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SELF-TUNING AND CONVERGENCE OF PARAMETER ESTIMATES IN MINIMUM VARIANCE TRACKING AND LINEAR MODEL FOLLOWING ***

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Abstract

We examine the problem of *self-tuning trackers*, for which the self-tuning property can be proved.

1. INTRODUCTION

Recently an algorithm for minimum variance control of ARMAX systems has been proved to be **self-tuning** for the **regulation** problem, see Becker, Kumar and Wei [1]. This is a central question of fundamental interest since it implies that the adaptive controller can be used as a mechanism for tuning to the parameters of an optimal control law.

(Recall that in the *regulation* problem one wants the output of the system to stay as close as possible to zero, whereas in the *tracking* problem one wants to track a given arbitrary trajectory. By *self-tuning* it is meant that the adaptive control *law* converges to the optimal control law). The work of [1] above thus complements the work of Goodwin, Ramadge and Caines [2] who have proved the *self-optimality* of some adaptive control algorithms for minimum variance regulation and tracking. (By *self-optimality* it is meant that the cost, the time average of the square of the tracking error, is minimal).

We resolve in this paper the more general problem of self-tuning for the **tracking** problem, which has remained open so far. For full details regarding the results presented here the reader is referred to the forthcoming paper [3].

An important feature which distinguishes the tracking problem from the regulation problem is the necessity of knowing the coefficients of the colored noise polynomial, see [2,1,4].

We show how one may estimate these unknown parameters and obtain self-tuning for the **general tracking problem**. This is done under the natural assumption that the reference trajectory is sufficiently rich of appropriate order.

Importantly, we also address the problem of obtaining self-tuning even when the reference trajectory is not so rich as to allow one to identify all the coefficients of the colored noise polynomial. For example, in the important class of *set-point problems*, the reference trajectory is a non-zero constant, which is sufficiently rich of order *one* only.

So motivated, we examine the problem of tracking trajectories which are generated by **linear models**. We show how one may adjust the *dimension* of the regression vector to the degree of excitation present in the reference trajectory, and then provide a proof of self-tuning of the resulting reduced dimension adaptive controllers.

2. The General Tracking Problem

Consider the ARMAX system

$$y(t) = \sum_{i=1}^{p} a_i y(t-i) + \sum_{i=1}^{q} b_i u(t-i) + \sum_{i=1}^{s} c_i w(t-i) + w(t) \quad (1)$$

where y, u and w are respectively the output, input and white noise. The parameters $(a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s)$ are unknown. The goal is to design an adaptive control law which converges to a controller minimizing the squared deviation between the output and a given bounded reference trajectory $\{y^*(t)\}$.

When $\{y^*(t)\}$ is a reference trajectory with no special properties, we will use the following adaptive controller. Note the notation $p \vee s := max(p, s)$.

$$\theta(t+1) = \theta(t) + \frac{\mu\phi(t)}{r(t)}[y(t+1) - y^*(t+1)],$$

where $\mu \neq 0$ is an arbitrary constant, and

$$r(t+1) := 1 + \sum_{k=0}^{t+1} \phi^T(k)\phi(k).$$

The regressor vector is

$$\phi(t) := (y(t), \ldots, y(t-p\vee s+1), u(t), \ldots, u(t-q+1), \quad (2)$$

$$-y^*(t+1), \ldots, -y^*(t-s+1))^T.$$

The control input is then chosen as,

$$u(t) := \frac{-1}{\beta_1(t)}[\sum_{i=1}^{p\vee s} \alpha_i(t) y(t-i+1) \quad (3)$$

$$+ \sum_{i=2}^{q} \beta_i(t) u(t-i+1) - \sum_{i=0}^{s} \gamma_i(t) y^*(t-i+1)]$$

where

$$(\alpha_1(t), \ldots, \alpha_{p\vee s}(t), \beta_1(t), \ldots, \beta_q(t), \quad (4)$$

$$\gamma_0(t), \ldots, \gamma_s(t))^T := \theta(t)$$

Note that (3) can equivalently be written as

$$\phi^T(t)\theta(t) = 0 \quad (5)$$

Motivation for adaptive controller

To understand the motivation behind this adaptive controller, rewrite the system (1) as.

