A generalized control law for uniform, global and exponential magnetic detumbling of rigid spacecraft

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Abstract: The problem of magnetic detumbling, also known as magnetic momentum unloading, is considered. The objective is to stabilize a rigid spacecraft angular velocity to zero only through magnetic actuation, i.e. magneto-torquers. We propose a generalized control law that is *uniformly, globally exponentially stabilizing* (UGES) and show that the celebrated *b-dot law* is one particular case of this law. Furthermore, we provide an explicit (time-varying) strict Lyapunov function to establish the stability claim.

Keywords: satellite control, time-varying systems, Lyapunov stability

1. INTRODUCTION

The last couple of decades have seen an astonishing series of achievements in aerospace science and technology, such as the increased deployment of reusable launch vehicles and nano-satellites, that have marked the beginning of an exciting new space era. These developments have led to the definition of new mission scenarios that necessitate more efficient hardware components and robust algorithms which however should not increase the overall system complexity and cost. Such requirements influence directly one of the most critical components for the precise operation of a spacecraft. This is the attitude control system (ACS) that ensures the active attitude stabilization and disturbance rejection. The first immediate task of the ACS after launch is to detumble the spacecraft, i.e. drive all angular velocities to zero.

Through the decades there has been a variety of approaches for the design of ACS. One of the most robust and efficient way has been through the use of electromagnetic actuators, which have shown to be very suitable for low Earth orbit (LEO) satellites. The underlying principle for the operation of these actuators hinges upon the interaction between a set of three orthogonal, current-driven magnetic coils (a.k.a. magneto-torquers) and the magnetic field of the Earth, see Wertz (1978). The exploitation of this physical phenomenon leads naturally to a simple solution to the problem of torque generation on board a spacecraft.

For such actuators, the only noticeable limiting factor appears to be the fact that the generated control torques are constrained to lie in the plane orthogonal to the magnetic field vector. This implies that complete controllability is possible only on the average and dependent on the variability of the geomagnetic field. This condition depends considerably on the inclination of the selected orbit; in equatorial orbit dynamics is almost uncontrollable as controllability characteristics tend to improve with orbit inclination, see for example Wiśniewski and Blanke (1999); Silani and Lovera (2005). Despite this drawback, for a number of missions the advantages of magnetotorquers are undeniable and have been already exposed in Stickler and Alfriend (1976); Avanzini and Giulietti (2012); Silani and Lovera (2005); Lovera and Astolfi (2004, 2005); Lovera and Astolfi (2006) among other works. Essentially, these are simple, cheap and reliable actuators as they do not have any moving or hydraulic parts, require only renewable electric power for (almost limitless) operation and do not present any unrecoverable failure modes. We refer to Ovchinnikov et al. (2018); Ovchinnikov and Roldugin (2019) for an exposition of recent use cases and to Zamorano et al. (2017); Rodríguez-Rojo et al. (2019) for particular application on CubeSats.

While there is an important literature on the attitude control of spacecraft with only magnetic actuators, see Stickler and Alfriend (1976); Wiśniewski and Blanke (1999); Psiaki (2001); Silani and Lovera (2005), there has been only recently a focus on the crucial detumbling phase. The most noticeable works are Lovera (2015); Avanzini and Giulietti (2012); Ahmed and Kerrigan (2014). These works essentially exploit (variations) of the well-known *b-dot* law which allows to detumble the spacecraft using only the available magnetic field measurement $b(t)$, and its approximate derivative $\dot{b}(t)$, while avoiding the use of any angular rate measurements. Despite this evident advantage of requiring a minimum amount of measured information, these controllers, and their corresponding closed-loop systems, have shown to be difficult to analyze from a stability viewpoint. This is essentially due to two facts: 1) the time-derivative of the magnetic field measurement $b(t)$ depends on the time-derivative of its corresponding vector $b_i(t)$ expressed in inertial coordinates; 2) the presence of the Coriolis term in the Euler equations. For the latter, works such as Lovera (2015); Avanzini and Giulietti (2012) assume that this term is negligible while for the former either that $\dot{b}_i$ is another negligible term or that it has a particular form. As such, these works provided proofs of (non-uniform) global asymptotic stability based on non-strict Lyapunov functions and application of appropriate versions of Barbalat's lemma (Avanzini and Giulietti (2012)) and LaSalle's invariance principle for periodic systems, along with averaging theory (Lovera (2015)).

