAPTIVE SYSTEMS: THE REGULAR CASE

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Abstract. We study adaptive systems in presence of small periodic forcing terms (reference signals + noise) and without any assumption on the plant order. Poincaré method is applied. A necessary condition for existence of periodic solutions is written in terms of existence of zeros for a bifurcation equation. This condition is sufficient if these zeros are non degenerate. In this latter case, called the regular case, a sufficient condition for (un)stabilty is also given. An example illustrates these results.

Keywords. Adaptive systems, Nonlinear systems, Stability, Time-varying systems, Discrete time systems

1. INTRODUCTION

Background: It is now well established that boundedness of all the solutions of adaptive linear schemes is qualitatively as a robust property as exponential stability is for linear feedback systems (Praly, 1982, 1983, 1986). But "bounded" does not imply "satisfactory". A more qualitative study of solutions of interest is needed. This justifies the current attention paid to the study of the local properties of adaptive linear systems.

Kosut and Anderson (1984) have proposed to linearize the system about a time function called the tuned solution and chosen in order to simplify the study the linearized system. Riedle and Kokotovic (1985) have completed this approach in the case of slow adaptation. Using averaging theory, they have derived sufficient conditions for stability and unstability of the linearized system (see also (Kokotovic et al. (1985), Riedle et al. (1986)). However a technical difficulty to extend this result to the truely nonlinear systems stands in the choice of the tuned solution mentionned above: the simplicity of the linearized system and the fact that the tuned solution is actually a solution of the nonlinear system are incompatible in general. In the limiting case where we have an approximation of a solution, the results of stability (but not unstability) is completed invoking a total stability argument as proposed by Anderson et al. (1986). Following this idea and under the restrictive assumption that the tuned solution is exactly a solution, assumption proposed by Riedle et al. (1985), a robust stability result for the nonlinear system has been derived by Praly, Rhode (1985).

However, using a different approach, Riedle and Kokotovic (1986) for the continuous time case and Praly (1985) for the discrete time case have obtained more satisfying results. The averaging theory is applied to a reduced order nonlinear system instead of the linearized system as above. This requires a coordinate transformation based on the existence of a locally attractive integral manifold. Introducing stationnarity assumptions from the begining, similar results are obtained using two time scale averaging technique as proposed by Ljung (1977) (see also Bodson et al. (1986)).

In this paper, we complete these results for the simpler case of a periodic forcing term. A necessary condition and a sufficient condition for existence and a sufficient condition for stability of a periodic solution is derived using the Poincaré method (see chap. VIII.5 in (Lefschetz, 1977), for example).

Problem formulation: Consider a time invariant finite dimensional linear system with state Y, input u, extraneous additive disturbance d, described by:

$$Y(k+1) = F\ Y(k) + G\ u(k) + H\ d(k) \tag{1.1}$$

in closed loop with a parameterized state feedback and reference signal r:

$$u(k) = -K(\theta(k))\ Y(k) + J(\theta(k))\ r(k) \tag{1.2}$$

whose parameters are adapted by:

$$\theta(k+1) = \theta(k) + L(Y(k),\theta(k),r(k),\lambda(k)) \tag{1.3}$$

The L function characterizes a family of adaptation laws indexed by $\lambda(k)$. The closed loop system can readily be written in:

$$\begin{aligned} Y(k+1) &= A(\theta(k))\ Y(k) &+ \quad & B(\theta(k))\ w(k) \\ \theta(k+1) &= \theta(k) &+ \quad & C(Y(k),\theta(k),w(k),\lambda(k)) \end{aligned} \tag{1.4}$$

with $w=(r,d)$. It turns out that most adaptive controllers in feedback with a linear time invariant system with arbitrary order and extraneous additive disturbance satisfy (1.4). For example if a least square algorithm with forgetting factor is used, the form (1.4) is obtained by incorporating the columns of the covariance matrix in the θ-vector. If an indirect pole placement were used, the function $A(\theta)$ would incorporate the operation of solving the linear system given by the Bezout identity... More interestingly, in all these cases, the C function satisfies (at least locally):

$$C(\sqrt{\epsilon}X, \theta, \sqrt{\epsilon}v, \lambda) = \epsilon\, C(X, \theta, v, \epsilon\lambda), \ \forall \epsilon \geq 0 \tag{1.5}$$