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$$y(t+1)-y^*(t+1) = [\sum_{i=1}^{p} a_i y(t+1-i) + \sum_{i=1}^{q} b_i u(t+1-i)$$

$$+ \sum_{i=1}^{s} c_i w(t+1-i) - y^*(t+1)] + w(t+1)$$

Provided one could observe the past of $w(\,.\,)$ at each time t, then an optimal control $u(t)$ could be chosen so that the term in [. .] on the right hand side above is zero, i. e.

$$u(t) = \frac{-1}{b_1}[\sum_{i=1}^{p} a_i y(t+1-i) + \sum_{i=2}^{q} b_i u(t+1-i)$$

$$+ \sum_{i=1}^{s} c_i w(t+1-i) - y^*(t+1)]\,.$$

This would result in ensuring that $y(t+1) - y^*(t+1) = w(t+1)$, which is clearly the best possible tracking error. However, the sequence $w(\,.\,)$ is *not* observed. So let us consider replacing $w(t)$ by $y(t) - y^*(t)$, which is what we hope it would be, at least asymptotically. This gives the implementable control law,

$$u(t) = \frac{-1}{b_1}[\sum_{i=1}^{p\vee s}(a_i + c_i) y(t+1-i) + \sum_{i=2}^{q} b_i u(t+1-i)$$

$$- \sum_{i=1}^{s} c_i y^*(t+1-i) - y^*(t+1)]\,.$$

It can be shown that this control law is actually optimal with respect to the long run average of the square of the tracking error; for more details, see Kumar and Varaiya [4]. Let us accordingly define,

$$\theta^\circ := (a_1+c_1, \ldots, a_{p\vee s}+c_{p\vee s}, b_1, \ldots, 1, c_1, \ldots, c_s)^T \tag{6}$$

(where, for convenience, we set $c_i := 0$ for $i > s$ and $a_i := 0$ for $i > p$ in (6)). Note that under *optimal control*, the system (1) can be represented as

$$y(t+1) - y^*(t+1) = \phi^T(t)\theta^\circ + w(t+1)\,,$$

while the optimal control law can be written as one which chooses $u(t)$ to satisfy,

$$\phi^T(t)\theta^\circ = 0\,.$$

The motivation behind our adaptive control law is clear. We are trying to estimate θ° when the system is being optimally controlled.

Remark :

The $(p\vee s+q+1)$-th component of θ° is 1, and hence is a *known* quantity. However, the estimator ignores this knowledge and estimates it anyway by $\gamma_0(t)$. Hence the dimension of the parameter estimator in this adaptive controller is one larger than that of Goodwin, Ramadge and Caines [2].

3. Assumptions

Define the polynomials

$$A(z) := 1 - \sum_{i=1}^{p} a_i z^i$$

$$B(z) := \sum_{i=1}^{q} b_i z^{i-1}$$

$$C(z) := 1 + \sum_{i=1}^{s} c_i z^i$$

We make the following assumptions on the system.

All the roots of $B(z)$ and $C(z)$ are strictly outside the unit circle.

$$\mathrm{Re}[C(e^{i\omega}) - \frac{\mu}{2}] > 0 \quad \text{for } 0 \leqslant \omega < 2\pi$$

$$b_1 \neq 0$$

$z^{-1}[C(z) - A(z)]$ and $B(z)$ are polynomials of degrees respectively equal to $(p\vee s-1)$ and $(q-1)$, which have no common factors.

$\{w(t)\}$ is a sequence of scalar random variables on a probability space $\{\Omega, F, P\}$, whose distributions are all mutually absolutely continuous with respect to Lebesgue measure.

Let $F_t := \sigma\{w(1), \ldots, w(t)\}$ be the sub-σ-algebra of F generated by $\{w(1), \ldots, w(t)\}$. We assume that there are $\sigma^2 > 0$ and $\delta > 0$ such that

$$E[w(t) \mid F_{t-1}] = 0 \quad a.s.$$

$$E[w^2(t) \mid F_{t-1}] = \sigma^2 \; a.s.$$

$$\sup_t E[\,|w(t)|^{2+\delta} \mid F_{t-1}] < +\infty \quad a.s.$$

$$\|\theta(0)\| > 0$$

$\{y^*(t)\}$ is bounded.

