ROBUSTNESS OF INDIRECT ADAPTIVE CONTROL BASED ON POLE PLACEMENT DESIGN

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Abstract. We study the robustness of an indirect adaptive control scheme based on pole placement design with respect to unmodeled dynamics, non linearities, time variations or to ill-modeled measurement disturbances. The known results about this problem show that classical adaptation mechanisms have to be modified. Here introducing a regularized normalized least squares algorithm with a projection, we state a boundedness property in presence of mismodeling quantified in terms of noise to signal ratio. However an extra condition about the controllability of the adapted model is required.

Keywords. Adaptive control; closed loop systems; control theory; discrete time systems; iterative methods; parameter estimation; pole placement; stability; system order reduction; robustness.

INTRODUCTION

Most of the proofs of stability of adaptive control algorithms available today have been established for linear time invariant plant with known order and well modeled disturbances (bounded or movering average). There still remains a significant gap between these theoretical methodologies and the potential applications. In particular it is important to determine the robustness of adaptive schemes with respect to unmodeled dynamics, non linearities, time variations or to ill-modeled measurement disturbances.

Several attempts have been made to formulate and analyse such problems (Kreisselmeier, 1982 ; Gawthrop, Lim, 1982 ; Praly, 1983 a, 1983 b ; Ortega, Landau, 1983). Among them let us mentionn Ioannou and Kokotovic (1982) who study a singularly perturbed continuous time MRAC scheme and using a Lyapounov formulation exhibit an upperbound of the admissible parasitic time constants in terms of initial conditions. Kosut and Friedlander (1982) study an MRAC scheme for a plant with known DC gain and relative degree less than one and apply I/O stability concepts for interconnected blocks to characterize plant uncertainty by conic sector.

However the assumptions required in these results are still too restrictive. In fact, as mentionned by Rohrs and co-workers (1982), one of the major difficulties is due to the existence in classical adaptation mechanisms of infinite gain operators (see Remark 2 below) : the operator $\mathcal{H}_p$ between the output error and the adapted parameters, the operator $\mathcal{H}_e$ between the output error and the estimation error. Unfortunately the inverse gain of $\mathcal{H}_e$ limits the admissible unmodeled effects (Ortega, Landau, 1983; Gawthrop, Lim, 1982). And, in the presence of unmodeled dynamics, $\mathcal{H}_p$ may produce unbounded adapted parameters. It may follow unboundedness of the complete system as mentionned by Egardt (1979) or it makes inaccurate the cornerstone assumptions used by Kreisselmeier (1982).

To limit the gain of $\mathcal{H}_p$, Egardt (1979) and Narendra, Kreisselmeier (1982) propose to keep the estimated parameters inside a compact set using a projection. This solution only requires an a priori bound of these parameters (necessarily introduced by computer) and does not modify the initial control objective. About the gain of $\mathcal{H}_e$, we have proposed to use a normalized least squares algorithm as adaptive mechanism (Praly, 1983 a, 1983 b). As a consequence the gain of $\mathcal{H}_e$ is bounded and the unmodeled effects are characterized in term of noise to signal ratio.

To show how projection and normalized least squares algorithm are sufficient to get robustness of adaptive schemes with respect to a very wide class of unmodeled effects, we will here study the indirect adaptive control scheme based on pole placement design proposed by Goodwin and Sin (1981). In Praly, 1983c) such a study is presented for a direct adaptive control scheme.

