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Relaxed conditions for uniform complete observability and controllability of LTV systems with bounded realizations [1]

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Abstract: Uniform complete observability and controllability are key concepts for the design of state observers and controllers for linear time-varying systems. While the definitions of these properties are simple, it is more often that not very difficult to show they hold for a particular system. This paper presents alternative simpler sufficient conditions. Examples are shown that illustrate the improvement over previous results, the fact that the new conditions are not necessary, and a practical application.

Keywords: uniform complete observability, uniform complete controllability, linear time-varying systems.

1. INTRODUCTION

As is evidenced by the massive stream of publications spread over the past decades, considerable amount of work has been devoted to the study of linear control systems and it is unsurprising that these are, by now, well understood. For linear systems, the tasks of full feedback stabilization and state reconstruction can be solved with numerous techniques, vastly exposed in textbooks. These tasks are dual, as is well-known. These techniques consider as central assumptions that the property of observability (and the dual property of controllability) holds. For linear time invariant systems observability is sufficient to guarantee the existence of observers with globally exponentially stable error dynamics. But, that is not the case for linear time-varying (LTV) systems, and stronger forms of observability are required, in particular uniform complete observability. While the concept of uniform complete observability itself is simple and straightforward to define, it is more often that not very difficult to show that a particular system is uniformly completely observable (UCO). This is a true limitation as the concept of uniform completely observability is instrumental in numerous studies. It permits to establish the convergence of the Kalman filter for LTV systems (see Besançon (2007) and references therein). It is also recurrent in adaptive control, see e.g. Ioannou and Sun (1995); Tsakalis and Ioannou (1993); Loría and Panteley (2002); Fang et al. (2015). The interested reader can also refer to Morgado et al. (2011) which exposes a practical applications where UCO is crucial in the design of observers Zhang and Leonard (2010). UCO also appears in the design of observers for nonlinear systems, see e.g. Batista et al. (2011a,b, 2012, 2013) and references therein, to mention just a few. Finally, the relation between UCO and uniform complete estimatability is also explored in Ikeda et al. (1975). Of course, the exact same discussions hold for the dual control problems, and the dual notion of uniform complete controllability (UCC).

As mentioned earlier, showing that a particular LTV system is UCO may prove cumbersome. There exist several results in the literature to ease the process. A very popular one states that UCO is preserved under bounded piecewise continuous output feedback, see (Ioannou and Sun, 1995, Lemma 4.8.1) and Zhang and Zhang (2012). An alternative "folk" result was shown in Bristeau et al. (2010). In this paper, we present relaxed sufficient conditions that allow to show, in some cases, that a system

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is UCO. The sufficient assumption considered here is that the maximum time elapsed between time instants where the observability matrix has a strictly positive pseudo-inverse should be bounded. This allows to cover cases of punctual or periodic non observability, among others. A result is established (Theorem 3). Along with its proof, which exploits the smoothness of variations of the system dynamics, this result constitutes the main contribution of the article. Theorem 3 has a dual formulation for uniform complete controllability which is given here (Theorem 5). Interestingly, this result could be applied in many of the works cited earlier and possibly also to Xu et al. (2007); Gadre (2007); Bryne et al. (2015); Thienel and Sanner (2007); Eudes and Morin (2014); Changey et al. (2013) that all relate to practical applications in the design of navigation systems. Some illustrative examples detailed in this article sketches the potential of this result.

2. PRELIMINARY DEFINITIONS

Consider the LTV system given by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$ are the system state, input, and output, respectively, and $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t) \in \mathbb{R}^{n\times m}$, and $\mathbf{C}(t) \in \mathbb{R}^{p\times n}$ are bounded continuous functions of time.

The following definitions introduce the concepts of observability Gramian and uniform complete observability for systems with bounded realizations Kalman (1960), Silverman and Anderson (1968), Jazwinski (1970), Silverman and Meadows (1967).

Definition 1. (Observability Gramian). The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on $[t_0, t_f]$ is defined as

$$\boldsymbol{\mathcal{W}}_o\left(t_0, t_f\right) := \int_{t_0}^{t_f} \boldsymbol{\phi}^T\left(t, t_0\right) \mathbf{C}^T(t)\mathbf{C}(t)\boldsymbol{\phi}\left(t, t_0\right) dt,$$

where $\boldsymbol{\phi}(t, t_0)$ is the state transition matrix associated with $\mathbf{A}(t)$ from t_0 to t.

Definition 2. (Uniform complete observability). The pair $(\mathbf{A}(t), \mathbf{C}(t))$ is uniformly completely observable (UCO) if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$,

$$\boldsymbol{\mathcal{W}}_o\left(t, t+\delta\right) \succeq \alpha\mathbf{I}.$$

Likewise, the dual definitions for control are as follows.