4. Results for the General Tracking Problem

The following self-optimality and self-tuning results can be proved for the above adaptive controller for general tracking.

Theorem

Self-optimality property

$$\lim_N \frac{1}{N}\sum_{t=1}^{N}[y(t)-y^*(t)]^2 = \sigma^2 \; a.s.$$

Convergence Property

Suppose that $\{y^*(t)\}$ is strongly sufficiently rich of order $(s+q)$, i.e. for the choice of $l := (s+q)$ there exists an n and an $\epsilon > 0$ such that

$$\sum_{k=t+1}^{t+n} (y^*(k-1), \ldots, y^*(k-l))^T (y^*(k-1), \ldots, y^*(k-l)) \geqslant \epsilon I_l$$

for all t large enough.

(Here I_l here is the lxl identity matrix). Then,

$$\lim_t \theta(t) = \xi\theta^\circ \quad a.s.$$

for some a.s. finite nonzero scalar random variable ξ.

Strong consistency property

Under the above excitation condition,

$$\lim_t \frac{1}{\gamma_0(t)}(\alpha_1(t)-\gamma_1(t), \ldots, \alpha_p(t)-\gamma_p(t), \beta_1(t), \ldots, \beta_q(t), \gamma_1(t), \ldots, \gamma_s(t))$$

$$= (a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_s)\; a.s. \; (\text{with } \gamma_i(t) := 0 \text{ for } i > s)\,.$$

Self-tuning property

Also under the above excitation property,

$$\lim_t \frac{1}{\beta_1(t)}(\alpha_1(t), \ldots, \alpha_{p\vee s}(t), \beta_2(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_s(t))$$

$$= \frac{1}{b_1}(a_1+c_1, \ldots, a_{p\vee s}+c_{p\vee s}, b_2, \ldots, b_q, 1, c_1, \ldots, c_s) \; a.s.$$

5. The Linear Model Following Problem

In many practical situations, the reference trajectory is generated, at least asymptotically, as the output of a homogeneous linear model. Suppose that there is a sequence $\{y_m(t)\}$ such that

$$y_m(t) = \sum_{i=1}^{l} h_i y_m(t-i) \tag{7}$$

and the trajectory to be tracked $y^*(t)$ is asymptotically close to $y_m(t)$ in that

$$\sum_{t=1}^{\infty}(y^*(t) - y_m(t))^2 < +\infty \tag{8}$$

Without loss of generality we can make the following two assumptions:

(i) There is no lower order difference equation satisfied by $\{y_m(t)\}$.

(ii) The roots of $H(z) := 1 - \sum_{i=1}^{l} h_i z_i$ are exactly on the unit circle and there are no repeated roots.

The condition (i) is without loss of generality since we could otherwise always write $y_m(t)$ as the solution of a homogeneous difference equation where *all* the modes are excited by the initial conditions. Regarding (ii) note first that since we intend to work only with *bounded* $\{y^*(t)\}$, and since all the modes of $H(z)$ are excited, we have to assume that $H(z)$ has roots on or outside the unit circle, and also that those roots which are on the unit circle are not repeated. Second, since we are only interested in the *asymptotic* behavior of $\{y^*(t)\}$, we can eliminate all the modes corresponding to roots of $H(z)$ which are strictly outside the unit circle since they decay geometrically to 0. This leaves us with only unrepeated modes on the unit circle. (As an example, note that a *constant* reference trajectory, which is important for the *set-point* problem, satisfies $(1-z)y_m(t) = 0$).

It is worth noting that (i) and (ii) together imply that

$$y_m(t) = d_0 + d_1(-1)^t + \sum d_i \sin(\omega_i t + \delta_i) .$$

Note that the degree of $H(z)$ is l ; it is also the degree of sufficient richness of $y^*(t)$. We will reduce the dimension of the parameter estimator by $(s+1-l)$ components by using the following regressor vector in place of (2-4).

$$\phi(t) := (y(t), \ldots, y(t-p \vee s+1), u(t), \ldots, u(t-q+1),$$

$$-y^*(t+1), \ldots, -y^*(t+2-l))^T ,$$

$$\theta(t) := (\alpha_1(t), \ldots, \alpha_{p \vee s}(t), \beta_1(t), \ldots, \beta_q(t), \qquad (9)$$

$$\gamma_0(t), \ldots, \gamma_{l-1}(t))^T .$$

and

$$u(t) = \frac{-1}{\beta_1(t)}\Big[\sum_{i=1}^{p \vee s} \alpha_i(t) y(t-i+1) + \sum_{i=2}^{q} \beta_i(t) u(t-i+1)$$

$$- \sum_{i=0}^{l-1} \gamma_i(t) y^*(t-i+1)\Big] ,$$

or equivalently by (5).