As already mentioned, these *b-dot* control laws require essentially only the measurement of the vector $b(t)$. Apart from the simplicity of such approach, the design was based on the idea that angular velocity information cannot be reconstructed from vector measurements and in particular from magnetic field measurements, as pointed out in Lovera (2015). However, recently there have been advances on angular velocity estimation directly from (single or multiple) vector observations obtained from Sun sensors, star trackers or magnetometers. In particular, the case of multiple (more than 2) vector measurements was studied in Magnis and Petit (2016) while the results on the single measurement scenario were reported in Magnis and Petit (2017). In this latter work, it is shown that the tumbling dynamics (free rotation) is, in almost all cases (i.e. except for initial conditions belonging to a set having measure zero), satisfying a persistency-of-excitation condition. In both these works, nonlinear observers are designed and semi-global uniform asymptotic and local exponential stability is established using (non strict) Lyapunov functions and results on uniform complete observability of linear time-varying (LTV) systems. In the case where attitude measurements are already available, it is of course well known from the seminal work Salcudean (1991) how a convergent angular estimate can be reconstructed. See also Berkane et al. (2016) and references therein for some recent designs under this scenario.

In this work we wish to exploit these recent developments on the estimation of a rigid body's angular velocity from vector observations, and in general the possible availability of the angular velocity, for the control design. The main contribution of this work is thus to propose a generalized control law for magnetic detumbling for which we can establish uniform global exponential stability (UGES) of the origin. We show that a particular case of our law is the celebrated *b-dot* law while for the other possible laws the implementation requires measurements of the angular velocity that can be considered available based on the advances mentioned previously. The stability proof hinges upon the explicit construction of a strict (time-varying) Lyapunov function as is done for classes of persistently excited nonlinear time-varying systems, see Malisoff and Mazenc (2009); Maghenem and Loría (2017).

To the authors' knowledge this is the first result in the literature that establishes such stability characteristics, even for the *b-dot* law. The paper runs as follows: the model used in the control design is described in section 2 along with our main working assumption; the proposed law is presented in section 3 along with a detailed Lyapunov analysis that establishes UGES. Finally, we discuss some extensions (saturation, regulation) and perspectives.

2. PROBLEM FORMULATION – MODEL

The measurement of the Earth's magnetic field in the body frame is denoted as $b(t)$ with a known bound $|b(t)| \leq c_b$. The corresponding vector in the inertial frame is given as $b_i(t)$. These vectors are related using the rotation matrix $R(t) \in \mathrm{SO}(3)$ through the relation

$$b_i(t) := R(t)b(t). \tag{1}$$

Naturally, the dynamics of the magnetic measurements can be expressed, using the attitude dynamics $\dot{R} = R\omega_\times$ [1], as

$$\dot{b} = b \times \omega + R^T \dot{b}_i. \tag{2}$$

In addition, we consider the dynamic evolution of the angular velocity through the Euler equations

$$J\dot{\omega} = (J\omega) \times \omega + b \times \tau, \tag{3}$$

with J the constant, symmetric inertia matrix, ω the angular velocity expressed in the body frame and τ the control input.

In summary, the dynamics under consideration are [2]

$$\dot{b} = b \times \omega + R^T \dot{b}_i \tag{4}$$

$$J\dot{\omega} = (J\omega) \times \omega + b \times \tau. \tag{5}$$

The following working assumption will be adopted for the magnetic field in the inertial frame.

Assumption 1. The considered orbit for the spacecraft satisfies

$$\frac{1}{T}\int_t^{t+T} b_{i\times}(\sigma) b_{i\times}^T(\sigma) d\sigma \succeq \mu I \succ 0, \tag{6}$$

for some $T > 0$, $\mu > 0$ and $\forall t \geq 0$.