ROBUSTNESS PROBLEM STATEMENT

Consider a plant with $u(t)$, $y(t)$ as scalar input and output respectively. We define a model by choosing an integer n and a vector θ

$$\theta = (-a_1 \; \dots \; -a_n b_1 \; \dots \; b_n)^T \qquad (1)$$

We call residuals the error $w(t)$ between the true output $y(t)$ and the modeled output :

$$w(t) = y(t) - \theta^T\Phi(t) \quad (2)$$

where $\Phi(t)$ is the following vector

$$\Phi(t) = (y(t-1) \ldots y(t-n)u(t-1) \ldots u(t-n))^T \quad (3)$$

Let $A(q^{-1}), B(q^{-1})$ be polynomials defined from θ as :

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n} \quad (4)$$

$$B(q^{-1}) = b_1 q^{-1} + \ldots + b_n q^{-n} \quad (5)$$

The following assumption about the plant will be used :

AP : Given an integer n, a vector θ_o and a positive scalar ρ_o, there exists unknown relatively prime polynomials $A^*(q^{-1}), B^*(q^{-1})$ such that :

i) $\|\theta^* - \theta_o\| \leq \rho_o$ (6)

ii) The corresponding residuals as defined by eq.(2) satisfy :

$$\left|\frac{w(t)}{s(t)}\right| \leq \eta_w \quad (7)$$

where :

$$s(t) = \sigma\, s(t-1) + \mathrm{Max}\{\|\Phi(t)\|, s\} \quad (8)$$

$$0 < \sigma < 1 \quad , s > 0 \quad (9)$$

Inequality (7) characterizes a very wide class of unmodeled effects : $w(t)$ may contain nonlinearities $f(y(t-i),u(t-j))$, higher order terms $(a_{n+i}y(t-n-i)+b_{n+j}u(t-n-j))$ or time variations $((a_i-a_i(t))y(t-i)+(b_j-b_j(t))u(t-j))$ ...

With this assumption the robustness problem may be formulated as follows : find an adaptive control law such that :

i) $\exists\eta > 0 : \eta_w < \eta \Rightarrow u(t), y(t)$ are uniformly bounded.

ii) If there is no residuals, the output $y(t)$ tracks some reference output $y^M(t)$ as "well" as possible.

Note that the second part of this problem deals with a tracking property with its inherent problems of delays and non minimum phase.

ADAPTIVE CONTROL

Following Goodwin and Sin (1981), let $A^M(q^{-1})$ be a strictly stable polynomial

$$A^M(q^{-1}) = 1 + a_1^M q^{-1} + \ldots + a_{2n-1}^M q^{-2n+1} \quad (10)$$

For any vector θ (with the primeness condition), we define a vector ψ :

$$\psi = (f_o \ldots f_{n-1}\, 1\, e_1 \ldots e_{n-1})^T \quad (11)$$

as solution of the following linear system (Diophantine equation) :

$$\begin{bmatrix} 0 & & & 1 & & \\ b_1 & 0 & & a_1 & 1 & \\ \cdot & \cdot & 0 & \cdot & \cdot & 1 \\ b_n & & b_1 & a_n & & a_1 \\ & \cdot & & & \cdot & \\ & & b_n & & & a_n \end{bmatrix} \psi = \begin{bmatrix} 1 \\ a_1^M \\ \cdot \\ \cdot \\ \cdot \\ a_{2n-1}^M \end{bmatrix} \quad (12)$$

We note symbolically

$$\mathcal{A}(\theta)\psi = \mathcal{B} \quad (13)$$

To solve the robustness problem, we propose the following indirect adaptive control scheme :

$$\nu(t) = y(t) - \theta(t-1)^T\Phi(t) \quad (14)$$

$$g(t) = \frac{\alpha(t)}{\mu s^2(t) + \Phi(t)^T P(t-1)\Phi(t)} \quad (15)$$

$$\theta(t-\tfrac{1}{2}) = \theta(t-1) + g(t)P(t-1)\Phi(t)\nu(t) \quad (16)$$

$$P(t-\tfrac{1}{2}) = P(t-1) - g(t)P(t-1)\Phi(t)\Phi(t)^T P(t-1) \quad (17)$$

$$\theta(t) = \theta_o + \mathrm{Min}\left\{1, \frac{\rho(t)}{\|\theta(t-\frac{1}{2})-\theta_o\|}\right\}(\theta(t-\tfrac{1}{2})-\theta_o) \quad (18)$$

