MIMO STOCHASTIC ADAPTIVE CONTROL: STABILITY AND ROBUSTNESS

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ABSTRACT This paper proves boundedness of indirect adaptive control schemes even when the residuals between the true plant and the assumed linear time invariant model are linearly dominated by the information vector. By the way we take into account such problems as higher order plant, non linearities, time variation effects, plant or measurement disturbances

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I INTRODUCTION

A commonly asked question concerning adaptive control systems is: while neglecting all other aspects of comparison, do they have at least the stability properties corresponding to those of linear control systems. Especially what happens when the true plant differs from the assumed linear time invariant model in that it is higher order, contains nonlinearities or is subject to plant or measurement disturbances?

For linear control, this problem has received considerable attention in the recent automatic control litterature (see [1] to [5] for example). Unfortunately this is not the case of adaptive control. It can be explained by the great difficulty of this problem since the unmodeled residuals between the plant and the assumed linear model are coupled through the controller.

In fact the adaptive control problem has to be restated in order to relaxe some requirements on the controller and in this way to allow fewer assumptions to be made regarding the plant. This approach leads to the study of bounded error adaptive control in which, instead of perfect output matching, a bounded error between plant and (implicit or explicit) reference model is all that is required

An interesting result is given in [6] an indirect adaptive scheme is shown to have stable domains even when there are the above mentionned residuals. However stability is proved locally. In [7] Narendra and Peterson show the advantages due to introduction of non linearities into classical adaptive laws suchas dead zone or projection into a bounded area (see also [11]). An other kind of nonlinearity is discussed in [8], here the variation of adaptive gain with respect to the input or output signals is noticed. In [9] Kokotovic and Ioannou expose the relevance of filtering the plant output in order to reduce the observability of the residuals. Such filters are also used to change the adaptive law gradient direction (with a positiveness condition) or together with parallel feed forward they play the role of pre or post-compensator modifying the apparent poles and zeros of the plant (see [12], [13]).

In [10] we have integrated most of these remarks in a specific direct adaptive scheme and proved global boundedness when the residuals are linearly dominated by input and output signals ie.

$$\|\omega_n\| \leq \epsilon \quad \mathrm{Max}\ \{\phi, \|\phi_n\|\} \qquad (1)$$

where w_n (resp ϕ_n) represents residuals (resp. input and output signals), ϕ is a positive constant and ε is a small positive constant whose upper bound is given by the closed loop characteristics (without residuals)

Here we state such a result for indirect adaptive schemes and thus we extend the results of [21] to a very wide class of ill-modeled systems. Then we may say that adaptive control has at least the same qualitative robustness properties as time invariant linear control.

In section II, we give a boundedness theorem for time varying systems. In section III we show how an adaptive law meets the conditions of this theorem Section IV explains the assumptions in terms of robustness and section V draws conclusions.

II LINEAR TIME VARYING CONTROL

II.1 MIMO System representation

Consider a MIMO (multi input - multi output) system for which, at time n, we let u_n be the control input vector (in $\mathbb{R}^m$) and y_n be the output vector (in $\mathbb{R}^\ell$). Let ϕ_n be the vector given as follows:

$$\phi_n^t = \left(y_{n-1}^t \cdots y_{n-N_a}^t \quad u_n^t \cdots u_{n-N_b}^t \right) \tag{2}$$

where N_a, N_b are time invariant integers.

We assume the following representation

There exists a time varying block matrix θ_n

$$\theta_n^t = \left(-A_n^1 \ \cdots \ -A_n^{N_a} \ B_n^0 \ \cdots \ B_n^{N_b} \right) \tag{3}$$

such that, if we let

$$\varepsilon_n = y_n - \theta_n^t \phi_n \tag{4}$$

we have

HS1: $\quad \| \theta_n \| < M_1$

HS2 There exists a sequence s_n of positive real numbers such that.

$$\operatorname{Max}\{s, \|\phi_n\|\} \leq s_n \leq d\, s_{n-1} + \operatorname{Max}\{s, \|\phi_n\|\} + S \quad (5)$$

with d, s, S such that

$$d < 1\,, \quad s > 0, \quad S > 0 \quad (6)$$

and

HS2.1: $\frac{\|\varepsilon_n\|}{s_n}$ has the property of mean η_1-smallness relatively to s_n (see definition below)

HS2.2 $\|\theta_n - \theta_{n-1}\|$ has the property of mean η_2-smallness relatively to s_n.

We have used the following definition (see [10]):

Definition A sequence v_n of positive real numbers is said to have the property of mean η-smallness relatively to s_n iff.

i) v_n is bounded

ii) $\exists (\Sigma, K)$: $\forall k > K$, $\forall q$
such that : $\forall n \in [q+1, q+k]$ $\quad s_n \geq \Sigma$
then $\frac{1}{k} \sum_{n=q+1}^{q+k} v_n < \eta$ (7)

II.2 Control law

Let Ψ_n be defined as the following block matrix

$$\Psi_n^t = \left(D_n^0 \ldots D_n^{N_a-1} \; I \; C_n^1 \ldots C_n^{N_b} \right) \tag{8}$$

where I is the identity matrix.

We formally compute the next stage control u_{n+1} as follows:

$$\Psi_n^t \, \phi_{n+1} = Q_n(b) \, y_n^* \tag{9}$$

here y_n^* is a bounded reference output

$Q_n(b)$ is a $(m \times \ell)$ polynomial matrix with b as the backward shift operator

$$b y_n^* = y_{n-1}^* \tag{10}$$

In practice Ψ_n, $Q_n(b)$ are obtained from θ_n using classical design method such as pole placement [14] or LQG [15], [16], ...