As mentioned in Lovera (2015), this assumption is only mildly restrictive and can be easily verified that it holds for most orbits of practical interest for LEO spacecraft. For example, in the case of a satellite orbiting the Earth, the assumption can be explored, depending on the various parameters of the orbit. Typical results obtained for a variety of orbital elements are reported in Table 1. The results were obtained using the simulation software package PROPAT Carrara (2015) which includes the IGRF11 model for Earth's magnetic field, defined by the IAGA (International Association of Geomagnetism and Aeronomy). As performance index, the following normalized parameter is considered: Index := $\frac{\lambda_{\min}(\int_0^{T_{\text{orbit}}} b_{i\times}(\sigma) b_{i\times}^T(\sigma) d\sigma)}{\int_0^{T_{\text{orbit}}} \|b_i(\sigma)\|^2 d\sigma}$. As clearly appears, inclined orbits with low-altitude are more favorable with respect to the assumption. In this model, an equatorial orbit features a low index. Obviously, the GEO geo-synchronous orbit, which features a constant magnetic field as the position of the orbiting elements is constant relative to the Earth, has a null performance index.

3. MAIN RESULT

In this work, and as opposed to the other works in the literature, our control design will be solely based on the Euler equations (3) and Assumption 1.

Consider now the control law

$$\tau := kb \times (M\omega) \tag{7}$$

[1] The matrix $y_\times$ is the skew-symmetric matrix associated to any vector $y \in \mathbb{R}^3$, and is defined simply through the cross-product $y_\times x := y \times x$ for any $x \in \mathbb{R}^3$.

[2] We drop the explicit dependence on time when it is clear from the context.

Table 1. Index relative to Assumption 1, for various orbits

Orbit type	semimajor axis (km)	eccentricity	inclination (deg)	Index
LEO Non-inclined	150	0	0	0.07
LEO Inclined	150	0	30	0.44
LEO Inclined	150	0	60	0.28
LEO Polar	150	0	90	0.22
LEO Sun-Sync	150	0	98	0.22
LEO Sun-Sync	900	0	98	0.22
LEO Inclined non-polar	340	0	51.8	0.34
LEO Near-polar non-sun-sync	450	0	80	0.22
LEO Near-polar non-sun-sync	800	0	80	0.22
MEO Semi synchronous	20000	0.01	55	0.32
GEO Geo-synchronous	35786	0	0	0.00

with the scalar gain $k > 0$. This law is parameterized by the positive-definite matrix $M = M^T \succ 0$, that is to be chosen accordingly. The closed-loop system (3),(7), reads

$$J\dot{\omega} = (J\omega) \times \omega + kb_\times^2(M\omega). \tag{8}$$

Taking the new momenta vector

$$\Omega := RJ\omega, \tag{9}$$

the resulting dynamics takes the form

$$\dot{\Omega} = kRb_\times^2 MJ^{-1}R^T\Omega, \tag{10}$$

which since $Rb_\times^2 R^T = -Rb_\times b_\times^T R^T = -Rb_\times R^T Rb_\times^T R^T = -Rb_\times R^T(Rb_\times R^T)^T = -(Rb)_\times (Rb)_\times^T = -b_{i\times}b_{i\times}^T$, results in the following closed-loop system

$$\dot{\Omega} = -kb_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega. \tag{11}$$

We will essentially define a class of control laws through the selection of the gain matrix M.

Remark 2. We refer to the aforementioned laws as a class of controllers as, similarly to what is done in the literature, and apart from the selection of M, we can obtain different scaled/normalized versions of these laws. Such a law is derived later when we consider control saturations.

The main result of this work is summarized in the following, under conditions on the persistence constant μ that will be explicited in the proof.

Proposition 3. The closed-loop dynamics (11) with $M := I$ or $M := J$ is UGES at the origin.

