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ROBUST MODEL REFERENCE ADAPTIVE CONTROLLERS, PART 1: STABILITY ANALYSIS*

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ABSTRACT

We study the stability given by a modified model reference adaptive controller. Modifications are projections of the adapted parameters into a compact set, normalization of the signals entering the adaptation law by a weighted ℓ_2-norm of the I/O signals. The a priori knowledge about the plant is implicit: order of a stabilizing regulator, compact set containing its coefficients. Global boundedness of the signals is established using both the error to signal approach of [10] and the operator theoretic approach of [8]. A conicity condition is involved but its robustness with respect to the graph topology of [20] is proved.

1. INTRODUCTION

Nominal adaptive controllers (as those of [13] for example) may lead to a nonrobust stability. The most evident symptom of this problem is the drift of the adapted parameters as noticed by Egardt in [1]. Therefore modifications have to be introduced. Either the signals (active modification) (see [3] for example), or the adapted parameters (passive modification) can be modified. Here only passive modifications will be used: projection of the parameters into a bounded area (following [1]), and normalization of the signals entering the adaptation law (see [1],[4]).

Operator theoretic approach: In [7], in order to study stability, the operator theoretic approach has been applied to the error model given by nominal direct adaptive controllers. Then plants can be defined for which the previously mentioned drift does not occur. They are such that their transfer function, in closed loop with a linear time invariant controller (lying among those reachable by the adaptation law), is strictly inside a cone. However, no proof is available that the radius of this cone does or does not vanish. As mentioned in [4],[10], normalization, the passive modification mentioned above, allows us to derive a lower bound for this radius, see [8]. However, to establish global boundedness in the presence of output disturbances, an active modification is also needed in order to meet a condition of persistent span of the parameter space. These results are still too restrictive. The conicity condition implies perfect knowledge of the plant delay. This is not robust. And, today, no proof is available that persistent excitation of the output reference leads to persistent span of the parameter space. Using a different approach similar though more conservative results have been obtained in [9]. Here we relax these assumptions. Due to space limitations, only results for discrete time plants have been presented (see [2],[5],[6], for example, for the continuous time case).

Error to signal ratio approach: In [10], the author has proposed to reflect the unmodeled effects outside the closed loop system as an open loop disturbance, the modeling error. To make the assumptions signal independent, this modeling error is normalized by an ℓ_2-norm of these signals. Then unmodeled effects are captured in terms of an error to signal ratio, equivalent to an operator gain. Thus we are motivated to normalize the signals entering the adaptation law. When both normalization and projection are used, global boundedness of the signals is established provided the error to signal ratio is sufficiently small. However, though a wide class of unmodeled effects (see [4]) is captured by this approach, it is essentially qualitative. Taking care of the transformation by feedback of the unmodeled effecs could make it more quantitative. This is the goal of this paper.

The paper is organized as for the simultaneous stabilization problem (see [11], for example). In Section 2 we define the adaptive controller. In Section 3, we restrict our attention to a class of plants. And, in Section 4, we state that these plants are stabilized by our controller. The following sections are devoted to the proof of this theorem.

Notations:

- The $\ell_2[T_0,T_1]$-norm of the sequence $x(t)$ is:

$$\|x(t)\|_{T_0}^{T_1} \triangleq \Big(\sum_{t=T_0}^{T_1} x(t)^2\Big)^{\frac{1}{2}}. \quad (1.1)$$

- Uniformly with respect to T_0,T_1 means that the bounds do not depend on T_0,T_1.

- The conditional gain γ of an operator G is (compare with [18]): for a given ξ (the conditioning bound)

$$\gamma(G) \triangleq \sup_{\|x\|_\infty \le \xi} \frac{\|Gx\|}{\|x\|}. \quad (1.2)$$

- The $\ell_2(\mu)[0,T]$-norm is:

$$\|x(t)\|_{T,\mu}^2 \triangleq \sum_{r=0}^{T} \mu^{-t} x(t)^2 \quad (1.3)$$

for a constant sequence or a delayed sequence, we have:

$$x = \text{constant} \Rightarrow \|x\|_{T,\mu}^2 \le x^2 \mu^{-(T+2)} \quad (1.4)$$

$$\|x(t-d)\|_{T,\mu}^2 = \mu^{-d} \|x(t)\|_{T-d,\mu}^2 \quad (1.5)$$

2. AN ADAPTIVE CONTROLLER

Let $y(t),u(t)$ be the output and input of the plant to be controlled. The adaptive controller we consider in this paper is a usual model reference scheme based on a least squares estimation incorporating both projection and normalization. It is parameterized by integers n_S, n_R, d, positive constants

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λ, λ_0, λ_1, μ, ρ, K, σ_0 ($\lambda_0 \le \lambda_1$, $\mu < 1$), a vector θ_c in $\mathbb{R}^n$ ($n = n_S + n_R + 2$)(first component of $\theta_c \ge \sigma_0$) and a polynomial $C(q^{-1})$. Its objective is to minimize the tracking error:

$$e(t) \triangleq C(q^{-1})(y(t)-y^m(t)) \qquad (2.1)$$

where $y^m(t)$ is a uniformly bounded output reference.