As here we are only concerned with boundedness, we will just give a full list of sufficient conditions Ψ_n and $Q_n(b)$ should meet (see section III.3 for discussion). For that we need the following notations:

From (3), (8) we let

$$A_n(b) = I + A_n^1 b + \dots + A_n^{N_a} b^{N_a} \qquad (11)$$

$$B_n(b) = B_n^0 + B_n^1 b + \dots + B_n^{N_b} b^{N_b} \qquad (12)$$

$$C_n(b) = I + C_n^1 b + \dots + C_n^{N_b} b^{N_b} \qquad (13)$$

$$D_n(b) = D_n^0 + D_n^1 b + \dots + D_n^{N_a-1} b^{N_a-1} \qquad (14)$$

$$\det \begin{vmatrix} A_n(b) & -b B_n(b) \\ D_n(b) & C_n(b) \end{vmatrix} = \hat{r}_n(b) \qquad (15)$$

where $\det|\cdot|$ denotes the determinant

Note that

$$\hat{r}_n(0) = 1 \qquad (16)$$

Let $r_n(b)$ be a sequence of polynomials with degree less than or equal to N_r

$$N_r = \ell . N_a + m\, N_b \tag{17}$$

and

$$r_n(b) = 1 + r_n^1\, b + \cdots + r_n^{N_r}\, b^{N_r} \tag{18}$$

Let $\hat{R}_n$ (resp R_n) be a vector in $\mathbb{R}^{N_r}$ whose coordinates are the coefficients of $\hat{r}_n(b)$ (resp. $r(b)$)

Let F_n be the companion matrix of $r_n(b)$

$$F_n = \begin{pmatrix} -r_n^1 & \cdots & \cdots & -r_n^{N_r} \\ 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0\ 1 & 0 \end{pmatrix} \tag{19}$$

We assume the following properties

HC1: $\| \Psi_n \| < M_2$

HC2. $\exists N_1 \quad \forall k > N_1 , \forall q$

such that $\displaystyle \frac{1}{k} \sum_{n=q+1}^{q+k} \| \theta_n - \theta_{n-1} \| \leq \eta_2$

then $\displaystyle \frac{1}{k} \sum_{n=q+1}^{q+k} \| \Psi_n - \Psi_{n-1} \| \leq \eta_3$

and $\displaystyle \frac{1}{k} \sum_{n=q+1}^{q+k} \| \hat{R}_n - R_n \| \leq \eta_3$

HC3 There exist consistent vectorial and matricial time varying norms, both noted $\|\cdot\|_n$ such that

$$\alpha \|\cdot\| \leq \|\cdot\|_n \leq \beta \|\cdot\|_n$$

$$\forall x \quad \|F_n x\|_{n+1} < \rho \|x\|_n \quad , \rho < 1$$

where $\|\cdot\|$ is a time invariant norm and α, β are strictly positive constants.

HC4. The coefficients of $Q_n(b)$ are locally bounded functions of θ_n, φ_n and its degree is upperbounded

II.3 Boundedness theorem

If HC1, HC3, HC4 and HS1 hold, then there exist strictly positive η_1, η_2, η_3 such that if HS2 and HC2 hold then $\|u_n\|$ and $\|y_n\|$ are bounded

Proof see appendix A

III INDIRECT ADAPTIVE CONTROL

III.1 Introduction

In the previous section we have stated a boundedness theorem for a class of linear time varying controls. Here we will show that indirect adaptive controls may be within this class.

First let us state the problem: Consider a time invariant MIMO system for which there exists an unknown matrix θ such that:

HP: $\|y_n - \theta^t \phi_n\|^2 \leq v_n s_n^2$

where ϕ_n (resp. s_n) is defined in (2) (resp (5)) and v_n is a sequence of positive real numbers which has the property of mean η_4-smallness relatively to s_n.

The problem of indirect adaptive control is to find an identification and a computation procedure whose outputs are respectively θ_n and $\psi_n, \varphi_n(6)$ such that HS1, HS2, HC1 to HC4 hold.

III.2 Satisfaction of HS1-HS2

To state identification procedures, the following a priori knowledge about the system is sufficient:

HI1: An upperbound of the number of entries of θ is known

HI2: ν_n is known and s_n can be computed

Then we may use the classical adaptive laws but modified as follows:

i) work with normalized entries, i.e.:

$$y_n - \theta_{n-1}^t \phi_n \rightarrow \frac{y_n - \theta_{n-1}^t \phi_n}{s_n} \qquad (20)$$

$$\phi_n \rightarrow \frac{\phi_n}{s_n} \qquad (21)$$

ii) use dead zone for the a priori error, i.e.:

$$\| y_n - \theta_{n-1}^t \phi_n \| < \nu_n s_n \Rightarrow \theta_n = \theta_{n-1} \qquad (22)$$

Note that (20), (21) induce no modification when in (5), $\|\phi_n\|$ is less than or equal to $\mathrm{Max}\{s, \|\phi_n\|\}$.

For instance the following modified least square algorithm will work [17]:

$$\theta_n = \theta_{n-1} + \alpha_n \frac{P_{n-1}\,\phi_n}{\phi_n^t P_{n-1}\phi_n + \mu_n s_n^2}\left(y_n - \theta_{n-1}^t \phi_n\right) \qquad (23)$$

with:

- P_n a sequence of positive definite matrices such that:

$$P_n = P_{n-1} - \alpha_n \frac{P_{n-1}\,\phi_n\,\phi_n^t\,P_{n-1}}{\phi_n^t P_{n-1}\phi_n + \mu_n s_n^2} + \alpha_n\, O_n \quad , P_0 = I \qquad (24)$$

where O_n is a sequence of positive symetric matrices chosen such that:

$$0 < \Lambda_0 \leq \lambda\mathrm{min}\, P_n \leq \lambda\mathrm{max}\, P_n \leq \Lambda_1 \qquad (25)$$

where λmin (resp. λmax) denotes minimum (resp. maximum) eigen value.

- μ_n is a sequence of positive real numbers such that

$$0 < \mu < \mu_n \leq M \qquad (26)$$

- α_n is such that if

$$\| y_n - \theta_{n-1}^t \, \phi_n \|^2 \geqslant \left(s_n^2 + \frac{\phi_n^t P_{n-1} \phi_n}{\mu_n} \right) v_n \qquad (27)$$

then

$$\alpha_n = 1 \qquad (28)$$

if not

$$\alpha_n = 0 \qquad (29)$$

Properties: see appendix B

III.2 Satisfaction of HC1 to HC4

The question is, given θ_n, compute Ψ_n and $Q_n(b)$ such that HC1 to HC4 could be fulfilled. This computation is classically derived from feedback design methods. Then a requirement for HC1 to HC4 to hold is at least uniform stabilizability (see [19] for a definition) of the couple $A_n(b), B_n(b)$ as given from θ_n by (3), (11), (12) (see [14], [16]). Theoretically this difficulty is got around if the assumed linear model θ do have this property and θ_n is kept in a neighbourhoud of θ (for this property see appendix B). In practice the problem may appear when there is an overparametrization of the system which can be corrected on line (see [18]).