Proof. The first step of the proof is to show that for the above choices we can establish uniform global stability of the origin. To this end consider the positive definite, radially unbounded function

$$V_1(\Omega, t) := \frac{1}{2}\Omega^T R(t)MJ^{-1}R^T(t)\Omega. \tag{12}$$

Calculating its time derivative along the closed-loop trajectories yields

$$\dot{V}_1 = \frac{1}{4}\Omega^T R\Big((J^{-1}R^T\Omega)_\times MJ^{-1} + MJ^{-1}(J^{-1}R^T\Omega)_\times^T\Big)R^T\Omega \\ - k\Omega^T RMJ^{-1}R^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega. \tag{13}$$

Now, in order to establish uniform boundedness of the trajectories $\omega(t)$, and without imposing any a priori bound on $\omega(t)$, it is necessary that the first term cancels out. This is exactly so for the two choices of the matrix M: 1) $M = I$, which recovers the classic *b-dot* law; 2) $M = J$, that generates another class of control laws.

For both of these cases, i.e. $M = I$ and $M = J$, we then obtain $\dot{V}_1 = -k\Omega^T RJ^{-1}MR^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega \le 0$, which establishes $V_1(\Omega(t), t) \le V_1(\Omega(t_0), t_0)$ and proves uniform global stability of the origin. Hence, we get boundedness of $\Omega(t)$, $|\Omega(t)| \le c_\Omega$ (c_Ω depending on the initial conditions and which is a priori practically fixed), and equivalently of $\omega(t)$ with $|\omega(t)| \le c_\omega$. The second step is to construct a strict Lyapunov function. Our approach hinges upon the construction of a strict Lyapunov function as is done following the recent works of Malisoff and Mazenc (2009) and Maghenem and Loría (2017) for persistently-excited linear time-varying systems.

To this end, we propose the positive definite, radially unbounded function

$$V(\Omega, t) := \rho_1 V_1 + \Omega^T RJ^{-1}MR^T P(t)RMJ^{-1}R^T\Omega \tag{14}$$

$$P(t) := (1 + c_b^2)TI - \frac{1}{T}\int_t^{t+T}\int_s^t b_{i\times}(\tau)b_{i\times}^T(\tau)d\tau d\sigma, \tag{15}$$

with $\rho_1 > 0$ to be properly selected. First of all, we can show after straightforward calculations that the positive definite matrix $P(t)$ satisfies

$$TI \preceq P(t) \preceq (1 + c_b^2)TI \tag{16}$$

$$\dot{P} = -\frac{1}{T}\int_t^{t+T} b_{i\times}(\tau)b_{i\times}^T(\tau)d\tau + b_{i\times}(t)b_{i\times}^T(t). \tag{17}$$

Its time-derivative along trajectories of the closed-loop system results in

$$\begin{aligned}\dot{V} \le &-\mu\Omega^T RJ^{-1}MMJ^{-1}R^T\Omega \\ &- (\rho_1 k - 1)\Omega^T RJ^{-1}MR^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega \\ &- 2k\Omega^T RJ^{-1}MR^T PRMJ^{-1}R^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega \\ &+ 2\Omega^T R\Big((J^{-1}R^T\Omega)_\times J^{-1}MR^T PRMJ^{-1} \\ &+ J^{-1}M(J^{-1}R^T\Omega)_\times^T R^T PRMJ^{-1}\Big)R^T\Omega.\end{aligned}$$

At this point notice that, for general (symmetric, positive definite) matrices M, the first two quadratic terms in $\dot{V}$ are negative (semi-)definite, the third term is quadratic in Ω and can be dominated by the first two, while the last term is problematic to establish uniform exponential/asymptotic stability of the origin. However, for the choices of interest, i.e. $M = I$ and $M = J$, we have already been able to prove uniform boundedness of $\Omega(t)$ and can incorporate this bound to establish the desired stability margins. Moreover, when that $M = J$ straightforward calculations show that the last term in $\dot{V}$ cancels out since

$$\Omega^T R\Big((J^{-1}R^T\Omega)_\times J^{-1}MR^TP(t)RMJ^{-1}$$
$$+ J^{-1}M(J^{-1}R^T\Omega)_\times^T R^TP(t)RMJ^{-1}\Big)R^T\Omega$$
$$= \Omega^T R\Big((J^{-1}R^T\Omega)_\times - (J^{-1}R^T\Omega)_\times\Big)R^TP(t)\Omega = 0.$$

As a result, for this case, it is not required to use the bound on $\Omega(t)$ to conclude uniform global exponential stability. This seems to be, at least from a pure stability viewpoint, an advantage of the modified control law with respect to the classic *b-dot* law.

