Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A continuation approach to state and adjoint calculation in optimal control applied to the reentry problem

Knut Graichen **Nicolas Petit**

Centre Automatique et Systèmes, Ecole des Mines de Paris, 75272 Paris, France (e-mail: { knut.graichen , nicolas.petit }@ensmp.fr)

Abstract: A well–known problem in indirect optimal control is to find a suitable initial guess for the adjoint states which is sufficiently close to the optimal solution. This paper presents a new homotopy approach to overcome this problem by deriving an auxiliary optimal control problem (OCP) for which the adjoint states are simply zero. A continuation method is employed to smoothly reach the original OCP. The auxiliary OCP can be derived with respect to any given initial trajectory of the system, for instance obtained by forward integration. The approach is applied to the space shuttle reentry problem, which represents a benchmark problem in optimal control due to its high numerical sensitivity with respect to the initial solution.

Keywords: Optimal control; Homotopy continuation method; Aerospace applications.

1. INTRODUCTION

Lately, optimal control problems (OCPs) have received much attention especially in the context of Model Predictive Control, which in turn has spurred interest in efficient numerical methods for real–time applications (Allgöwer et al., 1999; Kouvaritakis and Cannon, 2001; Diehl et al., 2002).

The numerical methods for solving OCPs can roughly be divided in two different classes. In the *direct approach*, the model equations of the considered system are discretized and the state and control trajectories are parametrized in order to obtain a finite–dimensional parameter optimization problem, see e.g. (Hargraves and Paris, 1987; Betts, 1998, 2001; Seywald, 1994; Nocedal and Wright, 1999). The well–known advantage of the direct approach is the good numerical robustness with respect to the initial guess as well as the handling of constraints. On the other hand, *indirect approaches* are based on Pontryagin's maximum principle (Pontryagin et al., 1962; Bryson and Ho, 1969) and require the solution of a two–point boundary value problem (BVP). Indirect methods are known to show a fast numerical convergence in the neighborhood of the optimal solution and to deliver highly accurate solutions, which makes them particularly attractive in aerospace industries. However, a main difficulty in the indirect method is the requirement of a good initial guess of the trajectories, especially of the adjoint states. If the initial guess is too far away from the optimal solution, the numerical solution of the BVP will in general fail to converge.

The problem of finding a suitable initial guess for the adjoint states has attracted much attention. In particular, von Stryk and Bulirsch (1992) proposed to use both direct and indirect methods combined in a hybrid scheme to overcome the initial guess problem and applied the method to the reentry problem. Further approaches to calculate the adjoint states based on trajectories obtained from direct methods are proposed e.g. by Martell and Lawton (1995) and Seywald and Kumar (1996). However, all these approaches still require the direct method to obtain initial near–optimal guesses for the indirect solution.

The main contribution of this paper is to present a new homotopy approach, which is based on an auxiliary OCP for which the adjoint states are simply zero. Starting from the auxiliary OCP, a continuation method is employed to smoothly reach the original OCP. The auxiliary OCP can be derived for any given initial trajectory of the system, e.g. obtained by forward integration. Hence, the homotopy approach can be seen as "self–contained", since it does not require any initial near–optimal trajectory from direct optimization approaches.

For the sake of illustration, the homotopy approach is applied to the space shuttle reentry problem, which is a frequently used benchmark in optimal control due to several challenging features like highly nonlinear dynamics and a high numerical sensitivity with respect to the initial guess of the trajectories. Direct optimization methods have been used for various reentry problems by Betts (2001); Bonnans and Launay (1998), and Neckel et al. (2003) in the context of inverse dynamic optimization. The indirect method based on Pontryagin's maximum principle has been applied to reentry problems e.g. by Pesch (1994); Kreim et al. (1996); Bonnard et al. (2003).

The paper is organized as follows: In Section 2, the homotopy approach based on the auxiliary OCP is introduced for a general class of OCPs. The reentry problem and the optimal control objective is described in Section 3. Finally, Section 4 shortly explains the collocation method, which is used to solve the 2–point BVP of the reentry problem and presents the reentry trajectories by using the homotopy approach via the auxiliary OCP.

2. HOMOTOPY APPROACH WITH AUXILIARY OCP

The considered OCP is to minimize a cost functional

$$J(x,u,t) = \varphi(x(T),T) + \int_0^T L(x,u,t)\,\mathrm{d}t \qquad (1)$$

with the free end time T, subject to the system equations and initial conditions

$$\dot{x} = f(x,u), \quad x(0) = x_0 \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{2a}$$

and the partial final conditions

$$x_i(T) = x_{T,i}, \quad i = 1,\ldots,q \tag{2b}$$

for the first q states x_i, $i = 1,\ldots,q$ of the state vector $x = (x_1,\ldots,x_n)^\mathsf{T}$. Due to the final conditions (2b), it is generally assumed that the final cost term $\varphi(x(T),T)$ in (1) only depends on the remaining $n-q$ states, i.e.