IV ROBUSTNESS PROPERTIES

Let us here study two cases where our boundedness theorem holds and which exhibits Robustness properties of adaptive control.

IV.1 Bounded disturbances or nonlinearities

Let the plant to be controlled be represented as follows:

$$y_n = \theta^T \phi_n + w_n \tag{30}$$

where w_n is a residual between the linear model θ and the true plant and is required to be only bounded for any (bounded or unbounded) ϕ_n and y_n:

$$\forall n \quad \|w_n\|^2 < W \tag{31}$$

Then let us take.

$$s_n^2 = \operatorname{Max}\left\{ \frac{W}{\gamma_4}, \|\phi_n\|^2 \right\} \tag{32}$$

we get:

$$\| y_n - \theta^t \phi_n \|^2 \leq \eta_4 \, s_n^2 \tag{33}$$

Hence HP holds. Then if W and the number of entries of θ are known, the assumptions of our theorem can be satisfied.

IV.2 Higher order plant plus bounded disturbances

Let the plant to be controlled be represented as follows:

$$y_n = \theta^t \phi_n + \left(\theta_n^{r\,t} \phi_n^r + w_n \right) \tag{34}$$

where the residual (time varying) matrix θ_n^r is such that:

$$\| \theta_n^r \| \leq K \tag{35}$$

the residual vector ϕ_n^r includes past inputs and outputs

$$\phi_n^r = \left(y^t_{n-N_a-1} \; \dots \; y^t_{n-N_a-d} \quad u^t_{n-N_b-1} \; \dots \; u^t_{n-N_b-d} \right)^t \tag{36}$$

where N_a (resp. N_b) is the number of outputs (resp. inputs) included in ϕ_n (see (2)) and d is a time invariant integer (which imposes the true plant to be rational, except w_n).

- w_n is bounded for any y_n, ϕ_n, ϕ_n^r:

$$\forall n \quad \|w_n\|^2 < W \tag{37}$$

Let us find s_n such that HP holds. First note that

$$\|\phi_n^r\| \leq \|\phi_{n-1}\| + \cdots + \|\phi_{n-d}\| \tag{38}$$

then let us try

$$s_n = \mathrm{Max}\left\{ s, \|\phi_n\| \right\} + \alpha\, s_{n-1} \quad , \alpha < 1 \tag{39}$$

we have

$$s_n \geq \alpha\, \mathrm{Max}\left\{ s, \|\phi_{n-1}\| \right\} + \cdots + \alpha^d\, \mathrm{Max}\left\{ s, \|\phi_{n-d}\| \right\} \tag{40}$$

$$s_n \geq \alpha^d\, \mathrm{Max}\left\{ d s, \|\phi_n^r\| \right\} \geq \alpha^d\, \|\phi_n^r\| \tag{41}$$

and

$$\|y_n - \theta^r \phi_n\| \leq W^{\frac{1}{2}} + \frac{K}{\alpha^d} s_n \tag{42}$$

$$\|y_n - \theta^r \phi_n\|^2 \leq 2\left(W + \left(\frac{K}{\alpha^d}\right)^2 s_n^2 \right) \tag{43}$$

hence let v_n be computed as follows

$$v_n = 2\left(\frac{W}{s_n^2} + \left(\frac{K}{\alpha^d}\right)^2 \right) \tag{44}$$

v_n will have the property of mean η_4 - smallness relatively to s_n if:

$$\|\theta_n^r\|^2 \leq \lambda^{2d}\left(\frac{\eta_4}{2} - \frac{W}{s^2}\right) < \frac{\eta_4}{2} \qquad (45)$$

$$s^2 < +\infty \qquad (46)$$

Then the boundedness theorem holds even when the order of the plant is underestimated but the neglected parameters satisfies an inequality such as (45). Note that the number of parameters is not important, since it is taken into account into s_n via d.

IV. 3 How to robustify?

Accordingly to the assumption HP, the robustness properties are characterized in terms of the sequence v_n or more simply in terms of the positive constant η_4. The boundedness theorem proclames the existence of strictly positive constants η_1, η_2 and the identification procedure relies η_1, η_2 upon η_4 as follows (see appendix B)

$$\eta_1^2 > k_1 \eta_4 \tag{47}$$

$$\eta_2^2 > k_2 \eta_4 \tag{48}$$

Thus to allow larger η_4, we must have smaller k_1, k_2 and larger η_1, η_2

Smaller k_1, k_2 yields the modification of the identification procedure as for instance by filtering inputs and outputs (see [9]) and/or by filtering the a priori model error (change of the gradient direction) (see [13])

Larger η_1, η_2 yields the modification of the control design as for instance by changing the closed loop poles (p in HC3) or by using apriori pre or post compensator together with parallel feed forward (see [12]). In peculiar the last technique is a very attractive way to use maximum of the apriori information about the plant

V CONCLUSION

We have stated boundedness properties of indirect adaptive control schemes even when the residuals between the true plant and the assumed linear time invariant model are linearly dominated by the information vector (containing inputs and outputs). This result has been obtained introducing non linearities in the adaptive laws such as dead zone and "asymptotic" normalization. In fact these changes were introduced to insure the coherency with the representation assumption.

Though this study gives only qualitative results, it defines parameters which would make a quantification of robustness possible. In fact much work has been done indirectly concerning this question but there was no available theory at that time. In view of the results presented here all these ideas may be revisited in terms of robustness and quantitatively compared. In this way we hope systematic procedures such as those used for linear control will be obtained

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Appendix A

PROOF OF THE BOUDEDNESS THEOREM

Introduction: Here we follow the scheme introduced in [14], which is essentially built upon the following lemma whose proof can be found in [20].