We continue now with the procedure of bounding $\dot{V}$. Taking from where we left off and by applying Young's inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ with $a := b_{i\times}^T RMJ^{-1}R^TPMJ^{-1}R^T\Omega$ and $b := kb_{i\times}^T RMJ^{-1}R^T\Omega$, with the bound from (6) and the known bounds on $\omega(t)$ (equivalently $\Omega(t)$), $b_i(t)$ and $P(t)$ we obtain

$$\begin{aligned}
\dot{V} &\leq -\mu\Omega^T RJ^{-1}MMJ^{-1}R^T\Omega \\
&\quad - (\rho_1 k - 1)\Omega^T RJ^{-1}MR^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega \\
&\quad - 2k\Omega^T RJ^{-1}MR^TPRMJ^{-1}R^T b_{i\times}b_{i\times}^T RMJ^{-1}R^T\Omega \\
&\quad + 2\Omega^T R\Big((J^{-1}R^T\Omega)_\times J^{-1}MR^TPRMJ^{-1} \\
&\quad + J^{-1}M(J^{-1}R^T\Omega)_\times^T R^TPRMJ^{-1}\Big)R^T\Omega \\
&\leq -\mu|MJ^{-1}R^T\Omega|^2 - (\rho_1 k - 1)|b_{i\times}^T RMJ^{-1}R^T\Omega|^2 \\
&\quad + \frac{k^2}{\epsilon}|b_{i\times}^T RMJ^{-1}R^T\Omega|^2 \\
&\quad + \epsilon c_b^2(1+c_b^2)^2T^2|J^{-1}M|^2|b_{i\times}^T RMJ^{-1}R^T\Omega|^2 \\
&\quad + 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1)|MJ^{-1}R^T\Omega|^2,
\end{aligned}$$

and finally,

$$\begin{aligned}
\dot{V} &\leq -\Big(\mu - \epsilon c_b^2(1+c_b^2)^2T^2 \\
&\quad - 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1)\Big)|MJ^{-1}R^T\Omega|^2 \\
&\quad - (\rho_1 k - 1 - \frac{k^2}{\epsilon})|b_{i\times}^T RMJ^{-1}R^T\Omega|^2 < 0, \quad \forall\Omega \neq 0,
\end{aligned}$$

for ϵ, ρ_1 freely chosen such that

$$\epsilon \ll \frac{1}{c_b^2(1+c_b^2)^2T^2} \tag{18}$$

$$\rho_1 \geq \frac{1}{k} + \frac{k}{\epsilon}, \tag{19}$$

while μ should satisfy

$$\mu > 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1). \tag{20}$$

We stress again the fact that for the case $M = J$ the third term from the first inequality in $\dot{V}$ vanishes and as such the condition on μ becomes

$$\mu > 0. \tag{21}$$

To obtain an explicit convergence rate we can make a practical selection

$$\epsilon := \frac{\mu}{2(c_b^2(1+c_b^2)^2T^2)} \tag{22}$$

$$\rho_1 := \frac{2}{k} + \frac{2k}{\epsilon} \approx \frac{2}{k} + \frac{4k}{\mu} \tag{23}$$

that results in

$$\dot{V} \leq -\Big(\frac{\mu}{2} - 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1)\Big)|\Omega|^2.$$

Using now the upper bound on V, that is

$$V(t,\Omega) \leq \Big(\frac{\rho_1}{2} + (1+c_b^2)T\Big)|MJ^{-1}R^T\Omega|^2,$$

we finally obtain

$$\dot{V} \leq -\frac{k\mu(\mu - 4(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1))}{2(\mu+2k^2)}V.$$

For the case $M = J$ in particular we would simply obtain

$$\dot{V} \leq -\frac{k\mu^2}{2(\mu+2k^2)}V,$$

since the term $4(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1))$ would not appear.