$$\varphi = \varphi(x_i(T),T), \quad i = q+1,\ldots,n$$

2.1 Necessary optimality conditions

The OCP (1), (2) can be solved with the indirect method in optimal control using the calculus of variations (Bryson and Ho, 1969). By defining the Hamiltonian

$$H(x,\lambda,u,t) = L(x,u,t) + \lambda^\mathsf{T} f(x,u),$$

the first–order necessary conditions for an optimal solution of the OCP (1), (2) concern the minimization of $H(x,\lambda,u,t)$ with respect to u,

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^\mathsf{T}\frac{\partial f}{\partial u} = 0, \tag{3}$$

and the adjoint equations

$$\dot{\lambda}^\mathsf{T} = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^\mathsf{T}\frac{\partial f}{\partial x}, \tag{4a}$$

with the final conditions

$$\lambda_i(T) = \left.\frac{\partial \varphi}{\partial x_i}\right|_T, \quad i = q+1,\ldots,n \tag{4b}$$

to be satisfied. The free (unknown) end time T is taken into account by the transversality condition

$$H(x,\lambda,u,t)|_T = -\left.\frac{\partial\varphi}{\partial t}\right|_T. \tag{5}$$

The ordinary differential equations (ODEs) and boundary conditions (2), (4), (5) together with the algebraic equation (3) form a two–point BVP. Its numerical solution yields the optimal state trajectories $x^*(t)$, $\lambda^*(t)$, the optimal control $u^*(t)$, $t \in [0,T^*]$ and the optimal end time T^*.

2.2 Auxiliary optimal control problem

A main obstacle for solving the above BVP (2)–(5) is the initial guess of the adjoint trajectories $\lambda(t)$, $t \in [0,T]$ and the guess of the free end time T. If $\lambda(t)$ and T are not sufficiently close to the optimal solution $\lambda^*(t)$ and T^*, the numerical solution of the BVP may not converge. As mentioned in the introduction, several approaches exist in the literature to address this problem, see e.g. (von Stryk and Bulirsch, 1992; Martell and Lawton, 1995; Seywald and Kumar, 1996). However, they typically require a near–optimal trajectory (usually by involving direct optimization methods) to calculate an initial trajectory for the adjoint state, which is sufficiently close to the optimal one. In contrast to this, the focus of this section is to construct an auxiliary OCP for a given (not necessarily near–optimal) trajectory of the system for which the optimal solution of the adjoint state can easily be derived. This auxiliary OCP is then used in a homotopy approach to reach the original OCP (1), (2).

In a first step, it is assumed that an initial trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$ with a certain final time T^0 is known, which satisfies the system equations and initial conditions in (2a):

$$\dot{x}^0 = f(x^0,u^0), \quad x^0(0) = x_0\,, \tag{6a}$$

In general, the partial final conditions (2b) will not be satisfied but evaluate to a residual

$$x^0(T^0) = x^0_T \tag{6b}$$

at the end time T^0. One possibility to derive the trajectory $(x^0(t),u^0(t))$ is, for instance, a numerical forward integration of the system over the time interval $t \in [0,T^0]$. The final time T^0 should be reasonably chosen within the range of the expected optimal final time T^* of the original OCP (1), (2), while keeping the distance between the final values in (2b) and (6b) sufficiently small.

The idea of the homotopy approach is to derive an auxiliary OCP for which the optimal solution is exactly the initial trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$. This can be achieved by minimizing the cost

$$J^0(u,t) = \varphi^0(T) + \int_0^T L^0(u,t)\,\mathrm{d}t \tag{7}$$

with the functions [1]

$$\varphi^0(T) = \frac{1}{2}(T-T^0)^2, \quad L^0(u,t) = \frac{1}{2}\left|u(t) - u^0\left(t\,T^0/T\right)\right|^2$$

and the free end time T, subject to the system (2a)

$$\dot{x} = f(x,u), \qquad x(0) = x_0 \tag{8a}$$

and the modified final conditions

$$x_i(T) = x^0_{T,i}, \quad i = 1,\ldots,q \tag{8b}$$

adapted from (6b). Obviously, the optimal solution to the OCP (7), (8) is the end time $T = T^0$ and the previous trajectory $(x^0(t),u^0(t))$ which results in the minimal cost $J^0(u,t) = 0$ and satisfies the final conditions (8b).