Lemma 1: If

L1: $\forall n,\ 0 \leq a_{n+1} < (\gamma + b_n)\, a_n + M_4 \quad , 0 \leq \gamma < 1$

L2: b_n is a sequence of positive real numbers which has the property of mean ξ-smallness relatively to a_n, with

$$\xi < 1 - \gamma$$

Then a_n is bounded.

Then we only need to establish L1, L2. For L2, the following lemma will be required.

Lemma 2: Let d be an integer and H, k, L be positive constants. If:

L4: $\forall n > d, \quad a_n \le \Gamma_1 \max_{n-d-1 \le i \le n} s_i$

L5: $\forall n \quad s_{i+1} \le \Gamma_2 s_i + \Gamma_3$

then we have:

$\forall \Sigma, \exists \Gamma, \exists N_2 > d: \quad \forall k > N_2, \forall q$

such that $: \forall n \in [q+1, q+k] \quad a_n \geqslant \Gamma$

then $\exists t \in [q+k-d, q+k]$ such that

$\forall j \in [q+1, t] \quad s_j \geqslant \Sigma$

Now we give our proof in four steps.

first step (L1): Let $X_n(b)$, $Y_n(b)$, $B_{1n}(b)$, $A_{1n}(b)$ be polynomial matrices given by the adjoint matrix of

$$\begin{pmatrix} A_n(b) & -b\,B_n(b) \\ D_n(b) & C_n(b) \end{pmatrix}$$

as follows (use definition (15)):

$$\begin{pmatrix} X_n(b) & b\,B_{1n}(b) \\ -Y_n(b) & A_{1n}(b) \end{pmatrix} \begin{pmatrix} A_n(b) & -b\,B_n(b) \\ D_n(b) & C_n(b) \end{pmatrix} = \hat{r}_n(b) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad (A.1)$$

With definitions (3), (8), (11) to (14), this expression (A.1) when applied to ϕ_n yields:

$$\sum_{i=0}^{N_x} X_n^i \left(y_{n-i} - \theta_n^t \phi_{n-i} \right) + \sum_{i=0}^{N_{b1}} B_{1n}^i \Psi_n^t \phi_{n-i} = \hat{r}_n(b)\, y_n \qquad (A.2)$$

$$- \sum_{i=0}^{N_y} Y_n^i \left(y_{n-i} - \theta_n^t \phi_{n-i} \right) + \sum_{i=0}^{N_{a1}} A_{1n}^i \Psi_n^t \phi_{n+1-i} = \hat{r}_n(b)\, v_{n+1} \qquad (A.3)$$

where X_n^i, Y_n^i, B_{1n}^i, A_{1n}^i are the matricial coefficients and N_x, N_y, N_{b1}, N_{a1} are the degrees of the polynomial matrices $X_n(b)$, $Y_n(b)$, $B_{1n}(b)$, $A_{1n}(b)$.

Then with help of (4), (9), we get:

$$\hat{r}_n(b)\, y_n = \begin{cases} X_n(b)\, \mathcal{E}_n + \sum_{i=1}^{N_x} X_n^i \left(\theta_{n-i} - \theta_n\right)^t \phi_{n-i} \\ + \sum_{i=0}^{N_{b1}} B_{1n}^i\, Q_{n-i-1}(b)\, y^*_{n-i-1} + \sum_{i=0}^{N_{b1}} B_{1n}^i \left(\psi_n - \psi_{n-1-i}\right)^t \phi_{n-i} \end{cases} \tag{A.4}$$

$$\hat{r}_n(b)\, u_{n+1} = \begin{cases} -Y_n(b)\, \mathcal{E}_n + \sum_{i=1}^{N_y} Y_n^i \left(\theta_n - \theta_{n-i}\right)^t \phi_{n-i} \\ + \sum_{i=0}^{N_{a1}} A_{1n}^i\, Q_{n-i}(b)\, y^*_{n-i} + \sum_{i=1}^{N_{a1}} A_{1n}^i \left(\psi_n - \psi_{n-i}\right)^t \phi_{n+1-i} \end{cases} \tag{A.5}$$

Now let d_n be the maximum of N_x, N_y, N_{a1}, N_{b1}. Using (A.1), we have:

$$\forall n, \quad d_n \leq d = (m+\ell)\ \mathrm{Max}\{N_a, N_b\} \tag{A.6}$$

Note that with (15), (17), we know that degrees of $r_n(b)$ and $\hat{r}_n(b)$ are less than or equal to N_r with:

$$N_r \leq d \tag{A.7}$$

Then let us introduce a new vector Ψ_n defined as follows

$$\Psi_n^t = \left(y^t_{n-1} \ \cdots \ y^t_{n-d-N_a} \quad u^t_n \ \cdots \ u_{n-d-N_b}\right) \tag{A.8}$$

Ψ_n is made up of the coordinates of $\phi_n, \phi_{n-1}, \ldots, \phi_{n-d}$.

Now (A.4), (A.5) may be rewritten in a state space representation:

$$\Psi_{n+1} = \left(F_n + \Delta F_n\right) \Psi_n + G_n Y_n^* + H_n E_n \tag{A.9}$$

where - F_n is the companion matrix of $r_n(b)$ as given by (19)

- ΔF_n incorporates the following differences:

$(\hat{r}_n(b) - r_n(b))$, $(\theta_n - \theta_{n-i})$, $(\psi_n - \psi_{n-i})$, $(\psi_n - \psi_{n-i-1})$

- G_n includes entry coefficients of $B^i_{1n} Q_{n-i-1}(b)$ and $A^i_{1n} Q_{n-i}(b)$.

- Y^*_n is a vector defined as follows:

$$Y^*_n = \left(y^{*t}_n \; \dots \; y^{*t}_{n-d'} \right)^t \qquad (A.10)$$

where d' is an upper bound of d + degree of $Q_n(b)$ (use HC4).