As usual, let $\phi(t)$,$\theta(t)$ be the following vectors in $\mathbb{R}^n$:

$$\phi(t) = (u(t) \ \dots \ u(t-n_S) \ y(t) \ \dots \ y(t-n_R)) \qquad (2.2)$$

$$\theta(t) = (s_0(t) \ \dots \ s_{n_S}(t) \ r_0(t) \ \dots \ r_{n_R}(t)), \qquad (2.3)$$

the algorithm is: usual update

$$g(t) = 1/(\lambda + \bar{\phi}(t-d)^T F(t-d)\bar{\phi}(t-d))$$

$$\theta'(t) = \theta(t-d) + g(t)F(t-d)\bar{\phi}(t-d)\bar{e}(t) \qquad (2.4)$$

$$F'(t) = F(t-d) - g(t)F(t-d)\bar{\phi}(t-d)\bar{\phi}(t-d)^T F(t-d)$$

matrix regularization

$$F(t) = (1-\lambda_0/\lambda_1)F'(t) + \lambda_0 I \qquad (2.5)$$

leading coefficient regularization

$$\theta''(t) = \theta'(t) + \mathrm{Max}(0, \ \sigma_0 - s_0'(t))F_{.1}(t)/F_{11}(t) \qquad (2.6)$$

Projection into the sphere (θ_c,K)

$$\theta(t) = \theta_c + (\theta''(t)-\theta_c)\mathrm{Max}(1, K/\|\theta(t)-\theta_c\|) \qquad (2.7)$$

control law (implicit in u(t), recall $s_0(t) \ge \sigma_0$):

$$C(q^{-1})y^m(t+d) = \theta(t)^T\phi(t) \qquad (2.8)$$

where $F_{.1}(t)$ is the first column of F(t), $F_{11}(t)$ is the first entry of F(t); $\bar{\phi}(t-d)$, $\bar{e}(t)$ are normalized signals as defined below. In the following, we call adaptation law Eqs. (2.4) to (2.7).

<u>Normalization procedure</u>: Before entering the adaptation law, the signals arriving from the plant are normalized as follows: let $\rho(t)$ be the output of a first order filter with $\bar{\phi}(t-d)^T\bar{\phi}(t-d)$ as input or more precisely:

$$\rho(t) = \mu\rho(t-1) + \max(\|\phi(t-d)\|^2, \rho), \qquad (2.9)$$

then a sequence x(t) is normalized as:

$$\bar{x}(t) = \rho(t)^{-\frac{1}{2}}x(t). \qquad (2.10)$$

In the following, we denote ($\bar{\cdot}$) the normalized signals and operators acting on them. Note that $\rho(t)^{-\frac{1}{2}}$, $\bar{\phi}(t) \in \ell_\infty$.

<u>Comment</u>: The algorithm presented here is a least square version of the "DSA-algorithm with projection" proposed by Egardt (p. 69, [1]). Our goal being to deal with uncertain plant structure, it incorporates four modifications compared with the nominal least squares algorithm:

1. Monitoring of the adapted parameters using projections (2.6),(2.7). This is an efficient remedy to the problem of bounded disturbances (see [1]).

2. Normalizations procedure: This causes the adaptation law to see the effects of unmodeled dynamics as a bounded disturbance (see [4],[10]).

3. Matrix regularization (2.5): this keeps alertness of the adaptation, a desirable property in the presence of mismodeling.

4. d-interlaced recursion algorithms. This is motivated only by analyzability in terms of the I/O theoretic approach. It is not needed in [10] for example.

The technical interest of the first three modifications is that they guarantee the following property mentioned in [4],[10].

<u>Conicity property of the adaptation law</u>: Let $\theta_\star$ be any vector with its first component > σ_0 and in the open sphere with center θ_c and radius $K\sqrt{\lambda_0/\lambda_1}$. Depending on $\theta_\star$, we define $\Psi(t)$ as:

$$\Psi(t) = (\theta(t-d) - \theta_\star)^T\phi(t-d). \qquad (2.15)$$

Then we may consider the adaptation law as an operator H_a with input ($\phi(t)$,e(t)), output $\Psi(t)$, and state ($\theta(t)-\theta_\star$), or in terms of normalized signals as $H_a : (\bar{\phi}(t),\bar{e}(t)) \to \bar{\Psi}(t)$. More interesting, independently of $\bar{\phi}(t)$ we have.