The derivation of the corresponding adjoint state solution $\lambda(t) = \lambda^0(t)$ requires a closer look at the optimality conditions. The Hamiltonian

$$H^0(x,\lambda,u,t) = L^0(u,t) + \lambda^\mathsf{T} f(x,u) \tag{9}$$

yields

$$\begin{aligned}\frac{\partial H^0}{\partial u} &= \frac{\partial L^0}{\partial u} + \lambda^\mathsf{T}\frac{\partial f}{\partial u} \\ &= \left(u - u^0\left(t\,T^0/T\right)\right)^\mathsf{T}\mathbb{I}_m + \lambda^\mathsf{T}\frac{\partial f}{\partial u} = 0,\end{aligned} \tag{10}$$

whereby $\partial L^0/\partial u$ simplifies due to $|u-u^0|^2 = \sum_{i=1}^m (u_i - u_i^0)^2$. The symbol $\mathbb{I}_m$ denotes the $(m\times m)$ unit matrix. The adjoint system is defined accordingly by

$$\dot{\lambda}^\mathsf{T} = -\frac{\partial H^0}{\partial x} = -\lambda^\mathsf{T}\frac{\partial f}{\partial x} \tag{11a}$$

with the homogeneous final conditions

$$\lambda_i(T) = \left.\frac{\partial\varphi^0}{\partial x_i}\right|_T = 0, \quad i = q+1,\ldots,n\,. \tag{11b}$$

In view of the fact that $x(t) = x^0(t)$ and $u(t) = u^0(t)$ is the optimal solution to the OCP (7), (8), it directly follows that the trivial adjoint state $\lambda(t) = \lambda^0(t) = 0$ satisfies the optimality conditions (10) and (11).

[1] The time normalization $t\,T^0/T$ in the running cost $L^0(u,t)$ in (7) is necessary, since $u^0(t\,T^0/T)$ is only defined on $t \in [0,T^0]$ and not on the (free) time interval $[0,T]$.

In addition, the transversality condition

$$H^0(x,\lambda,u,t)\big|_T = -\left.\frac{\partial \varphi^0}{\partial t}\right|_T = -(T-T^0) \tag{12}$$

must hold for the free end time T, which is consistent with the previous statement that $u(t) = u^0(t)$ is the optimal control with $T = T^0$: for $u(t) = u^0(t)$ and the trivial optimal adjoint state $\lambda(t) = \lambda^0(t) = 0$, the Hamiltonian (9) evaluates to zero and the transversality condition (12) yields $T = T^0$ as the optimal end time.

2.3 Continuation towards original OCP

The auxiliary OCP (7), (8) defined in the previous section can be used as starting point for a continuation scheme to eventually reach the original OCP (1), (2). Hence, the cost to be minimized

$$J^c(x,\lambda,u,t) = \varphi^c(x(T),T) + \int_0^T L^c(x,u,t)\,\mathrm{d}t \tag{13a}$$

with the cost functions

$$\begin{aligned}\varphi^c(x(T),T) &= c_1\,\varphi(x(T),T) + (1-c_1)\,\varphi^0(T)\,,\\ L^c(x,u,t) &= c_1\,L(x,u,t) + (1-c_1)\,L^0(u,t)\end{aligned} \tag{13b}$$

depends on a first continuation parameter $c_1 \in [0,1]$ which is used to smoothly transform $J^c(x,\lambda,u,t)$ from the auxiliary cost (7) for $c_1 = 0$ to the original one (1) for $c_1 = 1$. The system equations and initial conditions (2a) remain unchanged, i.e.

$$\dot{x} = f(x,u), \qquad x(0) = x_0, \tag{14a}$$

whereas the final conditions (2b) and (8b) are coupled by a second continuation parameter $c_2 \in [0,1]$:

$$x_i(T) = c_2\,x_{T,i} + (1-c_2)\,x^0_{T,i}, \quad i = 1,\ldots,q\,. \tag{14b}$$

Hence, both parameters $c = (c_1,c_2)$ separately affect the cost (13) and the final conditions (14b). Starting at $c = (0,0)$ corresponds to the auxiliary OCP (7), (8), whereas $c = (1,1)$ yields the original one (1), (2).

The optimality conditions are derived with the Hamiltonian

$$H^c(x,\lambda,u,t) = L^c(x,u,t) + \lambda^{\mathsf{T}} f(x,u)$$

and yield

$$\frac{\partial H^c}{\partial u} = \frac{\partial L^c}{\partial u} + \lambda^{\mathsf{T}}\frac{\partial f}{\partial u} = 0 \tag{15}$$

as well as the adjoint system

$$\dot{\lambda}^{\mathsf{T}} = -\frac{\partial H^c}{\partial x} = -\frac{\partial L^c}{\partial x} - \lambda^{\mathsf{T}}\frac{\partial f}{\partial x}, \tag{16a}$$

$$\lambda_i^{\mathsf{T}}(T) = \left.\frac{\partial \varphi^c}{\partial x_i}\right|_T = c_1 \left.\frac{\partial \varphi}{\partial x_i}\right|_T, \quad i = 1,\ldots,q\,, \tag{16b}$$

and the transversality condition

$$\begin{aligned}H^c(x,\lambda,u,t)\big|_T &= -\left.\frac{\partial \varphi^c}{\partial t}\right|_T\\ &= -\,c_1\left.\frac{\partial \varphi}{\partial t}\right|_T - (1-c_1)(T-T^0)\,.\end{aligned} \tag{17}$$

Note the simplification of the right–hand sides of (16b) and (17) due to the specific structure of the final cost $\varphi^c(x(T),T)$ in (13b).