$$P(t) \geq P(t-\tfrac{1}{2}) \quad (19)$$

$$\psi(t) = \mathcal{A}(\theta(t))^{-1}\mathcal{B} \quad (20)$$

$$\psi(t)^T\Phi(t+1) = E(t)A^M(q^{-1})y^M(t) \quad (21)$$

$$E(t) = \begin{cases} \dfrac{1}{B(t,1)} & \text{if } |B(t,1)| > \varepsilon \\ 1 & \text{if not} \end{cases} \quad (22)$$

where i) eq.(19) means that $P(t)$ is any positive symetric definite matrix greater than $P(t-\frac{1}{2})$ and such that :

$$0 < \Lambda_o \leq \lambda\min P(t) \leq \lambda\max P(t) \leq \Lambda_1 \quad (23)$$

For instance we can take

$$P(t) = \beta\, P(t-\tfrac{1}{2}) + (1-\beta)\,\Lambda_1\, I \quad (24)$$

ii) $\alpha(t)$, $\rho(t)$ are chosen such that

$$0 < \alpha \leq \alpha(t) \leq 1 \quad (25)$$

$$\left.\begin{matrix} \rho(t-1) - \tau\,\|\theta(t-\frac{1}{2})-\theta(t-1)\| \\ \dfrac{\Lambda_1}{\Lambda_o}\,\rho_o \end{matrix}\right\} \leq \rho(t) \leq \rho \quad (26)$$

$$\det\, \mathcal{A}(\theta(t)) \neq 0 \quad (27)$$

iii) $B(t,1) = \sum_{i=1}^{n} b_i(t)$ (28)

Remark 1 : Eq. (27) is always possible if $\det \mathcal{A}(\theta_o)$ and $\det \mathcal{A}(\theta(0))$ are different from zero.
We have the following :

Lemma 1 : subject to assumption AP, we have :

$$\|\theta(t)\| \leq M_\theta \qquad (E1)$$

$$\forall(q,k),\ \sum_{t=q+1}^{q+k} \frac{|\nu(t)|}{s(t)} \leq \sqrt{k}\, M_\nu + k\, L_\nu\, \eta_w \qquad (E2)$$

$$\forall(q,k),\ \sum_{t=q+1}^{q+k} \|\theta(t)-\theta(t-1)\| \leq \sqrt{k}\, \overline{M}_\theta + k\, L_\theta\, \eta_w \qquad (E3)$$

where M_θ, M_ν, $\overline{M}_\theta$ are positve constants independants of η_w and :

$$L_\nu^2 = 1 + \frac{\Lambda_1}{\mu} \qquad (29)$$

$$L_\theta^2 = (2+\tau)^2 \frac{\Lambda_1^2}{(\mu+\Lambda_1)\mu} \qquad (30)$$

Proof : see appendix.

Remark 2 : As discussed in introduction, the projection (18) limits the gain of $\mathcal{H}_p : \frac{\nu(t)}{s(t)} \rightarrow \theta(t)$ and the presence of $s(t)$ in eq. (15) limits the gain of $\mathcal{H}_e : \frac{\nu(t)}{s(t)} \rightarrow \frac{(\theta(t)-\theta^*)^T\Phi(t)}{s(t)}$. In particular, $\mathcal{H}_e$ is exterior to the conic sector with center - 1 and radius $\frac{1}{L_\nu}$ (compare with Ortega, Landau (1983)).

Lemma 2 : Subject to assumption AP, if there exists a strictly positive constant δ such that :

$$|\det \mathcal{A}(\theta(t))| \geq \delta \qquad (31)$$

then we have :

$$\|\psi(t)\| \leq M_\psi \qquad (C1)$$

$$\forall i \leq n\ ,\ \|\psi(t)-\psi(t-i)\| \leq L_\psi \|\theta(t)-\theta(t-i)\|. \qquad (C2)$$

Proof : with assumption (31) $\psi(t)$ is a differentiable function of $\theta(t)$.