- H_n includes entry coefficients of $X_n(b)$, $Y_n(b)$

- E_n is a vector defined as follows

$$E^t_n = \left(\varepsilon^t_n \; \dots \; \varepsilon^t_{n-d} \right) \qquad (A.11)$$

let use now use the time varying norms given by HC2

$$\| \Psi_{n+1} \|_{n+1} \leq \| F_n \Psi_n \|_{n+1} + \| \Delta F_n \Psi_n \|_{n+1} + \| G_n Y^*_n \|_{n+1} + \| H_n E_n \|_{n+1} \qquad (A.12)$$

and with help of HC3, we get:

$$\|\Psi_{n+1}\|_{n+1} \leq \left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\|\right)\|\Psi_n\|_n + \beta\|G_n\|\,\|Y_n^*\| + \beta\|H_n\|\,\|E_n\| \quad (A.13)$$

and with help of norm equivalence in finite dimensional vector space:

$$\|\Delta F_n\| < \|\hat{R}_n - R_n\| + \begin{cases} \max\limits_{1 \leq i \leq d} \left\{\|X_n^i\|, \|Y_n^i\|\right\} \sum\limits_{i=1}^{d} \|\theta_{n-i} - \theta_n\| \\ \max\limits_{0 \leq i \leq d} \left\{\|B_{1n}^i\|, \|A_{1n}^i\|\right\} \sum\limits_{i=1}^{d+1} \|\Psi_n - \Psi_{n-i}\| \end{cases} \quad (A.14)$$

$$\|G_n\| < \max_{\substack{0 \leq i \leq d \\ 0 \leq j \leq N_q}} \left\{\|B_{1n}^i\|\,\|Q_{n-i-1}^j\|, \|A_{1n}^i\|\,\|Q_{n-i}^j\|\right\} \quad (A.15)$$

$$\|H_n\| < \max_{0 \leq i \leq d} \left\{\|X_n^i\|, \|Y_n^i\|\right\} \quad (A.16)$$

$$\|E_n\| < \max_{0 \leq i \leq d} \left\{\|\varepsilon_{n-i}\|\right\} \quad (A.17)$$

But the entries of matrices $X_n^i, Y_n^i, A_{1n}^i, B_{1n}^i$ are multilinear functions of the entries of θ_n, Ψ_n. Hence using HS1, HC1 and the boundedness of $\|y_n^*\|$, there exist positive constants M_6, M_7, M_8, M_9 such that:

$$\|\Delta F_n\| < \|\hat{R}_n - R_n\| + M_6 \sum_{i=1}^{d} \|\theta_{n-i} - \theta_n\| + M_7 \sum_{i=1}^{d+1} \|\Psi_{n-i} - \Psi_n\| \quad (A.18)$$

$$\|G_n\|\,\|Y_n^*\| < M_8 \quad (A.19)$$

$$\|H_n\| < M_9 \quad (A.20)$$

Then (A.13) yields:

$$\|\Psi_{n+1}\|_{n+1} \leq \left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\|\right)\|\Psi_n\|_n + M_g \operatorname*{Max}_{0\leq i\leq d}\{\|\mathcal{E}_{n-i}\|\} + M_8 \tag{A.21}$$

Now let N be the index (which depends on n) such that:

$$\|\mathcal{E}_N\| = \operatorname*{Max}_{0\leq i\leq d}\{\|\mathcal{E}_{n-i}\|\} \tag{A.22}$$

and with s_n as defined in (5), we get

$$\|\Psi_{n+1}\|_{n+1} \leq \left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\|\right)\|\Psi_n\|_n + M_g \frac{\|\mathcal{E}_N\|}{s_N} s_N + M_8 \tag{A.23}$$

$$s_N \leq \operatorname{Max}\{s, \|\phi_N\|\} + d \; s_{N-1} + s \tag{A.24}$$

But for any j in $[n-d, n]$ we have

$$\|\phi_j\| \leq \|\Psi_n\| \leq \frac{1}{\alpha}\|\Psi_n\|_n \tag{A.25}$$

$$s_j \leq d^{\,j-n+d+1} s_{n-d-1} + \sum_{i=0}^{j-n+d} d^{\,i}\left(\operatorname{Max}\{s, \|\phi_{j-i}\|\} + s\right) \tag{A.26}$$

and with help of (6), this implies in peculiar

$$s_N \leq d \; s_{n-d-1} + (d+1)\left(\operatorname{Max}\left\{s, \frac{1}{\alpha}\|\Psi_n\|_n\right\} + \frac{s}{1-d}\right) \tag{A.27}$$

Thus (A.23), (A.24) may be rewritten in:

$$\left.\begin{array}{l} \|\Psi_{n+1}\|_{n+1} \leq \left\{\begin{array}{l}\left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\|\right)\|\Psi_n\|_n + M_8 \\ + M_g \frac{\|\mathcal{E}_N\|}{s_N}\left(d\, s_{n-d-1} + (d+1)\left(\operatorname{Max}\left\{s, \frac{1}{\alpha}\|\Psi_n\|_n\right\} + \frac{s}{1-d}\right)\right)\end{array}\right. \\ s_{n-d} \leq \operatorname{Max}\left\{s, \frac{1}{\alpha}\|\Psi_n\|_n\right\} + d\, s_{n-d-1} + s \end{array}\right\} \tag{A.28}$$

or since $\frac{\|\mathcal{E}_N\|}{s_N}$ is bounded (use HS2.1)

either

$$\frac{1}{\alpha}\|\Psi_n\|_n \leq s \qquad (A.29)$$

then

$$\left.\begin{aligned} \|\Psi_{n+1}\|_{n+1} &\leq \left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\|\right)\|\Psi_n\|_n + d M_g \frac{\|\mathcal{E}_N\|}{s_N} s_{n-d-1} + M_{10} \\ s_{n-d} &\leq \qquad\qquad\qquad\qquad\qquad d \qquad\qquad s_{n-d-1} + S+s \end{aligned}\right\} \qquad (A.30)$$

or

$$\frac{1}{\alpha}\|\Psi_n\|_n > s \qquad (A.31)$$

then

$$\left.\begin{aligned} \|\Psi_{n+1}\|_{n+1} &\leq \left(\rho + \frac{\beta^2}{\alpha}\|\Delta F_n\| + M_g \frac{d+1}{\alpha}\frac{\|\mathcal{E}_N\|}{s_N}\right)\|\Psi_n\|_n + d M_g \frac{\|\mathcal{E}_N\|}{s_N} s_{n-d-1} + M_{11} \\ s_{n-d} &\leq \qquad\qquad \frac{1}{\alpha} \qquad\qquad \|\Psi_n\|_n + d \qquad s_{n-d-1} + S \end{aligned}\right\} \qquad (A.32)$$