Remark 4. (Alternative proof for the case $M = J$). In this case in fact, we can establish UGES of the origin by referring to some well known results on the stability of linear time-varying systems.To do so, first observe that in this case the resulting closed-loop dynamics takes the form

$$\dot{\Omega} = kRb_\times^2 R^T\Omega, \tag{24}$$

which using the identity $Rb_\times^2R^T = -b_{i\times}b_{i\times}^T$, results in the following closed-loop system

$$\dot{\Omega} = -kb_{i\times}b_{i\times}^T\Omega. \tag{25}$$

This is the classical (linear time-varying) dynamical system studied widely in adaptive control theory that is ensured to be UGES with respect to the origin under the persistency-of-excitation condition of Assumption 1 and uniform boundedness of $b_i(t), \dot{b}_i(t)$, see Narendra and Anaswamy (1989); Loría and Panteley (2002).

Furthermore, notice that Assumption 1 can be relaxed using the results of the recent studies provided by Praly (2017); Barabanov and Ortega (2017). In these studies it is shown that it is possible to ensure (non uniform) convergence to zero even if the integral in (6) converges to zero, i.e., $\mu(t) \to 0$ as $t \to \infty$.

4. EXTENSIONS - PERSPECTIVES

4.1 Saturated control law

We wish now to modify the nominal controller of the previous section such that we respect the saturation bounds of the magnetic actuators and at the same time guarantee similar stability margins. To this end, consider now the modified, bounded control law

$$\tau := kb \times \frac{M\omega}{\sqrt{1+|M\omega|^2}} \tag{26}$$

with $k > 0$ and $|\tau| \leq kc_b$, where $|b(t)| \leq c_b$. In this case, the closed-loop system (3),(26) reads

$$J\dot{\omega} = (J\omega)\times\omega + kb_\times^2\frac{M\omega}{\sqrt{1+|M\omega|^2}}. \tag{27}$$

Similarly to the nominal case, using the rotated momenta Ω as state we arrive at the closed-loop system

$$\dot{\Omega} = -kb_{i\times}b_{i\times}^T\frac{RMJ^{-1}R^T\Omega}{\sqrt{1+|RMJ^{-1}R^T\Omega|^2}} \tag{28}$$

We can now state a result similar to the unsaturated case with a similar construction of a SLF and with appropriate conditions on the persistence constant μ that will appear in the proof given in the Appendix.

Proposition 5. The closed-loop dynamics (28) with $M = I$ or $M = J$ is UGAS at the origin.

4.2 Regulation to a constant, non-zero ω_d

The previous developments have considered the problem of regulation of the angular velocity to zero. Now, we study the problem of regulation to a given constant angular velocity and establish under which conditions our proposed nominal controller is effective. We will focus only to the case $M = J$ as it is easier to present.
For this case, the control law is chosen

$$\tau := kb \times \Big(J(\omega - \omega_d)\Big) \tag{29}$$

with the scalar gain $k > 0$. The closed-loop system (3)-(29) now reads

$$J\dot{\omega} = (J\omega) \times \omega + kb_\times^2 \Big(J(\omega - \omega_d)\Big). \tag{30}$$

To proceed with the stability analysis we need to define the tracking error

$$\omega_r := \omega - \omega_d. \tag{31}$$

The error dynamics then reads

$$J\dot{\omega}_r = (J\omega) \times \omega + kb_\times^2 (J\omega_r) \tag{32}$$

$$= (J\omega_r) \times \omega + (J\omega_d) \times \omega + kb_\times^2 (J\omega_r). \tag{33}$$