The ODEs and boundary conditions (14), (16), (17) together with the algebraic equation (15) define a two–point BVP for the states $x(t)$, $\lambda(t)$, the optimal control $u(t)$, and the end time T in dependence of the continuation parameters $c = (c_1,c_2)$.

The optimal solution $(x^*(t),\lambda^*(t),u^*(t),T^*)$ of the original OCP (1), (2) can be obtained for $c = (1,1)$ by following a homotopy map in N steps and successively increasing the two continuation parameters $c = (c_1,c_2)$ according to

$$\begin{aligned}0 \le c_1^1 \le \ldots \le c_1^{N-1} \le c_1^N = 1\,,\\ 0 \le c_2^1 \le \ldots \le c_2^{N-1} \le c_2^N = 1\,.\end{aligned}$$

The first run of the homotopy is initialized with the auxiliary trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$ and $\lambda^0(t) = 0$, whereas the subsequent steps use the solution of the previous run as initialization.

Since two continuation parameters $c = (c_1,c_2)$ separately affect the cost (13) and the boundary conditions (14b), some freedom exists concerning how c_1 and c_2 are increased and how many steps N are used until $c^N = (1,1)$ is reached. Moreover, the homotopy path as well as the number of necessary steps N also strongly depends on the system dynamics and the respective cost function and are connected to the properties of controllability and convexity.

The homotopy approach is simplified if the initial trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$ directly satisfies the desired final conditions (2b), i.e. $x^0_{T,i} = x_{T,i}$. In this case, the boundary conditions (14b) of the homotopy approach reduce to the original ones (2b). Hence, the second continuation parameter c_2 is not required and only the cost (13) has to be converted to the desired one (1) by means of c_1. This special case is always given if the considered system (2a) is flat, such that the boundary conditions in (2) and the initial trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$ can be expressed in terms of a flat output and its time derivatives (Fliess et al., 1995). The flatness approach readily solves the motion planning problem, i.e. (2b) can always be enforced and the corresponding trajectory $(x^0(t),u^0(t))$ is easily computed. Another method in the context of feedforward control design is developed in (Graichen et al., 2005; Graichen, 2006), which can be adapted to compute an initial trajectory $(x^0(t),u^0(t))$, $t \in [0,T^0]$ satisfying the boundary conditions in (2).

3. SPACE SHUTTLE REENTRY PROBLEM

The homotopy approach presented in the last section is applied to the space shuttle reentry problem, which is an ideal benchmark example for the homotopy method due to its high numerical sensitivity with respect to the initial guess of the trajectories. This section describes the model equations of the space shuttle before the optimal control problem is adapted to the homotopy BVP (14)–(17).

3.1 Equations of motion and optimal control objective

Several versions and problem formulations of the reentry optimal control problem exist in the literature, see e.g. (von Stryk and Bulirsch, 1992; Kreim et al., 1996; Betts, 2001). The reentry problem used in this contribution is due to Betts (2001). The equations of motion of the space shuttle are

$$\dot{h} = v \sin\gamma\,, \tag{18a}$$
$$\dot{v} = -\frac{D(h,v,\alpha)}{m} - g(h)\sin\gamma\,, \tag{18b}$$
$$\dot{\gamma} = \frac{L(h,v,\alpha)}{m\,v}\cos\beta + \cos\gamma\left(\frac{v}{R_e+h} - \frac{g(h)}{v}\right), \tag{18c}$$
$$\dot{\theta} = \frac{v}{R_e+h}\cos\gamma\cos\psi\,, \tag{18d}$$
$$\dot{\psi} = \frac{L(h,v,\alpha)}{m\,v\,\cos\gamma}\sin\beta + \frac{v}{R_e+h}\cos\gamma\sin\psi\sin\theta\,, \tag{18e}$$
$$\dot{\phi} = \frac{v}{R_e+h}\cos\gamma\sin\psi/\cos\theta \tag{18f}$$

with altitude h, velocity v, flight path angle γ, latitude θ, azimuth ψ, and longitude ϕ as the states of the system. The controls of the space shuttle are the angle of attack α and the bank angle β.

The gravity $g(h)$ and atmospheric density $\rho(h)$ are modeled by

$$g(h) = \mu/(R_e+h)^2, \quad \rho(h) = \rho_0\exp[-h/h_r] \tag{19}$$

and determine the lift and drag functions

$$L(h,v,\alpha) = \frac{1}{2}\,c_L(\alpha)S\rho(h)\,v^2\,, \qquad D(h,v,\alpha) = \frac{1}{2}\,c_D(\alpha)S\rho(h)\,v^2 \tag{20a}$$

with

$$c_L(\alpha) = a_0 + a_1\hat{\alpha}\,, \qquad \hat{\alpha} = 180\,\alpha/\pi\,, \qquad c_D(\alpha) = b_0 + b_1\hat{\alpha} + b_2\hat{\alpha}^2\,. \tag{20b}$$

The corresponding parameters are listed in Table 1.