To simplify notations let:

$$\Delta_1^n = \frac{\beta^2}{\alpha}\|\Delta F_n\| + M_g \frac{d+1}{\alpha}\frac{\|\mathcal{E}_N\|}{s_N} \qquad (A.33)$$

$$\Delta_2^n = d\, M_g \frac{\|\mathcal{E}_N\|}{s_N} \qquad (A.34)$$

let ρ_1, ρ_2 be strictly positive constants and let σmax denote the maximum singular value. We trivially have:

$$\sigma\max\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\begin{pmatrix} \rho+\Delta_1^n & \Delta_2^n \\ \frac{1}{\alpha} & d \end{pmatrix}\begin{pmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix} \tag{A.35}$$

$$\leq \sigma\max\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\begin{pmatrix} \rho & 0 \\ \frac{1}{\alpha} & d \end{pmatrix}\begin{pmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix} + \sigma\max\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\begin{pmatrix} \Delta_1^n & \Delta_2^n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix}$$

But since

$$d < 1, \quad \rho < 1 \tag{A.36}$$

one can choose p_1, p_2 such that

$$\sigma\max\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\begin{pmatrix} \rho & 0 \\ \frac{1}{\alpha} & d \end{pmatrix}\begin{pmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix} \leq \gamma < 1 \tag{A.37}$$

and then

$$\sigma\max\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\begin{pmatrix} \Delta_1^n & \Delta_2^n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{p_1} & 0 \\ 0 & \frac{1}{p_2} \end{pmatrix} \leq \Delta_1^n + \frac{p_1}{p_2}\Delta_2^n \tag{A.38}$$

This leads us to rewrite both (A.30), (A.32) in.

$$\left(p_1^2 \|\Psi_{n+1}\|_{n+1}^2 + p_2^2 s_{n-d}^2\right)^{\frac{1}{2}} \leq (\gamma + b_n)\left(p_1^2 \|\Psi_n\|_n^2 + p_2^2 s_{n-d-1}\right)^{\frac{1}{2}} + M_4 \tag{A.39}$$

where

$$b_n = \Delta_1^n + \frac{p_1}{p_2}\Delta_2^n = \frac{\beta^2}{\alpha}\|\Delta F_n\| + M_g\left(\frac{d+1}{\alpha} + \frac{p_1}{p_2}d\right)\frac{\|E_N\|}{s_N} \tag{A.40}$$

Thus if we let

$$a_{n+1}^2 = p_1^2 \|\Psi_{n+1}\|_{n+1}^2 + p_2^2 s_{n-d}^2 \tag{A.41}$$

we have established L 1.

Second step (L2.i) : Since θ_n, Ψ_n are bounded (with HS1, HC1), since R_n is bounded (with HC3), since the entries of $\vec{R}_n$ are multilinear functions of the entries of θ_n, Ψ_n and since $\frac{\|\varepsilon_n\|}{s_n}$ is bounded (with HS2.1), (A.40) and (A.18) yield boundedness of b_n.

Third step (L3_L4) : Since $\|\Delta F_n\|$ and $\frac{\|\varepsilon_n\|}{s_n}$ have the property of mean η-smallness relatively to s_n, to complete L2, we need to show that when a_n grows at a high level so does s_n.

From definitions (5), (2), (A.8) of s_n, ϕ_n and Ψ_n and using assumption HC3, we have

$$\|\Psi_n\|_n \leq \beta \|\Psi_n\| \leq \beta \operatorname*{Max}_{n-d\leq i\leq n} \{\|\phi_i\|\} \leq \beta \operatorname*{Max}_{n-d\leq i\leq n} \{s_i\} \tag{A.42}$$

Together with definition (A.41), this yields:

$$a_n \leq \rho_1 \|\Psi_n\|_n + \rho_2 \, s_{n-d-1} \leq \Gamma_1 \operatorname*{Max}_{n-d-1\leq i\leq n} \{s_i\} \tag{A.43}$$

Thus we have established L3.

To get L4, we rewrite (4), (9) in the following form

$$\begin{pmatrix} A_n(b) & -b\,B_n(b) \\ D_n(b) & C_n(b) \end{pmatrix} \begin{pmatrix} y_n \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_n \\ Q_n(b)\; y_n^* \end{pmatrix} \qquad (A.44)$$

And a state space representation of this relation yields for any time invariant norm:

$$\|\phi_{n+1}\| \leq L_n \|\phi_n\| + \|\mathcal{E}_n\| + \|Q_n(b)\, y_n^*\| \qquad (A.45)$$

where L_n incorporates upperbounds of the entries of $A_n^i, B_n^i, D_n^i, C_n^i$. But using HS1, HC1, L_n is bounded. Then using boundedness of y_n^*, HC4, HS2.1 and (5), we get:

$$\left.\begin{aligned} &\|\phi_{n+1}\| \leq M_{12} \|\phi_n\| + \frac{\|\mathcal{E}_n\|}{s_n} s_n + M_{13} \\ &\text{Max}\{s, \|\phi_n\|\} \leq s_n \leq \text{Max}\{s, \|\phi_n\|\} + d\, s_{n-1} + S \end{aligned}\right\} \qquad (A.46)$$

But this yields

$$\|\phi_{n+1}\| \leq M_{14}\, s_n + M_{15} \qquad (A.47)$$

$$s_{n+1} \leq \text{Max}\{s, M_{14}\, s_n + M_{15}\} + d\, s_n + S \qquad (A.48)$$

$$s_{n+1} \leq \Gamma_2\, s_n + \Gamma_3 \qquad (A.49)$$

This establishes L4 and allows us to use lemma 2.

<u>Fourth step (12.ii)</u>: We are going to state the second part of the property of mean ξ-smallness for b_n.