<u>Property 2.1</u>: With respect to the $\ell_2[T_0,T_1]$-norm and uniformly in T_0,T_1, the operator: $\bar{e}(t) \to \bar{\Psi}(t)$ is outside the cone with center -1 and radius $\sqrt{\lambda/\lambda+\lambda_1}$ (for a definition see [16]).

<u>Proof</u>: See [4],[17], for example.

This property has been recognized earlier [17]. However, without normalization there is no proof that the radius does not vanish, and without projection the result is established only for the $\ell_2[0,T]$-norm.

3. A CLASS OF STABILIZED PLANTS

With definitions (2.2),(2.3), the control law (2.9) may be rewritten as:

$$S(t,q^{-1})u(t) + R(t,q^{-1})y(t) = y^m(t+d) \qquad (3.1)$$

where $S(t,q^{-1})$,$R(t,q^{-1})$ are time varying polynomials in the unit delay operator q^{-1} with coefficients $s_i(t)$,$r_i(t)$, respectively. This is a linear control law. Then we restrict the plant to be "nearly" linear, or more precisely, letting any μ_0 such that

$$0 < \mu_0 < \mu. \qquad (3.2)$$

<u>Plant description assumption</u>: We assume that the following (unknown) relation exists between u(t) and y(t):

$$A(q^{-1})y(t) = B(q^{-1})u(t-1) + v(t) \qquad (3.3)$$

where $A(q^{-1})$ is a monic polynomial and $B(q^{-1})$ is a power series with coefficients b_i whose $\ell_1(\mu_0^{\frac{1}{2}})$-norm is finite:

$$\sum_{i=0}^{\infty} |b_i| \mu_0^{-i/2} < +\infty, \qquad (3.4)$$

and v(t), appearing as an extraneous signal, may incorporate nonlinearities or the effects of time variations, but is restricted to satisfy with v,V > 0 (see [14] for a discussion)

$$|v(t)| < v\rho(t+d-1)^{\frac{1}{2}} + V. \qquad (3.5)$$

In the following we note P any plant which satisfies this assumption and $\mathcal{P}$ the set of such plants.

Clearly P depends on μ, v, V.

Since we consider a controller with imposed order (n_S, n_R) it is reasonable to restrict our attention to plants for which there exists an implementable stabilizing linear time invariant controller of this order:

Set of stabilizing reduced order controllers: For ease of notation, let us define $S(q^{-1}), R(q^{-1})$ as any of the following polynomials:

$$S(q^{-1}) = s_0 + \cdots + s_{n_S} q^{-n_S} \quad (3.6)$$

$$R(q^{-1}) = r_0 + \cdots + r_{n_S} q^{-n_R} \quad (3.7)$$

and let θ be the corresponding vector

$$\theta = (s_0 \ldots s_{r_S} \; r_0 \ldots r_{n_R})^T. \quad (3.8)$$

For any plant $P \in \mathcal{P}$ we define $\Theta_S(P)$ as the following open subset of $\mathbb{R}^n$:

$$\Theta_S(P) = \{\theta \in \mathbb{R}^n / |S(q^{-1})A(q^{-1}) + q^{-1}R(q^{-1})B(q^{-1})| > 0, \quad \forall |q| \geq \mu_0^{\frac{1}{2}}\}. \quad (3.9)$$

Conversely, for any $\theta \in \mathbb{R}^n$ we define $\mathcal{P}(\theta)$ as the following subset of $\mathcal{P}$:

$$\mathcal{P}(\theta) = \{P \in \mathcal{P} / |S(q^{-1})A(q^{-1}) + q^{-1}R(q^{-1})B(q^{-1})| > 0, \quad \forall |q| \geq \mu_0^{\frac{1}{2}}\}. \quad (3.10)$$

$\Theta_S(P)$ is the set of controllers which simultaneously stabilize P (with poles within a disk of radius $\mu_0^{\frac{1}{2}}$). The dual set $\mathcal{P}(\theta)$ is the set of plants simultaneously stabilized by θ. In adaptive control, since the controller is not fixed, we are interested in union of sets $\mathcal{P}(\theta)$. However, implementation requires controllers with bounded gains. This was the goal of the projection in our adaptation law.