The shuttle reentry starts at the initial conditions

$$h(0) = 260000 \text{ ft}, \quad v(0) = 25600 \text{ ft/s}, \quad \gamma(0) = -1 \text{ deg}, \quad \theta(0) = 0 \text{ deg}, \quad \psi(0) = 90 \text{ deg}, \quad \phi(0) = 0 \text{ deg}\,. \tag{21}$$

The final point of the reentry trajectory occurs at the unknown end time T at the so–called terminal area energy management (TAEM), which is defined by the conditions (Betts, 2001)

$$h(T) = 80000\,\text{ft}, \;\; v(T) = 2500\,\text{ft/s}, \;\; \gamma(T) = -5\,\text{deg}. \tag{22}$$

The objective of the reentry problem is to maximize the cross–range, which is equivalent to maximizing the altitude $\theta(T)$.

Symbol	Value	Symbol	Value
μ	$0.1407654 \cdot 10^{17}$ ft/s^2	ρ_0	0.002378 lbs/ft^3
R_e	20902900 ft	h_r	23800 ft
S	2690 ft^2	m	6309.44 lbs
a_0	-0.20704	a_1	0.029244
b_0	0.07854	b_1	$-0.61592 \cdot 10^{-2}$
b_2	$0.621408 \cdot 10^{-3}$	c_0	1.06723181

Table 1. Parameters of the shuttle model taken from (Betts, 2001; Neckel et al., 2003).

3.2 Adaptation to the general OCP (1), (2)

The system model (18) can be put in the general form (2) with the state and control vectors

$$x = (h, v, \gamma, \theta, \psi), \qquad u = (\alpha, \beta)\,. \tag{23}$$

The longitude ϕ and the ODE (18f) are omitted since they are decoupled and do not affect the remaining ODEs in (18). The initial conditions in (2a) follow from (21). The partial final conditions (2b) are given by (22) for the first $q = 3$ states $x_1 = h$, $x_2 = v$, and $x_3 = \gamma$. The objective of the reentry problem is to maximize the cross-range, which corresponds to the final latitude $x_4(T) = \theta(T)$. Hence, the cost functions in (1) reduce to

$$\varphi(x(T),T) = -x_4(T) = -\theta(T), \quad L(x,u,t) = 0\,,$$

which yields the cost to be minimized

$$J(x) = -x_4(T)\,. \tag{24}$$

4. NUMERICAL SOLUTION OF THE SHUTTLE REENTRY PROBLEM

Various numerical methods can be used to solve two–point BVPs as they arise in indirect optimal control, see e.g. (Pytlak, 1999) for an overview. In the *shooting method*, the system equations (14a) and the adjoint system (16a) are iteratively integrated forward in time in order to meet the final conditions in (14b) and (16). However, a numerical problem of the shooting method is that integrating both original and adjoint systems simultaneously is an inherently unstable process (Bryson, 1999). This makes the shooting method especially ill–conditioned if the system equations are very sensitive as in the case of the reentry problem.

A variant of the shooting method is the *gradient method* which iteratively integrates the system equations in forward time and the adjoint system in backward time. Although this procedure naturally leads to a better numerical behavior, the gradient method is known to be slowly converging, see e.g. (Bryson, 1999).

A powerful alternative to solve the BVP of the optimal control problem is the *collocation method*, which is not affected by the drawbacks of the shooting and gradient methods. Moreover, it easily allows to account for additional algebraic equations like (15). In the following, the collocation code which is used to solve the reentry problem is described in some detail, before the numerical results for the reentry problem using the homotopy approach in Section 2 are given.

4.1 Collocation method

The basis for the numerical solution of the reentry problem is the standard MATLAB BVP solver bvp4c, which solves nonlinear 2–point BVPs by means of the collocation method (Shampine et al., 2000; Kierzenka and Shampine, 2001). However, to be applicable to optimal control problems, the bvp4c–code was adapted by the authors to additionally account for algebraic equations like (3) as they arise from the optimality conditions. This leads to the general BVP formulation of (index 1) differential–algebraic equations (DAE)

$$\dot{y} = f(y,z,t,p), \tag{25a}$$
$$0 = g(y,z,t,p)\,, \tag{25b}$$
$$0 = h(y(t_0), y(t_f), z(t_0), z(t_f), p) \tag{25c}$$

with the differential and algebraic equations (25a), (25b) for the dynamic and algebraic states $y(t), z(t)$ on the time interval $t \in [t_0, t_f]$, and the boundary conditions (25c).