Remark 3: The choice of $\alpha(t)$, $\rho(t)$ such that ineq. (27) is met, does not prevent $\lim_{t\rightarrow\infty} \inf \det \mathcal{A}(\theta(t))$ from being null.
Therefore assumption (31) is an extra condition to be considered for the forthcoming boundedness study.

With these bounds M_θ, M_ν, M_ψ and these gains L_θ, L_ν, L_ψ, we are in position to state our main result :

Theorem : Subject to assumption AP, if the adaptive scheme defined by eq. (14) to eq. (22) is applied and is such that ineq. (31) is met, then the robustness problem is solved :
i) if we have :

$$((M_\psi+L_\psi M_\theta)L_\theta + \Pi\, M_\psi L_\nu)\eta_w < \frac{(1-\xi)(1-\sigma)}{\gamma\, n^2} \qquad (32)$$

then $u(t)$, $y(t)$ are uniformly bounded. Here ξ is the spectral radius of $A^M(q^{-1})$, γ is a positive constant which depends on $A^M(q^{-1})$, Π is a positive constant which depends on n, σ, ξ.

ii) Moreover if η_w is equal to zero, then we have :

$$\lim_{t\rightarrow\infty} A^M(q^{-1})y(t) - \sum_{i=1}^{n} b_i(t)r(t-i) = 0 \qquad (33)$$

$$r(t) = E(t)A^M(q^{-1})y^M(t) \qquad (34)$$

Proof : see appendix.

Discussion : Let us study expression (32). About the control part of the scheme, we have the terms $(M_\psi+L_\psi M_\theta)$, M_ψ. Given our assumption AP (i.e. M_θ,σ), $A^M(q^{-1})$ (i.e. ξ,γ) should be chosen such that $(1-\xi)$ is greater and M_ψ, L_ψ are smaller. In this stability-robustness compromise , not only the amplitude of the controller parameters but also its sensitivity with respect to variations of θ appear.

About the adaptation part of the scheme, we have L_θ, L_ν. The less L_θ, L_ν are, the more robust the scheme is. However looking at eq. (29), (30) we see that L_θ, L_ν are smaller if $\frac{\Lambda_1}{\mu}$ is smaller i.e. if the adaptation ability is reduced. Therefore to the classical stability-robustness compromise an adaptation-robustness compromise is added for adaptive control schemes.

CONCLUSION

We have analysed stability of the indirect adaptive control scheme proposed by Goodwin and Sin (1981), when the residual between the plant and its assumed linear model is ill-modeled. More precisely we have shown the boundedness of the input-output signals when the residual to signal ratio meets :

$$\frac{|w(t)|}{s(t)} \leq \eta \qquad (35)$$

where $w(t)$ is the residual, $s(t)$ is the norm of the input-output signals passed through a first order filter and η is a bound which can be computed from the scheme characteristics.

To get this result we have been led to introduce a projection and a normalization in the adaptation algorithm. In particular we have shown that these modifications limit the gain of the infinite gain operators mentionned by Rohrs, and co-workers (1982) as leading to instability. This new algorithm reaches the initial control objective when there is no residual.

As an important consequence of our study, we have shown that not only the classical stability-robustness compromise, but also an adaptation-robustness compromise has to be made in adaptive control.

Note that for our result to hold, we need an extra condition about the adaptive scheme. It concerns the controllability of the estimated model.

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APPENDIX

Proof Of Lemma 1

The technique used here is by now standard and we only point out the major steps : Let $V(t)$ be defined as follows :

$$V(t) = (\theta(t)-\theta^*)^T P(t)^{-1} (\theta(t)-\theta^*) \quad (A1)$$

From eq.(2), eq.(14) to eq.(22), and projection property, the following relations can be derived :

$$V(t-\tfrac{1}{2}) = V(t-1) + g(t)\left(\frac{w(t)^2}{1-g(t)\Phi(t)^T P(t-1)\Phi(t)} - \nu(t)^2\right) \quad (A2)$$

$$V(t) \leq V(t-\tfrac{1}{2}) \quad (A3)$$

$$\|\theta(t)-\theta^*\| \leq \rho(t) + \rho_o \quad (A4)$$

$$\|\theta(t)-\theta(t-1)\| \leq (2+\tau) g(t) |\nu(t)| \, \|P(t-1)\Phi(t)\| \quad (A5)$$