Implementation condition: Let Θ_i be the intersection in $\mathbb{R}^n$ of the open sphere with center θ_C, radius $K\sqrt{\lambda_0/\lambda_1}$ and the open half space $s_0 > \sigma_0$. Θ_i is given with our adaptive controller. Then we define $\mathcal{P}_\ell$ as the set of plants having at least one stabilizing reduced order controller in Θ_i:

$$\mathcal{P}_\ell = \bigcup_{\theta \in \Theta_i} \mathcal{P}(\theta). \quad (3.11)$$

Then for a plant $P \in \mathcal{P}_\ell$, let $\Theta_{is}(P)$ be the open subset of stabilizing controllers within Θ_i, we have also:

$$\mathcal{P}_\ell = \{P \in \mathcal{P} \,|\, \Theta_{is}(P) = \Theta_i \cap \Theta_S(P) \neq \phi\}. \quad (3.12)$$

$\mathcal{P}_\ell$ depends on μ, n_S, n_R, θ_C, K, σ_0, λ_0, λ_1, parameters of our adaptive controller. Limitation of $\mathcal{P}_\ell$ by Θ_i would not be very restrictive if we did not have the constraint $s_0 > \sigma_0$. This is the usual assumption about the leading coefficient in the ideally modeled case.

Now, for a given plant $P \in \mathcal{P}_\ell$, let $\theta_\star$ be any element of $\Theta_{is}(P)$. It corresponds to the following invertible operator (use (3.4),(3.10) to apply Wiener theorem (Ex. 4, p. 251 [12]))

$$C_\star(q^{-1}) = S_\star(q^{-1})A(q^{-1}) + q^{-1}R_\star(q^{-1})B(q^{-1}). \quad (3.13)$$

And use of $\theta_\star$ as a time invariant controller in (3.1) leads to the following operator between $y^m(t)$ and $y(t)$

$$H_\star(q^{-1}) = C_\star(q^{-1})q^{d-1}C(q^{-1})B(q^{-1}). \quad (3.14)$$

This operator is the operator for which a conicity condition is imposed in [7],[8]. Here we remark that if the plant delay is smaller than (d-1), then $H_\star(q^{-1})$ is noncausal. This case is not treated in these earlier works. Therefore, we represent $H_\star(q^{-1})$ as the sum of a causal operator $H_\star^C(q^{-1})$ and a polynomial $\pi_\star(q)$, that is,

$$H_\star(q^{-1}) \triangleq (\pi_0 q^{d-1} + \cdots + \pi_{d-2}q) + H_\star^C(q^{-1}). \quad (3.15)$$

The presence of adaptation will lead us to introduce the following subset of $\mathcal{P}_\ell$.

Conicity relation condition: Let $\mathcal{P}_{\ell a}$ be the subset of $\mathcal{P}_\ell$ such that for each $P \in \mathcal{P}_{\ell a}$, there exists $\theta_\star \in \Theta_{is}(P)$ such that $H_\star^C((\mu q)^{-1})$ is strictly inside the cone with center $\frac{\lambda + \lambda_1}{\lambda_1}$ and radius $\frac{\sqrt{\lambda(\lambda+\lambda_1)}}{\lambda}$. We call $\Theta_{isc}(P)$ the set of such $\theta_\star$. It is an open subset of $\Theta_{is}(P)$(therefore not reduced to one point).

Given $\mathcal{P}_\ell$, $\mathcal{P}_{\ell a}$ depends only on d, $C(q^{-1})$, μ, λ, λ_1. With $\mathcal{P}_{\ell a}$, we are now in a position to state our result.

4. MAIN THEOREM

Main theorem: Our adaptive controller (2.1)-(2.8) stabilizes any plant P of $\mathcal{P}_{\ell a}$ (in the sense of global boundedness of the I/O signals), provided

i) v in (3.5) is sufficiently small.

ii) there exists $\theta_\star \in \Theta_{isc}(P)$ leading to a polynomial $\pi_\star(q)$ as defined by (3.15) such that $\pi_\star^2 = \sum_{i=0}^{d-2} \pi_i^2$ is sufficiently small.

The expression "sufficiently small" will be quantified in our proof.

Discussion: If a usual linear time invariant controller θ (with order n_S, n_R) were used instead of our adaptive controller, only the set $\mathcal{P}(\theta)$ (see (3.10)) would lead to stability. Therefore, use of adaptation allows us to extend stability to the larger set $\mathcal{P}_\ell$. Unfortunately this extension is not free since we actually have only the set $\mathcal{P}_{\ell a}$. No equivalent restriction appears for linear time invariant controllers. As we have mentioned $\mathcal{P}_{\ell a}$ is completely parameterized in terms of d, $C(q^{-1})$, μ, λ, λ_1. It is, therefore, important to clarify the role of each of these parameters. In order to limit the size of this paper, we restrict our attention to d.