The general collocation method and its implementation in bvp4c has been left unchanged as it was designed to be applicable and numerically robust for a wide range of

BVPs. The function bvp4c divides the time interval $[t_0, t_f]$ in subintervals and discretizes the differential equations (25a) along the time mesh. The resulting discretized system equations together with the boundary conditions (25c) and the additional algebraic equations (25b) evaluated at the time points results in a set of nonlinear algebraic equations, which is solved with a Newton iteration scheme.

In addition, bvp4c employs a robust mesh refinement strategy to adapt the time mesh and the number of grid points in each Newton step based on the residual along the discretized differential equations (25a). More details on bvp4c can be found in (Kierzenka and Shampine, 2001).

4.2 Numerical results

In order to apply the collocation method to the homotopy approach in Section 2, the BVP (14)–(17) has to be adapted to the DAE form (25). The original and adjoint systems in (14a) and (16) form the dynamics (25a) with the overall dynamic state $y^{\mathsf{T}} = (x^{\mathsf{T}}, \lambda^{\mathsf{T}})$. The input u denotes the algebraic variable $z = u$ with (15) corresponding to the algebraic equation (25b). The boundary conditions for x and λ in (14) and (16b) together with the transversality condition (17) are comprised in (25c). The free end time T is taken into account by means of the time transformation

$$t = \varepsilon\tau, \qquad T = \varepsilon, \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} \tag{26}$$

with the normalized time coordinate $\tau \in [0, 1]$. Hence, the scaling factor ε is treated as free parameter $p = \varepsilon$ in the DAE system (25) and the new time coordinate τ replaces $t \in [t_0, t_f]$ with the normalized interval boundaries $t_0 = 0$ and $t_f = 1$.

The analytic operations, e.g. derivation of the adjoint system (16), have been performed with the software package Mathematica. All functions and equations (14)–(17) for the homotopy solution of the reentry problem as well as their Jacobians are provided as Cmex–functions to Matlab. The simulations are performed on a PC equipped with an Intel CPU of type Pentium Core Duo 1.6 GHz and 2 GB memory.

In order to initialize the homotopy solution of the reentry problem outlined in Section 2, an initial trajectory $(x^0(t), u^0(t))$ is calculated by integrating the system equations (14a) over the time interval $t \in [0, T^0]$ with the final time $T^0 = 1000\,\mathrm{s}$ and the chosen constant input $u = (\alpha, \beta)^{\mathsf{T}} = (30, -30)^{\mathsf{T}}$ deg. The trajectory $(x^0(t), u^0(t))$ with the trivial adjoint state $\lambda^0(t) = 0$ is then used as initial guess for the homotopy approach. As mentioned before, some freedom exists concerning the number of steps N and how the two homotopy parameters $c = (c_1, c_2)$ are increased to $c^N = (1, 1)$. The best results for the reentry problem are obtained by firstly increasing c_1 in order to smoothly switch the cost function (13b) to the original one. This is done in 10 steps from $c_1^1 = 0.1$ to $c_1^{10} = 1$ while keeping $c_2 = 0$. Afterwards, c_2 is increased similarly from $c_2^{11} = 0.1$ to $c_2^{20} = 1$ in 10 steps to force the boundary conditions (14b) to the original values given in (22).

Figure 1 shows the reentry trajectories for several steps of the homotopy method. Clearly visible is the homotopy path that the trajectories follow by starting from the initial trajectory for $c^0 = (0, 0)$ and finally reaching the optimal solution for $c^{20} = (1, 1)$ in the 20th step. This is particularly interesting since the initial trajectory $(x^0(t), u^0(t))$ and the initial end time $T^0 = 1000\,\mathrm{s}$ are clearly far away from the optimal solution.

Table 2 summarizes some details of the successive numerical solutions by means of the collocation code described in the previous section. The homotopy approach is started with the initial trajectory $(x^0(t), u^0(t))$, $t \in [0, T^0]$ and 100 mesh points. During the steps 11 and 20 when the desired boundary conditions are reached, the mesh refinement increases the mesh size to 141 points. A final run leads to the optimal reentry trajectory with 301 mesh points, the final time $T^* = 2008.59\,\mathrm{s}$ and the maximum cross–range $\theta(T) = 34.1412\,\mathrm{deg}$, also see Figure 1. The overall required CPU time for the numerical solution amounts to 20.3 seconds.

The values of the final time T and the maximum cross–range $\theta(T)$ coincide with the reference values in (Betts, 2001) (up to the last digits given in (Betts, 2001)). This shows the accuracy of the indirect method in optimal control in connection with the homotopy approach and the collocation method. Compared to this methodology, the direct optimal control solution of the reentry problem in (Neckel et al., 2003) was more difficult to initialize and provided less accurate results.