Consequently, if the forcing term w satisfies:

$$\| w(k) \| \leq \sqrt{\epsilon}, \ \forall k \tag{1.6}$$

the following transformation:

$$\sqrt{\epsilon}\, X = Y, \ \sqrt{\epsilon}\, v = w, \ \gamma = \epsilon \lambda \tag{1.7}$$

leads to:

$$\begin{aligned} X(k+1) &= A(\theta(k))\, X(k) + B(\theta(k))\, v(k) \\ \theta(k+1) &= \theta(k) + \epsilon\, C(X(k), \theta(k), v(k), \gamma(k)) \end{aligned} \tag{1.8}$$

The smaller ϵ is (i.e. the forcing term is), the slower θ is adapted. In this circumstance, the actual system (1.8) is a small perturbation of the following system, called frozen system:

$$\begin{aligned} X_f(k+1) &= A(\theta_f(k))\, X_f(k) + B(\theta_f(k))\, v(k) \\ \theta_f(k+1) &= \theta_f(k) = \theta_f(0) \end{aligned} \tag{1.9}$$

This system can be considered as a family of linear systems indexed by $\theta_f(0)$. In particular, for v bounded, this system has a unique solution $(X_f(\theta_f(0),k), \theta_f(0))$, bounded on $(-\infty,\infty)$ associated with each $\theta_f(0)$ for which $A(\theta_f(0))$ has no eigenvalue on the unit circle. Moreover, this solution is periodic whenever v is periodic.

In the following, our problem is to study under which conditions this property holds for the actual system (1.8) with ϵ sufficiently small (i.e. small forcing term or forced slow adaptation).

The following assumptions will be used:
A1: v is N-periodic (i.e. periodic with period N).
A2: There exists an open set Γ such that $A(\theta)$, $B(\theta)$ are continuously differentiable on Γ.
A3: $C(X, \theta, v, \gamma)$ is continuously differentiable in X, θ, v, γ.
Assumption A1 requires that both the reference and the disturbance are N-periodic. It is motivated by our desire of investigating the local properties of the adaptive system around a particular solution. For ease of interpretation of this analysis, this solution should be stationnary. In this context, the periodic case is the simplest. Note however that by a total stability argument (Theorem 1.1 of Anderson et al. (1986) for example), the results extend to any v which can be approximated by a periodic sequence if the corresponding approximating solution is hyperbolic.
The regularity assumptions A2, A3 are generally satisfied. However behind this restriction to the set Γ is the problem of the leading coefficient going to zero in MRAC schemes or the identified model stabilizability in indirect pole placement schemes.

In section 2, we study how existence of M-periodic solutions of (1.8), for any M, is related to the existence of θ_f satisfying:

$$E(\theta_f) = \sum_{k=0}^{M-1} C(X_f(\theta_f, k), \theta_f, v(k), 0) = 0 \tag{1.10}$$

In section 3, we show how the stability of these particular solutions is given by the eigenvalues of $A(\theta_f)$ and $\frac{dE}{d\theta}(\theta_f)$. Each of these sections is illustrated in the example of section 4.

To simplify the following expressions, we omit the arguments v, γ in the C function since they are unimportant.