Though not often mentioned, this parameter plays an important role in the stability of adaptive schemes. d is commonly confused with the plant delay. In fact d is a parameter which can be chosen different or equal to an estimated plant delay. This choice may be guided by the conicity condition. Since we require $H_\star^C((\mu q)^{-1})$ to be strictly inside a cone which does not contain the origin, $H_\star^C((\mu q)^{-1})$ has to be minimum phase. Since

$$d = \text{plant delay} \Rightarrow H_\star^C(q^{-1}) = H_\star(q^{-1}), \quad (4.1)$$

we conclude that this choice is convenient only when the plant is minimum phase. More generally (as a complete answer to the question in [15]), if in determining the delay of a plant, there are very small responses at delays smaller or equal to d_1-1 and a larger response at delay d_1 following a step change in the input, then it is better to take d equal to d_1. More precisely, if we choose $C(q^{-1}) = 1$ and we assume that our choice of parameters allows us to find a $\theta_\star$

in the set $\Theta_{is}(P)$, corresponding to the plant $P \in \mathcal{P}_\ell$, such that

$$C_*(q^{-1}) = 1, \qquad (4.2)$$

then

$$H_*(q^{-1}) = q^{d-1}B(q^{-1}). \qquad (4.3)$$

Hence d should be chosen in order to make $H_*^c(q^{-1})$ (truncation of $B(q^{-1})$) minimum phase. Then, following our main theorem, if the rejected coefficients of $B(q^{-1})$ are sufficiently small, we will have boundedness. This means that the neglected zeros of $B(q^{-1})$ may be unstable but close to infinity in the complex plane. This clarifies the restricted use of model reference adaptive controllers. Since the objective of such controllers does not take the input into account, their use is inherently limited to plants for which output stability implies input stability. If this property may be strongly violated (fast sampling, for example) another scheme should be used.

Another consequence is that the conicity condition is about $H_*^c(q^{-1})$, i.e., a projection of $H_*(q^{-1})$. A condition on $H_*(q^{-1})$ could be nonrobust since it implies the nonrobust (with respect to useful topology) assumption: d = plant delay. On the contrary, our condition on $H_*^c(q^{-1})$ is robust. To state this result precisely, let us restrict our attention to plants such that v=0 in (3.5). Then $\mathcal{P}$ can be represented as the set of operators $(A(q^{-1})^{-1}B(q^{-1}))$ (which admit coprime factorization in terms of operators with $\ell_1(\mu_0^{\frac{1}{2}})$-finite impulse response).

Lemma 4.1: $\mathcal{P}_{\ell a}$ is an open subset of $\mathcal{P}$ supplied with the graph topology of [20].

Proof: By definition of the graph topology the application: $P \to q^{1-d}H_*$ is continuous. The same property holds for the projection: $q^{1-d}H_*(q^{-1}) \to H_*^c(q^{-1})$. To conclude we notice that "$H_*^c((\mu q)^{-1})$ is strictly inside a cone" defines an open ball in our topology. Therefore, $\mathcal{P}_{\ell a}$ is the preimage of an open set by a continuous application.

With the same arguments: the subset of $\mathcal{P}_\ell$ for which π_* is sufficiently small is open. The interpretation of this result is as follows: given an adaptive controller, if it leads to bounded signals for a plant in $\mathcal{P}_{\ell a}$, then there exists a neighborhood (in the sense of the graph topology) of plants in $\mathcal{P}_{\ell a}$ for which the signals are also bounded.

An open problem for future research is to have a more concrete idea on how $\mathcal{P}_{\ell a}$ is restrictive compared to $\mathcal{P}_\ell$. Moreover, one can easily check that filtering either $\phi(t)$ or $e(t)$ before entering the adaptation law modify $H_*(q^{-1})$ by multiplying it by this filter. Hence we have a mean to change the "size" of $\mathcal{P}_{\ell a}$. How to incorporate a priori knowledge on the plant in order to increase this size?

Proof of the Theorem: In the following sections we give the proof of our theorem. In Section 5 we give a description of the feedback systems in terms of operator. This clearly reveals two interconnected loops: the signal loop and the error loop which incorporates $H_*(q^{-1})$ in its feedback path. To solve the problem of its noncausal part we establish properties guaranteed by normalization and projection. In Section 6 we first derive bounds on the normalized error loop using the conic-relation stability theorem, this allows us to prove stability of the signal loop using a small gain like theorem