Step	$c = (c_1, c_2)$	points	T	$\theta(T)$	CPU time
Start	$c^0 = (0, 0)$	100	1000 s	2.4 deg	—
1–10	$c^{10} = (1, 0)$	100	912.4 s	5.0 deg	5.2 s
11–20	$c^{20} = (1, 1)$	141	2008.3 s	34.1 deg	11.2 s
final refinement		301	2008.59 s	34.1412 deg	3.9 s

Table 2. Numerical statistics for the homotopy solution of the shuttle reentry problem.

5. CONCLUSION

A well–known difficulty with the indirect method in optimal control is the requirement of a good initial guess especially of the adjoint states. To overcome the problem of finding a near–optimal initial trajectory, a homotopy approach is presented which starts from an auxiliary optimal control problem and eventually reaches the original one by using a continuation scheme. An interesting property of the auxiliary OCP is that it can be constructed for any trajectory of the system which is, e.g., obtained by an initial numerical integration.

The applicability of the homotopy approach is demonstrated for the space shuttle reentry problem, which is an appropriate benchmark example in this context due to the high sensitivity of its numerical solution. Moreover, the single continuation steps and the final optimal reentry trajectory show the robustness and accuracy of the modified collocation method based on the Matlab solver bvp4c, which is employed to solve the differential–algebraic equations of the reentry problem.

The proposed homotopy approach seems to have potential in the field of optimal control, because it is easy to implement and proved efficient on a reportedly difficult OCP. It appears to be well suited for aerospace trajectory optimization problems and is currently used, along with ideas of nonlinear geometric control, on various ascent and ascent–reentry problems.

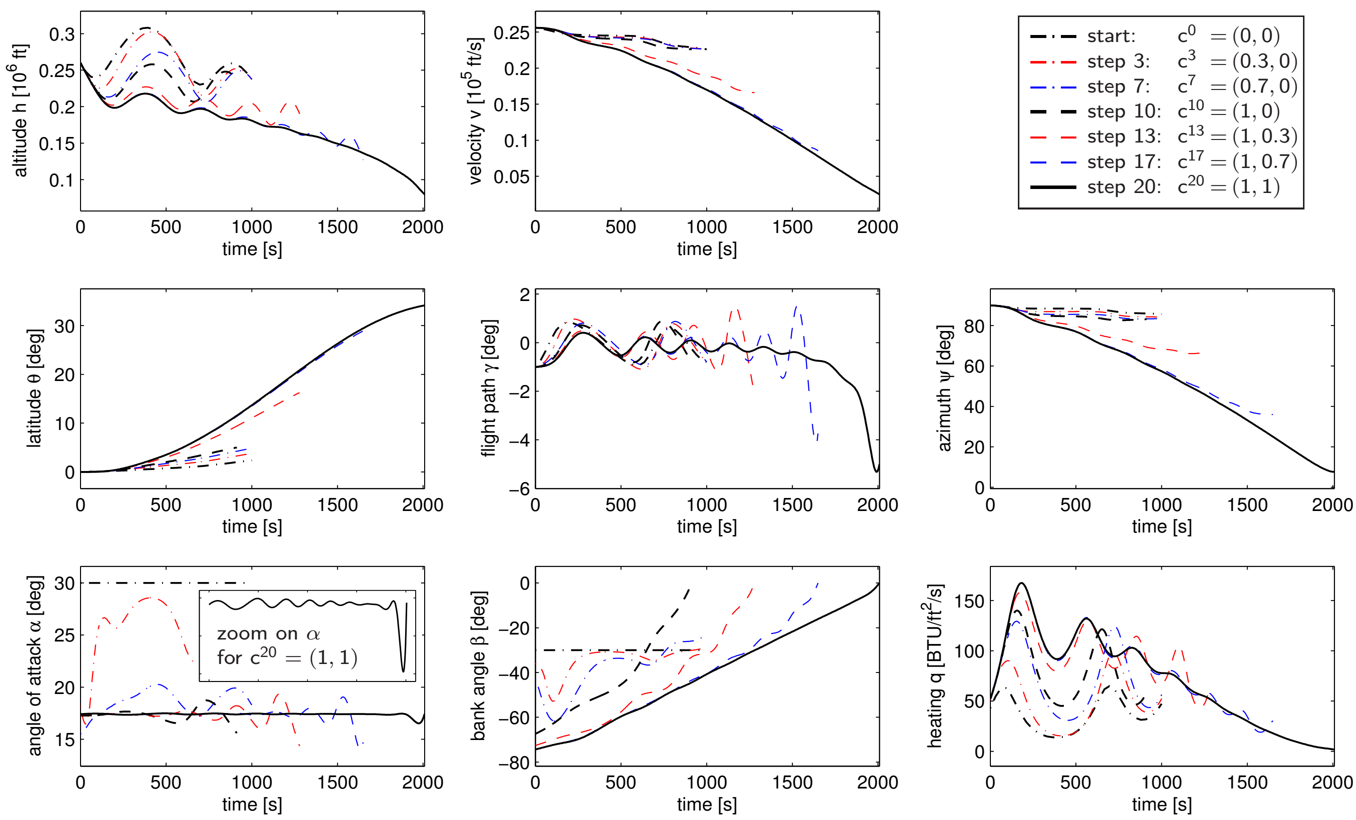

Fig. 1. Trajectories for the shuttle reentry problem using the homotopy approach with $N = 20$ steps.