2. EXISTENCE OF A PERIODIC SOLUTION

Let us first consider the frozen system (1.9). Two cases are to be considered:
i) The regular case: ψ is such that $A(\psi)^N - I$ is non singular. Then (1.9) has a unique N-periodic solution $(X_{fN}(\psi,k), \theta_{fN}(\psi,k))$ satisfying:

$$\begin{aligned} X_{fN}(\psi,k) &= A(\psi)^N\, X_{fN}(\psi,k) + \sum_{j=0}^{N-1} A(\psi)^{N-j-1} B(v) v(j+k) \\ \theta_{fN}(\psi,k) &= \psi \end{aligned} \tag{2.1}$$

Notice that if ψ is in Γ, $X_{fN}(\psi,k)$ is (locally) continuously differentiable in ψ, uniformly in k.
ii) The singular case: ψ is such that $A(\psi)^M - I$ is singular, with M multiple of N, but $A(\psi)$ has no eigenvalue in the spectrum of v. Then (1.9) has a linear manifold of M-periodic solutions $(X_{fM}(\psi,k), \theta_{fM}(\psi,k))$, spanned by the kernel of $A(\psi)^M - I$.

In these two cases, the initial conditions $(X_{fM}(\psi,0), \theta_{fM}(\psi,0))$ of these M-periodic solutions are fixed points of the map transforming the initial condition $(X_f(0), \theta_f(0))$ into the values at time M $(X_f(M), \theta_f(M))$, using (1.9) recursively. Similarly, let T_M be this so called M-advance map associated with the actual system (1.8), i.e.:

$$(X(M), \theta(M)) = T_M\left(X(0), \theta(0)\right) \tag{2.2}$$

If $(X(0),\theta(0))$ is the initial condition of an M-periodic solution of (1.8), it is a fixed point of T_M. The converse is true when M is a multiple of N:

Lemma 2.1: *(Assertion 1 of th.3.28 of Arnold (1978)): $(X(k),\theta(k))$ is an M-periodic solution of (1.8), with M multiple of N, if and only if its initial condition $(X(0),\theta(0))$ is a fixed point of T_M.*

Instead of looking for fixed points of T_M, we can equivalently (for $\epsilon \neq 0$) look for zeros of $Z_M(X,\theta,\epsilon)$ defined by:

$$Z_M = \begin{pmatrix} Z_{Mx} \\ Z_{M\theta} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \frac{I}{\epsilon} \end{pmatrix} \circ (T_M - I) \tag{2.3}$$

i.e. by:

$$Z_M(X(0),\theta(0),\epsilon) = \begin{pmatrix} X(M) - X(0) \\ \sum_{k=0}^{M-1} C(X(k),\theta(k)) \end{pmatrix} \tag{2.4}$$

Let $(X_M(0,\epsilon),\theta_M(0,\epsilon),\epsilon)$ denote a zero of Z_M. By definition of Z_M, $(X_M(0,\epsilon),\theta_M(0,\epsilon))$ is an initial condition of an M-periodic solution of (1.8) we denote by $(X_M(k,\epsilon),\theta_M(k,\epsilon))$. Let (X^*,ψ^*) be one of the accumulation points for ϵ going to zero (if it exists) of the initial conditions $(X_M(0,\epsilon),\theta_M(0,\epsilon))$, i.e. there exists a sequence of ϵ converging to zero such that the corresponding sequence $(X_M(0,\epsilon),\theta_M(0,\epsilon))$ converges to (X^*,ψ^*). We have the following necessary condition for existence:

Theorem 2.1: *Under assumptions A1 to A3, for any integer M, if ψ^*, as defined above, is in Γ, there exists an M-periodic solution of the frozen system $(X_{fM}(\psi^*,k),\theta_{fM}(\psi^*,k))$, with initial condition (X^*,ψ^*), which is an accumulation point of $(X_M(k,\epsilon),\theta_M(k,\epsilon))$, solution of the actual system and ψ^* satisfies:*

$$\sum_{k=1}^{M} C(X_{fM}(\psi^*,k),\psi^*) = 0 \tag{2.5}$$

Remark 2.1: i) Note that $A(\psi^*)-I$ may be singular.