5. OPERATOR DESCRIPTION

Given a plant $\mathcal{P}_\ell$, let θ_* be any (unknown) element in its corresponding $\Theta_{is}(P)$. Let $\phi_*(t)$, $e_*(t)$ be the signals equivalent to $\phi(t)$, $e(t)$, respectively, we should obtain if θ_* were used as linear time invariant controller instead of the actual adapted $\theta(t)$. $\phi_*(t)$ and $e_*(t)$ depend on $y^m(t+d), v(t)$ via linear time invariant causal operators. Then simple computations lead to a representation of the closed loop system as in Fig. 1. $H_{\phi\Psi}(q^{-1})$ and $H_{e\Psi}(q^{-1})$ may be noncausal and in particular with definition (3.14), we have

$$H_{e\Psi}(q^{-1}) = H_*(q^{-1}). \qquad (5.1)$$

Moreover, if we write $\phi(t)$ without its component $u(t)$ as

$$\phi^r(t) = (u(t-1) \dots u(t-n_S)\ y(t) \dots y(t-n_R)), \qquad (5.2)$$

then correspondingly, we get with $q^{1-d}H_{\phi\Psi}^r(q^{-1})$ causal:

$$\phi^r(t) = -H_{\phi\Psi}^r(q^{-1})\Psi(t) + \phi_*^r(t). \qquad (5.3)$$

Since θ_* is a stabilizing regulator, each $H_{..}(q^{-1})$ is a stable operator with in particular an impulse response with finite $\ell_1(\mu_0^{\frac{1}{2}})$-norm. Hence, noting that, for $t \le i$:

$$\rho(t)^{-1}\rho(i) \le \mu^{i-t} < \mu_0^{i-t}. \qquad (5.4)$$

It follows with (3.5) the existence of positive constants ϕ_1, ϕ_2, e_1, e_2 such that (use the $\ell_2(\mu_0)$-norm):

$$\|\phi_*^2(t)\| \le \phi_1 + \phi_2 v\, \rho(t+d-1)^{\frac{1}{2}} \qquad (5.5)$$

$$|e_*(t)| \le e_1 + e_2 v\, \rho(t+d-1)^{\frac{1}{2}}. \qquad (5.6)$$

The signal $\bar{e}_*(t)$ is the input of the normalized error loop as shown by Fig. 2 when from Fig. 1 we show explicitly the normalization operation. However, due to the difficulty in handling noncausal operators, here we reflect the effects of the (possible) noncausal part of $\bar{H}_*$ as an external signal $\bar{\eta}_*(t)$:

$$\eta_*(t) = -\pi_*(q)\Psi(t). \qquad (5.7)$$

If we establish that $\bar{\eta}_*(t) \in \ell_\infty$ whatever the internal signal $\bar{\Psi}(t)$ is, then internal ℓ_∞-stability of the normalized error loop will be preserved. As shown below, both projection and normalization guarantee this key property.

Property due to projection and normalization: Whatever the adaptation law and the signals are, if $\theta(t)$, $1/s_0(t) \in \ell_\infty$, if $u(t), y(t)$ are related by both (2.8) and (3.3) and if (3.4), (3.5) are satisfied, then

i) $\bar{\Psi}(t)$ is uniformly bounded.

ii) There exists positive constants K_u, L_u, K_ρ, L_ρ, K_p, γ_p such that:

$$|u(t)| \le K_u\|\phi^r(t)\| + L_u \qquad (5.8)$$

$$\rho(t+1) \le K_\rho\, \rho(t) + L_\rho \qquad (5.9)$$

$$\bar{\eta}_*(t) \le \pi_*(\gamma_p + K_p\, \rho(t)^{-\frac{1}{2}}) \qquad (5.10)$$

$$\pi_*^2 = \sum_{i=0}^{d-2} \pi_i^2. \qquad (5.11)$$

Proof: i) is straightforward since $\theta(t)$, $\bar{\phi}(t) \in \ell_\infty$. (5.8) is derived in writing the control law in terms of $u(t)$.

(5.9): From definition of $\rho(t)$, we have

$$\sum_{j=0}^{\infty} \mu^j (u(t-d-j)^2 + y(t-d-j)^2) \le \rho(t). \qquad (5.12)$$

Then from (3.3), using this inequality and the Cauchy-Schwartz inequality, we get

$$|(y(t-d+1)|^2 \le 2(\sum_{j=0}^{\infty} \mu^{-j}(a_{j+1}^2 + b_j^2))\rho(t) + 2v(t-d+1)^2. \qquad (5.13)$$

Then (5.9) follows from (3.4), (3.5), and (5.8).

(5.10): If in the definition of $\eta(t)$ we write $\Psi(t)$ in terms of $\theta(t), \phi(t)$ and we apply the Cauchy-Schwartz inequality, we get

$$|\bar{n}_*(t)|^2 \le \pi_*^2 (\sup_t |\theta(t)-\theta_*|^2)\rho(t)^{-1} \sum_{i=1}^{d-1} |\phi(t-d+i)|^2 . \quad (5.14)$$

The conclusion follows from definition of $\rho(t)$ and recursive use of (5.9).