Ineq. (A4) directly leads to E1 and with ineq.(23) yields the boundedness of $V(t)$. Then eq. (A2) and ineq. (A3),(A5),(23) lead to :

$$\left.\begin{array}{l} \frac{\mu}{\alpha}\left(1+\frac{\Lambda_1}{\mu}\right)(V(t-1)-V(t)) \\ + \left(1+\frac{\Lambda_1}{\mu}\right)\left(\frac{w(t)}{s(t)}\right)^2 \end{array}\right\} \geq \left(\frac{\nu(t)}{s(t)}\right)^2 \quad (A6)$$

$$\left.\begin{array}{l} \Lambda_1 \frac{\mu}{\mu+\Lambda_1}(V(t-1)-V(t)) \\ + \frac{\Lambda_1^2}{\mu(\mu+\Lambda_1)}\left(\frac{w(t)}{s(t)}\right)^2 \end{array}\right\} \geq \frac{\|\theta(t)-\theta(t-1)\|^2}{(2+\tau)^2} \quad (A7)$$

Use Schwartz inequality to get E2, E3.

Proof Of Theorem

Notations : Let $\|.\|$ be the usual euclidian norm and $|||.|||$ be any other equivalent norm. We have

$$\gamma_1 \|.\| \leq |||.||| \leq \gamma_2 \|.\| \quad (A8)$$

Lemma A : Let $\varphi(t)$, $\xi(t)$ be sequences of positive real numbers such that :

$$\varphi(t+1) \leq \zeta(t)\,\varphi(t) + M_\varphi \quad (A9)$$

$$\forall (q,k), \sum_{t=q+1}^{q+k} \zeta(t) \leq \sqrt{k}\, M_\zeta + k\eta_\zeta \quad (A10)$$

If we have

$$0 \leq \eta_\zeta < 1 \quad (A11)$$

Then $\varphi(t)$ is uniformly bounded.

Proof : From assumption (A9), it follows

$$\left.\begin{array}{l}(\prod_{t=0}^{q}\zeta(t))\varphi(0)\\ +(1+\sum_{k=1}^{q}\prod_{t=q+1-k}^{q}\zeta(t))M_\varphi\end{array}\right\} \geq \varphi(q+1) \quad (A12)$$

But with assumption (A10), let

$$\lambda = \exp-\left(\frac{1-\eta_\zeta}{2}\right) \quad (A13)$$

We have

$$k > \left(\frac{2M_\zeta}{1-\eta_\zeta}\right)^2 = K \Rightarrow \prod_{t=q+1-k}^{q}\zeta(t) < \lambda^k \quad (A14)$$

$$k \leq K \Rightarrow$$

$$\prod_{t=q+1-k}^{q}\zeta(t) \leq \exp\left(\frac{2M_\zeta^2}{1-\eta_\zeta}\right) = M \quad (A15)$$

It follows :

$$\varphi(q+1) \leq \lambda^{q+1}\varphi(0) + (1+KM+\lambda^{K+1}\frac{1-\lambda^{q-K}}{1-\lambda})M_\varphi \quad (A16)$$