ii) In the theory of critical systems (see Miller, Michel (1982)), equation (2.5) is called the bifurcation equation. To obtain this equation in our case, we have first to evaluate $X_{fM}(\psi,k)$, for each ψ. For this, one can use the first comments of this section. Second, with C given by the adaptation law, we evaluate the sum of (2.5). This is usually done using Parseval's Theorem.

iii) As known from the averaging theory (see Miller, Michel (1982) for example), the bifurcation equation is also the condition for ψ^* to be an equilibrium point of the following "averaged" system obtained by replacing $X(k)$ by $X_{fM}(\theta_{av}(k),k)$ in the second equation of (1.8):

$$\theta_{av}(k+1) = \theta_{av}(k) + \epsilon\, C_{av}(\theta_{av}(k))$$
$$C_{ac}(\theta) = \frac{1}{M} \sum_{k=0}^{M-1} C(X_{fM}(\theta,k),\theta) \tag{2.6}$$

This point of view has been considered by Ljung (1977), Bodson et al. (1986).

iv) The problem of existence of solutions ψ^* is of main interest. In the case of model reference adaptive controllers, Pomet (1986) has established that, generically, the corresponding bifurcation equation has solutions (see also (Riedle, 1986)).

v) Theorem 2.1 gives us all the possible accumulation sequences of M-periodic solutions of (1.8) as ϵ goes to zero. In particular, if the bifurcation equation has no solution ψ^*, then either (1.8) has no M-periodic solution for ϵ in a neigborhoud of zero or $\theta_M(0,\epsilon)$ has no accumulation points in the regularity domain Γ.

In section 4, we propose an example to illustrate the use of this Theorem.

Proof: < Since ψ^* is in Γ, we can use assumptions A2, A3 for ϵ in a neigborhood of zero. The solution $(X_M(k,\epsilon),\theta_M(k,\epsilon))$ with initial condition $(X_M(0,\epsilon),\theta_M(0,\epsilon))$ depends continuously (at least for finite k) on the parameter ϵ and this initial condition. Consider a sequence of ϵ converging to zero such that $(X_M(0,\epsilon),\theta_M(0,\epsilon))$ converges to (X^*,ψ^*). A limit $(X(k,0),\theta(k,0))$ exists also for all k and by continuity and choice of $(X_M(0,\epsilon),\theta_M(0,\epsilon))$, it is an M-periodic solution of the frozen system (1.9), i.e.:

$$X_M(k,0) = X_{fM}(\psi^*,k)\ ,\ \ \theta_M(k,0) = \psi^*\ ,\ \ \forall k \tag{2.7}$$

The conclusion follows with continuity of C. >

To obtain a sufficient condition in the regular case and M=N, we introduce the

Bifurcation Equation assumption: The functions $A(\theta)$, $B(\theta)$, $C(X,\theta)$ and the sequence v are such that there exists a vector ψ^*, belonging to Γ and satisfying :

$$E(\psi^*) = \sum_{k=0}^{N-1} C(X_{fN}(\psi^*,k),\psi^*) = 0 \tag{2.8}$$

with ψ^* non degenerate, i.e. the matrix:

$$\Sigma(\psi^*) = \frac{dE}{d\psi}(\psi^*) \tag{2.9}$$

and $A(\psi^*)^N-I$ are non singular.

Theorem 2.2: *Under assumptions A1 to A3 and the bifurcation equation assumption, there exists ϵ_* such that for all ϵ, $|\epsilon| \le \epsilon_*$, the system (1.8) has a (locally unique) N-periodic solution $(X_N(k,\epsilon),\theta_N(k,\epsilon))$, continuously differentiable in ϵ, satisfying:*

$$X_N(k,0) = X_{fN}(\psi^*,k)\ ,\ \ \theta_N(k,0) = \psi^*\ ,\ \ \forall k \tag{2.10}$$

Consequently the vectors $\theta_N(k,\epsilon)$ stay uniformly in an ϵ-neighborhoud of the point ψ^* and in particular $\theta_N(k,\epsilon)$ belongs to Γ for all k.