It follows from (5.10),(5.6) and recursive use of (5.9) that the input of the normalized error loop can be bounded as:

$$|\bar{e}_*(t) + \bar{\pi}_*(t)|^2 \le \gamma(\pi_*^2 + v^2) + K\rho(t)^{-1}. \quad (5.15)$$

6. PROOF OF THE MAIN THEOREM

Normalized Error Stability Theorem

In fact we know already that the signals of this loop belong to ℓ_∞, then only conditional properties of operators are needed (see notations). As a direct consequence of the conic-relation stability theorem of [16] (except that we have to extend it for $\ell_2[T_0,T_1]$-norm and check uniformity), we have:

Theorem 6.1: If, with respect to the $\ell_2[T_0,T_1]$-norm and uniformly in $[T_0,T_1]$, the operator H_*^{c} is conditionally strictly inside the cone with center $\lambda+\lambda_1/\lambda_1$ and radius $\sqrt{\lambda(\lambda+\lambda_1)}/\lambda_1$, then the operator, $\bar{e}_*(t) + \bar{n}_*(t) \to \bar{\psi}(t)$ is conditionally $\ell_2[T_0,T_1]$-finite gain uniformly in T_0,T_1.

Corollary 6.1: Under this condition it follows from (5.15) that there exist positive constants β_e, γ_e, K_e such that

$$(\|\bar{\psi}(t)\|_{T_0}^{T_1})^2 \le \gamma_e(\rho^2+v^2)(T_1-T_0) + K_e(\|\rho(t)^{-\frac{1}{2}}\|_{T_0}^{T_1})^2 + \beta_e. \quad (6.1)$$

As a consequence we consider the adaptation law as an operator: $\rho(t)^{\frac{1}{2}} \to \psi(t)$ for which (neglecting K_e for the time being), (6.1) provides a bound on the average value of its instantaneous gain $\psi(t)/\rho(t)^{\frac{1}{2}}$. This corollary completely describes the behavior of the normalized error loop. Unfortunately, its assumption is signal dependent since H_*^c is signal dependent. However, the following result can be proved:

Lemma 6.1: If an operator H has an impulse response with finite $\ell_1(\mu_0^{\frac{1}{2}})$-norm and if it lies strictly inside the cone with center c, radius r with respect to the $\ell_2(\mu)$-norm, then $\bar{H}$ is exponentially stable and conditionally strictly inside the same cone with respect to the $\ell_2[T_0,T_1]$-norm and uniformly in T_0,T_1.

Proof: See [19].

We conclude that Theorem 6.1 holds for any plant $P \in P_{\ell a}$, choosing θ_* in $\Theta_{isc}(P)$.

Signal Loop Stability Theorem

From Corollary 6.1 we can represent the signal loop as shown in Fig. 3. We will apply a small gain theorem on this loop. However, since we have information only on average instantaneous gain we have to derive a special theorem.

Bounds on the feedback path: From (5.3),(5.5) and the $\ell_2(\mu)$-stability of $H_{\phi\psi}^r(q^{-1})$, it follows that constants γ and β_1 exist such that

$$\|\phi^r(t)\|_{T,\mu} \le \gamma_1 \|\psi(t+d-1)\|_{T,\mu} + \phi_1 v \|\rho(t+d-1)^{\frac{1}{2}}\|_{T,\mu} + \phi_2 \mu^{-T-2} + \beta_1 \quad (6.2)$$

and if we take (5.8) into account we may write:

$$\|\phi(t)\|_{T,\mu} \le \gamma_2 \|\psi(t+d-1)\|_{T,\mu} + \gamma_3 v \|\rho(t+d-1)^{\frac{1}{2}}\|_{T,\mu} + \beta_2 \mu^{-T}. \quad (6.3)$$

To complete our bounds, we note with the definition of $\rho(t)$ that

$$\mu^{-T}\rho(T) \le \rho(0) + \sigma\mu^{-T-2} + \|\phi(t-d)\|_{T,\mu}^2. \quad (6.4)$$

Hence we have obtained the following theorem.