Step 1. (A closedloop state space representation) : Let $(-a_i(t), b_i(t))$ (resp. $(f_i(t), e_i(t))$ be the i components of $\theta(t)$ (resp. $\psi(t)$). From the Diophantine equation (20) applied to $u(t)$ and $y(t)$, it follows :

$$\left.\begin{array}{l}\sum_{i=0}^{n-1} e_i(t)(y(t-i)-\theta(t)^T\Phi(t-i))\\ +\sum_{i=1}^{n} b_i(t)\psi(t)^T\Phi(t+1-i)\end{array}\right\} = A^M(q^{-1})y(t) \quad (A17)$$

$$\left.\begin{array}{l}\sum_{i=0}^{n-1} f_i(t)(y(t-i)-\theta(t)^T\Phi(t-i))\\ -\sum_{i=0}^{n} a_i(t)\psi(t)^T\Phi(t+1-i)\end{array}\right\} = A^M(q^{-1})u(t) \quad (A18)$$

and with eq. (14), (21), (34) this yields :

$$\left.\begin{array}{l}\sum_{i=0}^{n-1} e_i(t)\,v(t-i)\\ +\sum_{i=1}^{n} e_i(t)(\theta(t-i-1)-\theta(t))^T\Phi(t-i)\\ +\sum_{i=0}^{n} b_i(t)r(t-i)\\ +\sum_{i=1}^{n} b_i(t)(\psi(t)-\psi(t-i))^T\Phi(t+1-i)\end{array}\right\} = A^M(q^{-1})y(t) \quad (A19)$$

$$\left.\begin{array}{l}\sum_{i=0}^{n-1} f_i(t)v(t-i)\\ \sum_{i=0}^{n-1} f_i(t)(\theta(t)-\theta(t-i-1))^T\Phi(t-i)\\ \sum_{i=0}^{n} a_i(t)r(t-i)\\ \sum_{i=1}^{n} a_i(t)(\psi(t)-\psi(t-i))^T\Phi(t+1-i)\end{array}\right\} = A^M(q^{-1})u(t) \quad (A20)$$

Then let $X(t)$ be the following vector

$$X(t) = (y(t-1).y(t-2n+1)u(t-1).u(t-2n+1))^T \quad (A21)$$

We can rewrite eq.(A19), (A20) in :

$$X(t+1) = (F+\Delta F_t)X(t)+\Psi_t\Delta(t)+\theta_t R(t) \quad (A22)$$

where F is a companion matrix with characteristic polynomial $A^M(q^{-1})^2$; Ψ_t includes the controller parameters $e_i(t), f_i(t)$; θ_t includes the estimated parameters $a_i(t), b_i(t)$; ΔF_t incorporates the following differences :

$e_i(t)(\theta(t-i-1)-\theta(t))$, $b_i(t)(\psi(t)-\psi(t-i))$, $f_i(t)(\theta(t)-\theta(t-i-1))$, $a_i(t)(\psi(t)-\psi(t-i))$.

$$\Delta(t) = (v(t)\ \ldots\ v(t-n+1))^T \quad (A23)$$

$$R(t) = (r(t)\ \ldots\ r(t-n))^T \quad (A24)$$

With the strict stability of $A^M(q^{-1})$, there exists a norm $|||.|||$ such that :

$$|||X(t+1)||| \leq \left\{\begin{array}{l}\xi|||X(t)|||\\ +|||\Delta F_t X(t)+\Psi_t\Delta(t)+\theta_t R(t)|||\end{array}\right. \quad (A25)$$

with

$$\xi < 1 \quad (A26)$$

Step 2. (some inequalities) : Using E1, C1 of lemma 1,2 and the norm equivalence (A8), we have :

$$|||\Delta F_t X(t)||| \leq \frac{\gamma_2}{\gamma_1}\|\Delta F_t\|.|||X(t)||| \quad (A27)$$

$$|||\Psi_t\Delta(t)||| \leq \gamma_2 M_\psi \|\Delta(t)\| \quad (A28)$$

$$|||\theta_t R(t)||| \leq \gamma_2 M_\theta \|R(t)\| \quad (A29)$$

From the definitions of $\Phi(t)$, $X(t)$, we have :

$$|||X(t)||| \geq \gamma_1\|\Phi(t)\| \quad (A30)$$

Introducing this inequality in the definition of $s(t)$ yields

$$s(t) \leq \sigma\, s(t-1) + \frac{1}{\gamma_1}|||X(t)||| + s \quad (A31)$$

On the other hand, from the definition of $\Delta(t)$ and property E2 of lemma 2, we have :

$$\forall(q,k),$$

$$\sum_{t=q+1}^{q+k}\frac{\|\Delta(t)\|}{s(t)} \leq \frac{1}{\sigma^{n-1}}\,\frac{1-\sigma^n}{1-\sigma}(\sqrt{k}M_v+kL_v\eta_w) \quad (A32)$$