Remark 2.2: i) On the contrary of Ljung (1977) or Bodson et al. (1986), no stability assumption is needed for existence. From Poincaré, we know that non degeneracy of the fixed point ψ^* is sufficient.

ii) Practically, this Theorem tells us that it is sufficient to find a non degenerate zero for the bifurcation equation. Usually, this non degeneracy is equivalent to a persistent spanning condition (see section 4).

iii) (2.10) gives an approximation of the periodic solution of (1.8) simply in terms of $(X_{fN}(\psi^*,k),\psi^*)$, periodic solution of the frozen system.

Proof: < Thanks to the bifurcation equation assumption, we know that $(X_{fN}(\psi^*,0),\psi^*,0)$ is a zero of Z_N, with ψ^* in Γ. From the regularity properties given by assumption A2, A3, we can use the implicit function theorem (see theorem 6.1.1.of (Miller, Michel, 1982). The result will follow if:

$$\det \begin{pmatrix} \frac{\partial Z_{Nx}}{\partial X} & \frac{\partial Z_{Nx}}{\partial \theta} \\ \frac{\partial Z_{N\theta}}{\partial X} & \frac{\partial Z_{N\theta}}{\partial \theta} \end{pmatrix} (X_{fN}(\psi^*,0),\psi^*,0) \neq 0 \tag{2.11}$$

To compute this determinant, notice that for ϵ=0, we have the frozen system for which $(X_{fN}(\theta,0),\theta)$ is known to be the initial condition of an N-periodic solution, i.e. for all θ such that $A(\theta)^N-I$ is non singular, we have:

$$Z_{Nx}(X_{fN}(\theta,0),\theta,0) = 0\ ,\quad Z_{N\theta}(X_{fN}(\theta,0),\theta,0) = E(\theta) \tag{2.12}$$

Hence, let us introduce a new variable:

$$\chi = X - X_{fN}(\theta,0) \tag{2.13}$$

This variable has been used previously by Riedle and Kokotovic (1985) to rewrite the linearized system in a form suitable for application of the averaging theory.

We decompose the Z_N map at ϵ=0 into:

$$(X,\theta) \rightarrow (\chi,\theta) \rightarrow Z_N(X_{fN}(\theta,0)+\chi,\theta,0)$$

The first map has derivative:

$$\begin{pmatrix} I & -\frac{\partial X_{fN}}{\partial \theta}(\theta,0) \\ 0 & I \end{pmatrix}$$

The second map has derivative at χ=0:

$$\begin{pmatrix} A(\theta)^N-I & 0 \\ * & \Sigma(\theta) \end{pmatrix}$$

The top left block comes from (2.4) which gives for ϵ=0:

$$Z_{Nx}(X,\theta,0) = (A(\theta)^N-I)\,(X_{fN}(\theta,0)+\chi) + \sum_{k=0}^{N-1} A(\theta)^{N-k-1}B(\theta)v(k)$$

The top and bottom right block follow from (2.12) and the bottom left block is unimportant. The result follows since we have established:

$$\nabla Z_N(X_{fN}(\theta.0).\theta.0)=\begin{pmatrix}A(\theta)^N-I & 0\\ \cdot & \Sigma(\theta)\end{pmatrix}\begin{pmatrix}I & -\frac{\partial X_f}{\partial\theta}(\theta,0)\\ 0 & I\end{pmatrix} \quad (2.15)>$$

3. STABILITY OF THE PERIODIC SOLUTION

Having obtained necessary condition and sufficient condition for existence of periodic solutions $(X_N(k,\epsilon), \theta_N(k,\epsilon))$, we are now interested in their (un)stability property. We have:

Theorem 3.1: *Under assumptions A1 to A3 and the bifurcation equation assumption there exists ϵ_* such that for all ϵ, $0<\epsilon\le\epsilon_*$, the N-periodic solution given by theorem 2.2 is:*

i) uniformly asymptotically stable if the eigenvalues of $A(\psi^*)$ have modulus strictly less than one and the real part of the eigenvalues of $\Sigma(\psi^*)$ are strictly negative.

ii) unstable if at least one eignevalue of $A(\psi^*)$ has a modulus larger than one or one eigenvalue of $\Sigma(\psi^*)$ has a positive real part.

Comment: This theorem establishes that stability of the periodic solution holds if:

i) ψ^*, solution of the bifurcation equation, is a stabilizing parameter, i.e. the spectral radius of $A(\psi^*)$ is strictly smaller than 1.

ii) ψ^* is an exponentially stable equilibrium of:

$$\frac{d\psi}{dt}=E(\psi) \quad (3.1)$$

Proof: < From continuity of a solution with respect to its initial condition (at least on finite time intervals), we have:

Lemma 3.1 *(Assertion 2 of Theorem 3.28 of Arnold (1978)): The N-periodic solution has the same (un)stability property as the corresponding fixed point of the N-advance map.*

On the other hand, the existence of invariant manifolds for a map (see Theorem 5.1 and corollary 5.1 (may be used in reverse time for unstability) of (Hartman, 1982) for example) implies that a sufficient condition for (un)stability of a fixed point is given by the position of the eigenvalues of the Jacobian matrix of this map, evaluated at the fixed point. Consequently we are lead to study the matrix $\nabla T_N(X_N(0,\epsilon), \theta_N(0,\epsilon),0)$.

First, we notice that from (2.10) and the continuous differentiability of A, B, C, $X_N(0,\epsilon)$, $\theta_N(0,\epsilon)$, we can use Hadamard Lemma (see Aubin, Ekeland, 1984) to obtain the existence of a function $\Delta(\epsilon)$, bounded on a neighborhoud of zero and satisfying:

$$\nabla Z_N(X_N(0,\epsilon),\theta_N(0,\epsilon),\epsilon)=\nabla Z_N(X_f(0,\psi^*),\psi^*,0)+\epsilon\,\Delta(\epsilon) \quad (3.2)$$

Secondly, ∇T_N is equivalent to:

$$I+S=\begin{pmatrix}I & -\frac{\partial X_f}{\partial\theta}\\ 0 & I\end{pmatrix}\nabla T_N\begin{pmatrix}I & \frac{\partial X_f}{\partial\theta}\\ 0 & I\end{pmatrix} \quad (3.3)$$

Finally, ∇T_N being related to ∇Z_N through (2.3), with (2.15), we obtain:

$$S(X_N(0,\epsilon),\theta_N(0,\epsilon),\epsilon)=\begin{pmatrix}A(\psi^*)^N-I+\epsilon\Delta_1(\epsilon) & \epsilon\Delta_2(\epsilon)\\ \epsilon\Delta_3(\epsilon) & \epsilon\Sigma(\psi^*)+\epsilon^2\Delta_4(\epsilon)\end{pmatrix} \quad (3.4)$$

where $\Delta_i(\epsilon)$, $i=1,4$ are bounded on a neighborhoud of zero. Now, we apply Lemma 1 of (Kokotovic, 1975): since $A(\psi^*)^N-I$ is non singular, there exists a function $L(\epsilon)$ bounded on a neighborhoud of zero such that S is equivalent to (omitting ϵ as argument):

$$\begin{pmatrix}A(\psi^*)^N-I+\epsilon(\Delta_1+\epsilon L\,\Delta_3) & 0\\ \epsilon\Delta_3 & \epsilon\Sigma(\psi^*)+\epsilon^2(\Delta_4-\Delta_3 L)\end{pmatrix}$$

It follows that the eigenvalues of $\nabla T_N(X_N(0,\epsilon),\theta_N(0,\epsilon),\epsilon)$ are:

$\lambda\{A(\psi^*)^N\}+o(1)$

$1+\epsilon\ \mathrm{Re}\ \lambda\{\Sigma(\psi^*)\}+o(\epsilon)$

where $o(1)$ and $\frac{o(\epsilon)}{\epsilon}$ are continuous functions of ϵ which tend to zero as ϵ tends to zero.