Theorem 6.2: If the plant belongs to P_ℓ, then there exist constants γ_o, β_o, γ such that

$$\mu^{-T}\rho(T) \le \gamma_o \|\psi(t)\|_{T-1,\mu}^2 + \gamma v^2 \|\rho(t)^{\frac{1}{2}}\|_{T-1,\mu}^2 + \beta_o \mu^{-T}. \quad (6.5)$$

To complete the proof of our main theorem, we need the following result: if (5.9),(6.1),(6.5) hold then $\rho(t) \in \ell_\infty$. This result is true provided

$$\gamma_o \gamma_e(\pi_*^2 + v^2) + \gamma v^2 < \log \frac{1}{\mu}. \quad (6.6)$$

This explains why π_* and v must be sufficiently small. The proof of this result relies on the Bellman-Gronwall lemma (see p. 254 [12]). It can be found in [19].

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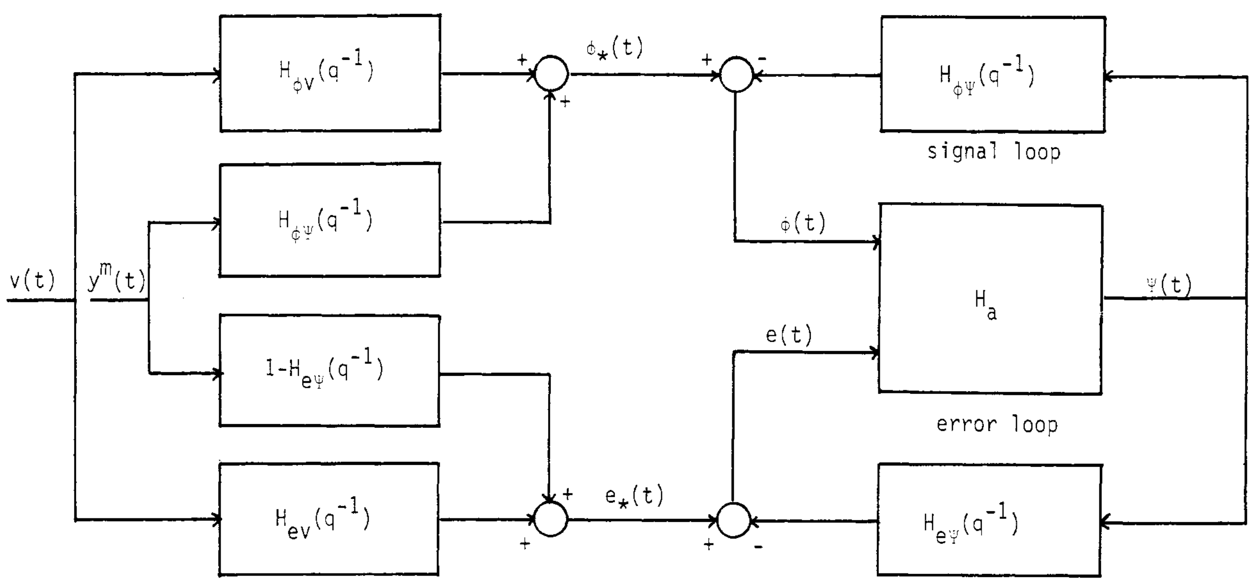

Figure 1

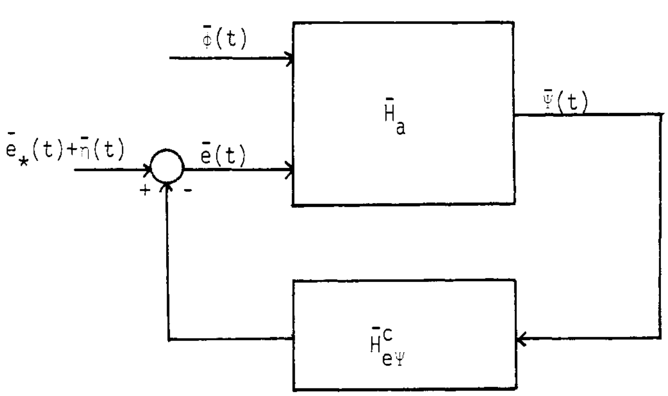

Figure 2

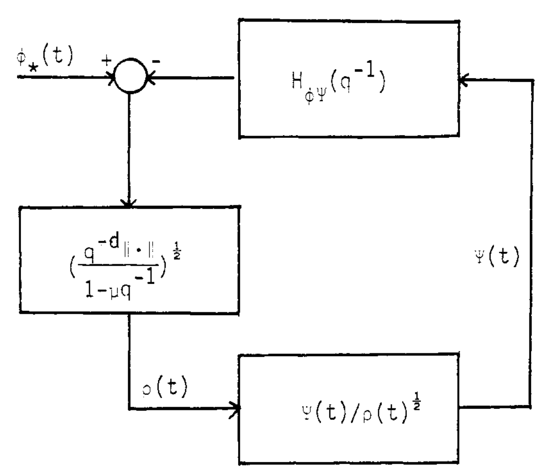

Figure 3