Taking the new (rotated) momenta vector

$$\Omega_r := RJ\omega_r, \tag{34}$$

the resulting rotated error dynamics takes the form

$$\dot{\Omega}_r = kRb_\times^2 R^T \Omega_r + R\Big((J\omega_d) \times \omega\Big) \tag{35}$$

$$= kb_{i\times}^2 \Omega_r + R\Big((J\omega_d) \times \omega\Big) \tag{36}$$

Based on some usual assumptions in the literature, we will examine three cases for the stability analysis:
1) Case of negligible gyroscopic terms, that is

$$(J\omega) \times \omega \approx 0. \tag{37}$$

In this case, the (rotated) error dynamics is given as

$$\dot{\Omega}_r = kb_{i\times}^2 \Omega_r, \tag{38}$$

and UGES of $\Omega_r = 0$ can be straightforwardly established using our previous results.
2) Case of a spherically symmetric mass distribution, i.e an inertia matrix satisfying

$$J := J_0 I, \quad J_0 > 0. \tag{39}$$

In such scenario, we can use the property

$$(J\omega_d) \times \omega_d = 0, \tag{40}$$

to re-express the (rotated) error dynamics as

$$\dot{\Omega}_r = kRb_\times^2 R^T \Omega_r + R\Big((J\omega_d) \times \omega\Big) \tag{41}$$

$$= kb_{i\times}^2 \Omega_r + R(J\omega_d) \times (J^{-1}\Omega_r). \tag{42}$$

Now, we can proceed *mutatis mutandis* as in the previous sections since the last term is linear in Ω_r. However, we will get slightly different conditions as we will need to account for the contribution of this extra term.
3) Case of a general inertia matrix. Since $\dot{\omega}_d = 0$, we can write

$$(J\omega_d) \times \omega_d = c_d, \tag{43}$$

with $c_d \in \mathbb{R}^3$ defining a certain constant vector. In this case, we write

$$\dot{\Omega}_r = kRb_\times^2 R^T \Omega_r + R\Big((J\omega_d) \times \omega\Big) \tag{44}$$

$$= kb_{i\times}^2 \Omega_r + R(J\omega_d) \times (J^{-1}\Omega_r) - Rc_d. \tag{45}$$

Following similar Lyapunov arguments as previously, we can show that the system is input-to-state stable (ISS) with c_d as input. As c_d is constant, hence uniformly bounded, we can immediately conclude boundedness of trajectories $\Omega_r(t)$, equivalently of ω_r and ω. Intuitively, we can see that at a certain extent we can theoretically minimize the effect of c_d thanks to the gain k, and eventually the constant μ from Assumption 1, in order to establish a semi-global result.

5. CONCLUSIONS

We have presented a class of control laws that ensure the global magnetic detumbling for a rigid spacecraft. It is shown that the classic *b-dot* controller is a particular law of this more general class of laws. Our main contribution was to prove analytically that for this set of laws we can ensure uniform global exponential stability of the origin. As such, this seems to be the first reported result in the literature which proves that the *b-dot* law can accomplish these stability margins. Furthermore, for a set of control gains different from *b-dot*, we prove that the corresponding closed-loop system can operate with less restrictions in the control gains.
Current work focuses on the comparison of the different control laws through extensive simulations for realistic mission scenarios, for which we will consider a time-varying inertia matrix, as in Filipe et al. (2014); Weiss et al. (2011). Finally, the combination of magnetic with other actuators, e.g. air drag Sutherland et al. (2019), will be another line of research.

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Appendix A. STABILITY PROOF FOR SATURATED LAW

As per the nominal case, we define

$$V_1(\Omega,t) := \frac{1}{2}\Omega^T R(t) M J^{-1} R^T(t)\Omega, \qquad \text{(A.1)}$$

but now we define

$$V(\Omega,t) := \rho_1 V_1 + \frac{\Omega^T R J^{-1} M R^T P(t) R M J^{-1} R^T \Omega}{\sqrt{1+|RMJ^{-1}R^T\Omega|^2}} \qquad \text{(A.2)}$$

$$P(t) := (1+c_b^2)TI - \frac{1}{T}\int_t^{t+T}\int_s^t b_{i\times}(\tau) b_{i\times}^T(\tau)\, d\tau\, d\sigma, \qquad \text{(A.3)}$$

with $\rho_1 > 0$ to be properly selected. And we remind that $P(t)$ satisifes

$$TI \preceq P(t) \preceq (1+c_b^2)TI \qquad \text{(A.4)}$$

$$\dot{P} = -\frac{1}{T}\int_t^{t+T} b_{i\times}(\tau) b_{i\times}^T(\tau)\, d\tau + b_{i\times}(t) b_{i\times}^T(t), \qquad \text{(A.5)}$$