4. EXAMPLE

To illustrate the results of the previous sections, let us consider a disturbed first order plant with an unknown pole:

$$y(k+1)=a\ y(k)+u(k)+\sqrt{\epsilon}\ \mathrm{Re}(dz_d^k) \quad (4.1)$$

in closed loop with a deadbeat adaptive controller (see (Goodwin, Sin, 1984)):

$$\begin{aligned}\theta(k+1) &= \theta(k) + \frac{y(k)\,(y(k+1)-u(k)-\theta(k)y(k))}{1+y(k)^2}\\ u(k) &= -\theta(k)\,y(k) + \sqrt{\epsilon}\ \mathrm{Re}(rz_r^k)\end{aligned} \quad (4.2)$$

where z_r, z_d are roots of $z^N=1$ and d, r are complex numbers. To simplify, we assume that z_r, z_d are distinct and different from ±1 and that $\mathrm{Re}(z_d)$ is positive.

The closed loop system can be written in the form (1.4), with:

$$A(\theta)=a-\theta\quad B(\theta)=(1\ 1)^T\quad C(y,\theta,w,\lambda)=\frac{y\,(y\,(a-\theta)+d)}{1+\lambda y^2} \quad (4.3)$$

Clearly assumptions A1 to A3 are satisfied with $\Gamma=\mathbf{R}$.

The frozen system: The set of periodic solutions is completely described by:

$$\begin{aligned}y_f(\psi,k) &= \sqrt{\epsilon}\left(\mathrm{Re}\left(\frac{rz_r^{k-1}}{1+\psi\bar z_r}+\frac{dz_d^{k-1}}{1+\psi\bar z_d}\right)+\alpha(-1)^k+\beta\right)\\ \theta_f(\psi,k) &= \psi+a\end{aligned} \quad (4.4)$$

where α, β are non zero in the singular case, i.e.:

$$\alpha\ne0\ \ iff\ \ \psi=1\ ,\quad \beta\ne0\ \ iff\ \ \psi=-1 \quad (4.5)$$

Consequently, they are all M-periodic with M=N if N is even and M=2N if N is odd. Using Parseval's Theorem to evaluate $E(\psi)$ defined in (2.8), we obtain:

$$E(\psi)=\frac{M}{2}\left(\frac{|d|^2\,\mathrm{Re}(z_d)}{|1+\psi\bar z_d|^2}-\frac{|r|^2\,\psi}{|1+\psi\bar z_r|^2}-2(\alpha^2-\beta^2)\right) \quad (4.6)$$

The actual system: We know, with Theorem 2.1, that the set of accumulation points of initial conditions of periodic solutions of (4.2) is completely contained in:

$$\left\{(y_f(\psi,0),\theta_f(\psi,0))\ /\ E(\psi)=0\right\} \quad (4.7)$$

In particular this gives us all the possible limits of periodic solutions of (4.2) which would be continuous in ϵ.