From (A.40) we deduce:

$\forall k \geq d, \ \forall q:$

$$\sum_{n=q+d}^{q+k} b_n = \frac{\beta^2}{\alpha} \sum_{n=q+d}^{q+k} \|\Delta F_n\| + M_3\left(\frac{d+1}{\alpha} + \frac{P_1}{P_2} d\right) \sum_{n=q+d}^{q+k} \frac{\|\mathcal{E}_N\|}{S_N} \quad (A.50)$$

But with help of (A.18), we have

$$\sum_{n=q+d}^{q+k} \|\Delta F_n\| \leq \left\{ \begin{array}{l} \displaystyle\sum_{n=q+1}^{q+k} \|\hat{R}_n - R_n\| + d^2 M_6 \sum_{n=q+1}^{q+k} \|\theta_n - \theta_{n-1}\| \\ \displaystyle + (d+1)^2 M_7 \sum_{n=q+1}^{q+k} \|\Psi_n - \Psi_{n-1}\| + (d+1) M_7 \|\Psi_q - \Psi_{q-1}\| \end{array} \right. \quad (A.51)$$

and with HC1

$$\forall \nu_1 > 0, \ \exists N_3 : \ \forall k > N_3 \Rightarrow \frac{(d+1) M_7}{k} \|\Psi_q - \Psi_{q-1}\| < \nu_1, \ \forall q \quad (A.52)$$

With help of (A.22), we have

$$\sum_{n=q+d}^{q+k} \frac{\|\mathcal{E}_N\|}{S_N} \leq d \sum_{n=q+1}^{q+k} \frac{\|\mathcal{E}_n\|}{S_n} + \frac{\|\mathcal{E}_q\|}{S_q} \quad (A.53)$$

and with HS2.1

$$\forall \nu_2 > 0, \ \exists N_4 : \ \forall k > N_4 \Rightarrow \frac{1}{k} \frac{\|\mathcal{E}_q\|}{S_q} < \nu, \ \forall q \quad (A.54)$$

Now with (A.52), (A.54) and HS2, HC2 there exists an integer N_5 and a positive constant Σ such that:

$$\left.\begin{aligned}
&\forall k > N_5 \geq \operatorname{Max}\{N_1, N_3, N_4, 2d\} \\
&\text{such that } \forall n \in [q+1, q+k] \quad s_n \geq \Sigma \\
&\text{then } \quad \frac{1}{k+1-d} \sum_{n=q+d}^{q+k} b_n \leq \frac{k}{k+1-d} \left\{ \begin{aligned} &\frac{\beta^2}{\alpha}\left(1+(d+1)^2 M_7\right)\eta_3 + d^2 M_6 \eta_2 + \nu_1 \\ &+ M_9\left(\frac{d+1}{\alpha} + \frac{p_1}{p_2} d\right)\left(d\eta_1 + \nu_2\right) \end{aligned} \right.
\end{aligned}\right\} \text{(A.55)}$$

And there exist strictly positive $\eta_1, \eta_2, \eta_3, \nu_1, \nu_2$ such that; with conditions of (A.55):

$$\frac{1}{k+1-d} \sum_{n=q+d}^{q+k} b_n \leq \zeta < 1-\gamma \tag{A.56}$$

But with help of the second step, this means that b_n has the property of mean ζ-smallness relatively to s_n.

Now with lemma 2:

$$\left.\begin{aligned}
&\exists (\Gamma, N_2) &&: \forall k' > N_2, \ \forall q \\
&\text{such that} &&: \forall n \in [q+d, q+k'] \quad a_n \geq \Gamma \\
&\text{then} &&: \exists k \in [k'-d, k'] \text{ such that} \\
& && \quad \forall j \in [q+1, q+k] \quad s_j \geq \Sigma
\end{aligned}\right\} \text{(A.57)}$$

But with help of boundedness of b_n, we have:

$$\forall k \in [k'-d,\ k'] \quad \sum_{n=q+k+1}^{q+k'} b_n < (k'-k)\, M_{16} < d\, M_{16} \qquad (A.58)$$

and

$$\exists N_6 \ : \ \forall k' > N_6 \ \Rightarrow \ \frac{d\, M_{16}}{k'+1-d} < \frac{1-\gamma-\xi}{2} \qquad (A.59)$$

then

$$\left.\begin{array}{lll} \exists (\Gamma, N_7) & : & \forall k' > N_7,\ \forall q \\ \text{such that} & : & \forall n \in [q+d,\ q+k'] \quad a_n \geq \Gamma \\ \text{then} & : & \dfrac{1}{k'+1-d} \displaystyle\sum_{n=q+d}^{q+k'} b_n \leq \xi < 1-\rho \end{array}\right\} (A.60)$$

which states that b_n has the property of mean ξ-smallness relatively to s_n.

<u>Conclusion</u>: From lemma, we conclude that a_n is bounded which implies that ψ_n and u_n, y_n are bounded

Appendix B

PROPERTIES OF A MODIFIED LEAST SQUARE ALGORITHM

Introduction: Consider the algorithm given in section III.2 and let us introduce the following notations

$$\tilde{\theta}_n = \theta_n - \theta \tag{B.1}$$

$$V_n = \mathrm{tr}\ \tilde{\theta}_n^T P_n^{-1} \tilde{\theta}_n \tag{B.2}$$

$$\Pi_n = P_{n-1} - \alpha_n \frac{P_{n-1}\ \phi_n \phi_n^T P_{n-1}}{\phi_n^T P_{n-1} \phi_n + \mu_n s_n^2} \tag{B.3}$$

$$\nu_n = y_n - \theta_{n-1}^T\ \phi_n \tag{B.4}$$

where tr denotes trace.

Note that, from (25), P_n is always strictly non singular and so is Π_n since:

$$\Pi_n^{-1} = P_{n-1}^{-1} + \alpha_n \frac{\phi_n \phi_n^T}{\mu_n s_n^2} \tag{B.5}$$

Moreover we have, in the sense of symetric positive matrices:

$$P_n^{-1} \leq \Pi_n^{-1} \qquad (B.6)$$

or:

$$V_n \leq \mathrm{tr}\, \tilde{\theta}_n^t \, \Pi_n^{-1} \, \tilde{\theta}_n \qquad (B.7)$$

At end, note that

$$\alpha_n \frac{\Pi_n^{-1} P_{n-1} \phi_n}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} = \alpha_n \frac{\phi_n}{\mu_n s_n^2} \qquad (B.8)$$

We have the following properties.