Proceeding similarly to the nominal control case, we examine the time derivative of V in (A.2) along the closed-loop trajectories of (28) that, using again Young's inequality with the constanst ϵ, ϵ_1, results in

$$\begin{aligned}
\dot{V} \le & -\mu\frac{\Omega^T R J^{-1} M M J^{-1} R^T \Omega}{\sqrt{1+|RMJ^{-1}R^T\Omega|^2}} \\
& - (\rho_1 k - 1)\frac{\Omega^T R J^{-1} M R^T b_{i\times} b_{i\times}^T R M J^{-1} R^T \Omega}{\sqrt{1+|RMJ^{-1}R^T\Omega|^2}} \\
& - 2k\frac{\Omega^T R J^{-1} M R^T P R M J^{-1} R^T b_{i\times} b_{i\times}^T R M J^{-1} R^T \Omega}{1+|RMJ^{-1}R^T\Omega|^2} \\
& + 2\Omega^T R\Big((J^{-1}R^T\Omega)_\times J^{-1} M R^T P R M J^{-1} \\
& + J^{-1}M(J^{-1}R^T\Omega)_\times^T R^T P R M J^{-1}\Big)\frac{R^T\Omega}{1+|RMJ^{-1}R^T\Omega|^2} \\
& - \frac{\Omega^T R J^{-1} M R^T P(t) R M J^{-1} R^T \Omega}{(1+|RMJ^{-1}R^T\Omega|^2)^2}\cdot \\
& \cdot\ \Omega^T R J^{-1} M M J^{-1} (J^{-1}R^T\Omega)_\times R^T\Omega \\
& + k\frac{\Omega^T R J^{-1} M R^T P(t) R M J^{-1} R^T \Omega}{(1+|RMJ^{-1}R^T\Omega|^2)^{3/2}}\cdot \\
& \cdot\ \Omega^T R J^{-1} M M J^{-1} R^T b_{i\times} b_{i\times}^T R M J^{-1} R^T \Omega \\
\le & -\Big(\mu - \epsilon c_b^2(1+c_b^2)^2 T^2 \\
& - 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1) \\
& - (1+c_b^2)Tc_\omega|J^{-1}M||M^{-1}J| \\
& - \frac{k(1+c_b^2)Tc_b^4|J^{-1}M|}{2\epsilon_1}\Big)|MJ^{-1}R^T\Omega|^2 \\
& - (\rho_1 k - 1 - \frac{k^2}{\epsilon} - \frac{k(1+c_b^2)Tc_b^2\epsilon_1}{2})\cdot \\
& \cdot\ |b_{i\times}^T R M J^{-1} R^T \Omega|^2 < 0, \quad \forall \Omega \neq 0,
\end{aligned}$$

for $\epsilon, \rho_1, \epsilon_1$ freely chosen such that

$$\epsilon \ll \frac{1}{c_b^2(1+c_b^2)^2T^2} \qquad \text{(A.6)}$$

$$\epsilon_1 \gg k(1+c_b^2)Tc_b^4|J^{-1}M| \qquad \text{(A.7)}$$

$$\rho_1 \ge \frac{1}{k} + \frac{k}{\epsilon} + \frac{k(1+c_b^2)Tc_b^2\epsilon_1}{2}, \qquad \text{(A.8)}$$

and the persistence constant μ must be such that the first term becomes negative, that is essentially

$$\mu > 2(1+c_b^2)Tc_\omega(|J^{-1}M||M^{-1}J|+1) + (1+c_b^2)Tc_\omega|J^{-1}M||M^{-1}J|.$$

Again, straightforward calculations can show that in the case where $M = J$ the conditions on the free parameters and μ are significantly less restrictive.