REFERENCES

F. Allgöwer, T.A. Badgwell, J.S. Qin, J.B. Rawlings, and S.J. Wright. Nonlinear predictive control and moving horizon estimation – An introductory overview. In P.M. Frank, editor, *Advances in Control, Highlights of ECC'99*, pages 391–449. Springer, 1999.

J.T. Betts. Survey of numerical methods for trajectory optimization. *J. Guid. Contr. Dyn.*, 21:193–207, 1998.

J.T. Betts. *Practical Methods for Optimal Control Using Nonlinear Programming*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

J.F. Bonnans and G. Launay. Large–scale direct optimal control applied to a re–entry problem. *J. Guid. Contr. Dyn.*, 21:996–1000, 1998.

B. Bonnard, L. Faubourg, G. Launay, and E. Trélat. Optimal control with state constraints and the space shuttle re–rentry problem. *J. Dyn. Contr. Sys.*, 9:155–199, 2003.

A.E. Bryson. *Dynamic Optimization*. Addison–Wesley, Menlo Park, CA, 1999.

A.E. Bryson and Y.-C. Ho. *Applied Optimal Control*. Ginn & Company, Waltham, Massachusetts, 1969.

M. Diehl, H.G. Bock, J.P. Schlöder, R. Findeisen, Z. Nagy, and F. Allgöwer. Real–time optimization and nonlinear model predictive control of processes governed by differential–algebraic equations. *J. Proc. Contr.*, 12:577–585, 2002.

M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Contr.*, 61:1327–1361, 1995.

K. Graichen. *Feedforward Control Design for Finite–Time Transition Problems of Nonlinear Systems with Input and Output Constraints*. Doctoral Thesis, Universität Stuttgart. Shaker Verlag, Aachen (Germany), 2006. http://elib.uni-stuttgart.de/opus/volltexte/2007/3004.

K. Graichen, V. Hagenmeyer, and M. Zeitz. A new approach to inversion–based feedforward control design for nonlinear systems. *Automatica*, 41:2033–2041, 2005.

C. Hargraves and S. Paris. Direct trajectory optimization using nonlinear programming and collocation. *J. Guid. Contr. Dyn.*, 10:338–342, 1987.

J. Kierzenka and L.F. Shampine. A BVP solver based on residual control and the MATLAB PSE. *ACM TOMS*, 27:299–316, 2001.

B. Kouvaritakis and M. Cannon, editors. *Non–Linear Predictive Control: Theory and Practice*. IEE Publishing, 1st edition, 2001.

H. Kreim, B. Kugelmann, H.J. Pesch, and M.H. Breitner. Minimizing the maximum heating of a reentry space shuttle: an optimal control problem with multiple control constraints. *Opt. Contr. Appl. Meth.*, 17:45–69, 1996.

C.A. Martell and J.A. Lawton. Adjoint variable solutions via an auxliliary optimization problem. *J. Guid. Contr. Dyn.*, 18:1267–1271, 1995.

T. Neckel, C. Talbot, and N. Petit. Collocation and inversion for a reentry optimal control problem. In *Proc. 5th Int. Conf. Launcher Techn.*, Madrid (Spain), 2003.

J. Nocedal and S.J. Wright. *Numerical Optimization*. Springer, New York, 1999.

H.J. Pesch. A practical guide to the solution of real–life optimal control problems. *Control and Cybernetics*, 23:7–60, 1994.

L.S. Pontryagin, V.G. Boltyansky, V. Gamkrelidze, and E.F. Mischenko. *Mathematical Theory of Optimal Processes*. Wiley–Interscience, New York, 1962.

R. Pytlak. *Numerical Methods for Optimal Control Problems with State Constraints*. Springer, Berlin, Germany, 1999.

H. Seywald. Trajectory optimization based on differential inclusion. *J. Guid. Contr. Dyn.*, 17(3):480–487, 1994.

H. Seywald and R.R. Kumar. Method for automatic costate calculation. *J. Guid. Contr. Dyn.*, 19:1252–1261, 1996.

L.F. Shampine, J. Kierzenka, and M.W. Reichelt. Solving boundary value problems for ordinary differential equations in MATLAB with bvp4c. http://www.mathworks.com/bvp_tutorial, 2000.

O. von Stryk and R. Bulirsch. Direct and indirect methods for trajectory optimization. *Annals Op. Res.*, 37:357–373, 1992.