<u>P1</u>: V_n is a convergent sequence

<u>Proof</u>. Using (23), we get

$$\mathrm{tr}\, \tilde{\theta}_n^t \, \Pi_n^{-1} \, \tilde{\theta}_n = \mathrm{tr}\, \tilde{\theta}_{n-1}^t \, \Pi_n^{-1} \, \tilde{\theta}_{n-1} + 2\alpha_n \frac{\nu_n^t \, \tilde{\theta}_{n-1}^t \, \phi_n}{\mu_n s_n^2} + \alpha_n \frac{\|\nu_n\|^2}{\mu_n s_n^2} \, \frac{\phi_n^t P_{n-1} \phi_n}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \qquad (B.9)$$

and

$$\mathrm{tr}\, \tilde{\theta}_{n-1}^t \, \Pi_n^{-1} \, \tilde{\theta}_{n-1} = V_{n-1} + \alpha_n \frac{\| \tilde{\theta}_{n-1}^t \phi_n \|^2}{\mu_n s_n^2} \qquad (B.10)$$

hence

$$V_n \leq V_{n-1} + \frac{\alpha_n}{\mu_n s_n^2} \| \nu_n + \tilde{\theta}_{n-1}^t \phi_n \|^2 - \alpha_n \frac{\| \nu_n \|^2}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \qquad (B.11)$$

and with assumption HP:

$$V_n \le V_{n-1} + \alpha_n \left(\frac{v_n}{\mu_n} - \frac{\| v_n \|^2}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \right) \tag{B.12}$$

then, since

$$\alpha_n \left(\frac{v_n}{\mu_n} - \frac{\| v_n \|^2}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \right) \le 0 \tag{B.13}$$

V_n is a positive non increasing sequence which converges.

P2: Parameter boundedness

Using Cauchy-Schwarz inequality in (B.2) and with help of (25), we get

$$\Lambda_0 \, \mathrm{tr} \, \tilde{\theta}_n^t \theta_n \le V_n \tag{B.14}$$

hence the conclusion with help of V_n boundedness

P3: $\frac{\| \varepsilon_n \|}{s_n}$ has the property of mean η_1-smallness relatively to s_n, with η_1 any positive real number such that:

$$\eta_1^2 > \left(1 + \frac{\Lambda_1}{\mu}\right) \eta_4 \tag{B.15}$$

Proof : First note that:

$$\varepsilon_n = y_n - \theta_n^t \phi_n = \nu_n \frac{(1-\alpha_n)\, \phi_n^t P_{n-1} \phi_n + \mu_n s_n^2}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \tag{B.16}$$

Now from (B.12), we get

$$\alpha_n \frac{\| \nu_n \|^2}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \leq V_{n-1} - V_n + \alpha_n \frac{v_n}{\mu_n} \tag{B.17}$$

hence

$$\alpha_n \frac{\| \varepsilon_n \|^2}{s_n^2} \leq \frac{\left((1-\alpha_n)\, \phi_n^t P_{n-1} \phi_n + \mu_n s_n^2\right)^2}{s_n^2 \left(\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2 \right)} \left(V_{n-1} - V_n + \alpha_n \frac{v_n}{\mu_n} \right) \tag{B.18}$$

and with help of (5), (25), (26) we get:

$$\alpha_n = 1 \Rightarrow \frac{\| \varepsilon_n \|^2}{s_n^2} \leq M \left(V_{n-1} - V_n \right) + \frac{M^2}{\Lambda_0 + M^2} v_n \tag{B.19}$$

$$\alpha_n = 0 \Rightarrow \begin{cases} \dfrac{\| \varepsilon_n \|^2}{s_n^2} = \dfrac{\| \nu_n \|^2}{s_n^2} \leq \dfrac{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2}{\mu_n s_n^2} v_n \leq \dfrac{\Lambda_1 + \mu}{\mu} v_n \\ V_n = V_{n-1} \end{cases} \tag{B.20}$$

Hence for any k, any q we have

$$\sum_{n=q+1}^{q+k} \frac{\| \varepsilon_n \|^2}{s_n^2} \leq \mathrm{Max}\{1, M\} \left(V_{q+k} - V_q \right) + \left(1 + \frac{\Lambda_1}{\mu} \right) \sum_{n=q+1}^{q+k} v_n \tag{B.21}$$

and using Cauchy-Schwarz inequality

$$\left(\frac{1}{k} \sum_{n=q+1}^{q+k} \frac{\| \varepsilon_n \|}{s_n} \right)^2 \leq \mathrm{Max}\{1, M\} \left(V_{q+k} - V_q \right) + \left(1 + \frac{\Lambda_1}{\mu} \right) \frac{1}{k} \sum_{n=q+1}^{q+k} v_n \tag{B.22}$$

and since V_n is bounded, using properties of v_n we get

our conclusion.

<u>P4</u>: $\|\theta_n - \theta_{n-1}\|$ has the property of mean η_2-smallness relatively to s_n, with η_2 any positive real number such that:

$$\eta_2 \geqslant \frac{\Lambda_1}{\mu}\,\eta_1 \tag{B.23}$$

<u>Proof</u>: From (23), we get

$$\|\theta_n - \theta_{n-1}\| = \alpha_n \left(\phi_n^t P_{n-1}^2 \phi_n\right)^{\frac{1}{2}} \frac{\|\nu_n\|}{\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} = \alpha_n \frac{\left(\phi_n^t P_{n-1}^2 \phi_n\right)^{\frac{1}{2}} \|\varepsilon_n\|}{(1-\alpha_n)\phi_n^t P_{n-1} \phi_n + \mu_n s_n^2} \tag{B.24}$$

hence

$$\|\theta_n - \theta_{n-1}\| \leq \frac{\Lambda_1}{\mu} \frac{\|\varepsilon_n\|}{s_n} \tag{B.25}$$

which yields our result.