Hence, to go further, we look for:

the zeros of E. i) For $\alpha=\beta=0$, the numerator of $E(\psi)$ is a third order polynomial which is positive for all negative ψ and negative for ψ going to $-\infty$. Moreover its second derivative is zero for some negative ψ. It follows that $E(\psi)$ has one and only one zero lying in $[0,\infty)$ and with negative derivative. On the other hand:

$$E(1) = \frac{M}{2}\left(\frac{|d|^2 \operatorname{Re}(z_d)}{2(1+Re(z_d))} - \frac{|r|^2}{2(1+Re(z_r))}\right) \tag{4.8}$$

Hence, this zero lies in $[0,1]$ iff:

$$\frac{|r|^2}{|d|^2} \geq \frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)} \tag{4.9}$$

ii) For $\alpha \neq 0$, we have to evaluate E at $\psi=1$. $E(1)$ is zero iff α satisfies:

$$2\alpha^2 = \frac{|d|^2 \operatorname{Re}(z_d)}{2(1+Re(z_d))} - \frac{|r|^2}{2(1+Re(z_r))} \tag{4.10}$$

which is possible iff:

$$\frac{|r|^2}{|d|^2} \leq \frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)} \tag{4.11}$$

iii) For $\beta \neq 0$, we have to evaluate E at $\psi=-1$. In this case, if $\operatorname{Re}(z_d) \geq 0$, there is no β satisfying $E(-1)=0$.

Conclusion: From Theorem 2.2, the system (4.1)-(4.2), has an N-periodic solution for ϵ small enough if:

$$\frac{|r|^2}{|d|^2} \neq \frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)} \tag{4.12}$$

From Theorem 3.1, this solution is a stable node if:

$$\epsilon > 0 \quad \text{and} \quad \frac{|r|^2}{|d|^2} \geq \frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)} \tag{4.13}$$

and is an unstable node if:

$$\epsilon < 0 \quad or \quad \frac{|r|^2}{|d|^2} \leq \frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)} \tag{4.14}$$

From Theorem 2.1, there is no other periodic solutions continuous in ϵ if (4.13) holds. On the other hand, if (4.14) holds, the only other possible periodic solutions continuous in ϵ are M-periodic and satisfy:

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{y(k,\epsilon)}{\sqrt{\epsilon}} &= \operatorname{Re}\left(\frac{r z_r^{k-1}}{1+\overline{z}_r} + \frac{d z_d^{k-1}}{1+\overline{z}_d}\right) + \alpha\,(-1)^k \\ \lim_{\epsilon\to 0} \theta(k,\epsilon) &= 1 + a \end{aligned} \tag{4.15}$$

with α given by (4.10).

In fact, it seems from our simulations that these solutions do exist, are foci both stable when N is odd, one stable and one unstable when N is even. The following figures are phase portraits $(\theta-a\ ,\ y)$ of (4.1)-(4.2). To simplify, we plot only 1 point out of M so that periodic solutions appear as fixed points. The data are:

$fig.\ 1$: $d=1$ $z_d=e^{\frac{2i\pi}{6}}$ $\mu=1.20$ $z_r=e^{\frac{2i\pi}{4}}$ $\sqrt{\epsilon}=0.10$ $N=12$

$fig.\ 2$: $d=1$ $z_d=e^{\frac{2i\pi}{6}}$ $\mu=0.98$ $z_r=e^{\frac{2i\pi}{4}}$ $\sqrt{\epsilon}=0.05$ $N=12$

$fig.\ 3$: $d=1$ $z_d=e^{\frac{2i\pi}{5}}$ $\mu=0.98$ $z_r=e^{\frac{2i\pi}{3}}$ $\sqrt{\epsilon}=0.05$ $N=15$

with

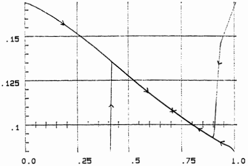

fig.1

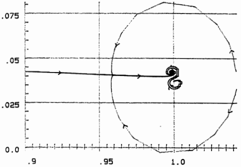

fig.2

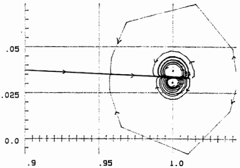

fig.3

$$r = \mu \sqrt{\frac{\operatorname{Re}(z_d)\,(1+Re(z_r))}{1+Re(z_d)}}$$

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