Motivation for adaptive controller.

The idea underlying the above adaptive control law is as follows. If the parameters were known the minimum variance adaptive control law would be,

$$u(t) = \frac{-1}{b_1}\Big[\sum_{i=1}^{p \vee s} (a_i + c_i) y(t-i+1) + \sum_{i=2}^{q} b_i u(t-i+1)$$

$$- y^*(t+1) - \sum_{i=1}^{s} c_i y^*(t-i+1)\Big] ,$$

see [4] for details. Note that the only terms featuring y^* which are important to the above control law are $y^*(t+1) + \sum_{i=1}^{s} y^*(t-i+1) = C(z) y^*(t+1)$ So the control law only requires knowledge of $C(z)y^*(t)$. Let $G(z) := \sum_{i=0}^{l-1} g_i z^i$ and $F(z) := \sum_{i=0}^{s-l} f_i z^i$ be polynomials satisfying :

$$C(z) = F(z)H(z) + G(z)$$

The polynomials $G(z)$ and $F(z)$ are the remainder and quotient respectively when the polynomial $C(z)$ is divided by the polynomial $H(z)$. Note that asymptotically at least we have the equality $C(z)y^*(t) = [F(z)H(z) + G(z)]y^*(t) = F(z)H(z)y^*(t) + G(z)y^*(t) = G(z)y^*(t)$, since by (7,8), $H(z)y^*(t) = 0$ holds asymptotically. So we find that we only need knowledge of $G(z)y^*(t)$ in order to implement the true minimum variance control law. Define

$$\theta^\circ := (a_1+c_1, \ldots, a_{p \vee s}+c_{p \vee s}, b_1, \ldots, b_q, g_0, g_1, \ldots, g_{l-1})^T .$$

and we can interpret the parameter estimate (9) as trying to estimate it by $\theta(t)$.

Remarks :

(i) The adaptive controller need not be provided with the precise information about what the polynomial $H(z)$ is. It only needs knowledge of the *degree* of $H(z)$.

(ii) Note that if $l \geq s+1$ then no savings in dimensionality can be achieved by using this special control law over the one given for the general case, and so one should use that control law. The special control law is only designed for use when there is insufficient excitation.

6. Results for the Linear Model Following Problem

We can obtain the following results regarding self-optimality and self-tuning.

Theorem

Self-optimality property

$$\lim_N \frac{1}{N} \sum_{t=1}^{N} [y(t) - y^*(t)]^2 = \sigma^2 \quad a.s.$$

Convergence Property

$$\lim_t \theta(t) = \xi \theta^\circ \quad a.s.$$

for some a.s. finite nonzero scalar random variable ξ.

Self-tuning property

$$\lim_t \frac{1}{\beta_1(t)}(\alpha_1(t), \ldots, \alpha_{p \vee s}(t), \beta_2(t), \ldots, \beta_q(t), \gamma_0(t), \ldots, \gamma_{l-1}(t))$$

$$= \frac{1}{b_1}(a_1+c_1, \ldots, a_{p \vee s}+c_{p \vee s}, b_2, \ldots, b_q, g_0, \ldots, g_{l-1})$$

setting $a_i := 0$ for $i > p$ and $c_i := 0$ for $i > s$.

7. Concluding Remarks

Among the outstanding problems still left unresolved are the following:

(i) Does the least squares based parameter estimation algorithm also self-tune ? This is of importance because the rate of convergence of least squares based algorithms has been observed to be superior to the type of parameter estimation algorithm considered here.

(ii) What robustness properties do these types of self-tuning adaptive control laws possess?

8. References

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