In the following, let us note :

$$x(t) = |||X(t)||| \quad (A33)$$

$$\gamma = \frac{\gamma_2}{\gamma_1} \quad (A34)$$

Step 3. (use of Lemma A) : let us put together ineq. (A25), (A31) to get the following system :

$$x(t+1) \leq \begin{cases} (\xi+\gamma\|\Delta F_t\|)x(t) \\ +\gamma_2(M_\psi \frac{\|\Delta(t)\|}{s(t)} s(t)+ M_\theta\|R(t)\|) \end{cases} \quad (A35)$$

$$s(t) \leq \sigma\, s(t-1) + \frac{1}{\gamma_1}\, x(t) + s$$

With (A32) and property E1, C1 of lemma 1, 2, $\frac{\|\Delta(t)\|}{s(t)}$ and $\|\Delta F_t\|$ are bounded. Then there exists M_x, M_s such that :

$$x(t+1) \leq \begin{cases} (\xi+\gamma\|\Delta F_t\|+\gamma M_\psi \frac{\|\Delta(t)\|}{s(t)})x(t) \\ + \sigma\, \gamma_2\, M_\psi \frac{\|\Delta(t)\|}{s(t)}\, s(t-1) + M_x \end{cases} \quad (A36)$$

$$s(t) \leq \frac{1}{\gamma_1}\, x(t) + \sigma\, s(t-1) + M_s$$

Let $\varphi(t)$ be defined as :

$$\varphi(t) = x(t) + \gamma_1\, \sigma(1-\xi)s(t-1) \quad (A37)$$

We get from (A36) :

$$\zeta(t) = \xi+\gamma\|\Delta F_t\|+ \frac{\gamma}{1-\xi}M_\psi \frac{\|\Delta(t)\|}{s(t)} + \sigma(1-\xi) \quad (A38)$$

Note that from the definition of ΔF_t and property E1, C1 of lemma 1, 2, we have :

$$\left. \begin{array}{l} M_\psi \sum_{i=0}^{n-1} \|\theta(t)-\theta(t-i-1)\| \\ + M_\theta \sum_{i=1}^{n} \|\psi(t)-\psi(t-i)\| \end{array} \right\} \geq \|\Delta F_t\| \quad (A39)$$

Then from property E3, C2, we get

$$\sum_{t=q+1}^{q+k} \|\Delta F_t\| \leq n^2(M_\psi+L_\psi M_\theta)(\sqrt{k}\overline{M}_\theta + k\, L_\theta \eta_w) \quad (A40)$$

Hence to meet assumption (A9) of lemma A, using using ineq. (A38), (A32),(A40) we let :

$$M_\xi = \begin{cases} + \gamma n^2(M_\psi+L_\psi M_\theta)\overline{M}_\theta \\ + \frac{\gamma}{1-\xi}\, \frac{1}{\sigma^{n-1}}\, \frac{1-\sigma^n}{1-\sigma}\, M_\psi\, M_\nu \end{cases} \quad (A41)$$

$$\eta_\zeta = \xi+\sigma(1-\xi) + \gamma\eta_w[(M_\psi+L_\psi M_\theta)n^2 L_\theta + \frac{1-\sigma^n}{\sigma^{n-1}(1-\xi)(1-\sigma)} M_\psi L_\nu] \quad (A42)$$

Then we conclude that $\varphi(t)$ is bounded if assumption (A10) is met. In this condition $s(t)$ is bounded and if η_w is equal to zero we obtain eq. (33) from properties E2, E3 and eq. (A19).