Definition 3. (Controllability Gramian). The controllability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$ on $[t_0, t_f]$ is defined as

$$\boldsymbol{\mathcal{W}}_c\left(t_0, t_f\right) := \int_{t_0}^{t_f} \boldsymbol{\phi}\left(t, t_0\right) \mathbf{B}(t)\mathbf{B}^T(t)\boldsymbol{\phi}^T\left(t, t_0\right) dt,$$

where $\boldsymbol{\phi}(t, t_0)$ is the state transition matrix associated with $\mathbf{A}(t)$ from t_0 to t.

Definition 4. (Uniform complete controllability). The pair $(\mathbf{A}(t), \mathbf{B}(t))$ is uniformly completely controllable (UCC) if there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$,

$$\boldsymbol{\mathcal{W}}_c\left(t, t+\delta\right) \succeq \alpha\mathbf{I}.$$

Finally, a vector norm inequality is introduced next. Let $\mathbf{x} \in \mathbb{R}^n$. Then,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\| \leq \sqrt{n}\,\|\mathbf{x}\|_\infty, \tag{2}$$

where $\|\mathbf{x}\|$ and $\|\mathbf{x}\|_\infty$ are the Euclidean and infinity norms of $\mathbf{x}$, respectively.

3. MAIN RESULT

The LTV system (1) is observable on $[t_0, t_f]$ if and only if the observability Gramian $\boldsymbol{\mathcal{W}}_o(t_0, t_f)$ is invertible. The following theorem corresponds to an alternative well known result on observability that does not require the computation of the observability Gramian.

Theorem 1. (Rugh (1995)). Suppose that q is a positive integer such that, for all $t \geq t_0$, $\mathbf{C}(t)$ is a q times continuously differentiable matrix and $\boldsymbol{A}(t)$ is a $(q-1)$ times continuously differentiable matrix. Define

$$\boldsymbol{\mathcal{L}}(t) = \begin{bmatrix} \mathbf{L}_0(t) \\ \vdots \\ \mathbf{L}_q(t) \end{bmatrix},$$

where

$$\begin{cases} \mathbf{L}_0(t) = \mathbf{C}(t) \\ \mathbf{L}_i(t) = \mathbf{L}_{i-1}(t)\mathbf{A}(t) + \dot{\mathbf{L}}_{i-1}(t),\ i = 1, \ldots, q. \end{cases}$$

Then, the linear system (1) is observable on $[t_0, t_f]$ if, for some $t_a \in [t_0, t_f]$, $\operatorname{rank}(\boldsymbol{\mathcal{L}}(\mathrm{t})) = \mathrm{n}$.

For uniform complete observability of a LTV system with bounded realization, we will consider the following "folk" result, related to Theorem 1.

Theorem 2. (Bristeau et al. (2010)). The bounded LTV system (1) is UCO if there exists a positive constant $\alpha > 0$ and an integer $q \in \mathbb{N}$ such that, for all $t \geq t_0$,

$$\boldsymbol{\mathcal{L}}^T(t)\boldsymbol{\mathcal{L}}(t) \succeq \alpha\mathbf{I}, \tag{3}$$

with $\boldsymbol{\mathcal{L}}(t) \in \mathbb{R}^{w\times n}$, $w \geq n$.

The first result of this paper is a relaxation of the condition of Theorem 2 so that the lower bound (3) does not need to hold for all t but only for some time instants, provided that the maximum time elapsed between these time instants is bounded. This is established in the following theorem.

Theorem 3. (main result). Suppose that $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are bounded and sufficiently smooth matrices such that $\boldsymbol{\mathcal{L}}(t)$ is well-defined and bounded for some $q \geq 0$. Further suppose that $\mathbf{L}_q(t)$ satisfies the Lipchitz condition

$$\left\|\mathbf{L}_q\left(t_1\right) - \mathbf{L}_q\left(t_2\right)\right\| \leq C_q\left|t_1 - t_2\right|,\ C_q > 0, \tag{4}$$

for all $t_1 \geq t_0$ and $t_2 \geq t_0$. If there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$, it is possible to choose $t_i \in [t, t+\delta]$ such that

$$\boldsymbol{\mathcal{L}}^T\left(t_i\right)\boldsymbol{\mathcal{L}}\left(t_i\right) \succeq \alpha\mathbf{I}, \tag{5}$$

with $\boldsymbol{\mathcal{L}}(t) \in \mathbb{R}^{w\times n}$, $w \geq n$, then the pair $(\mathbf{A}(t), \mathbf{C}(t))$ is UCO.

Proof. Let $\mathbf{d} \in \mathbb{R}^n$ be a unit vector, i.e., $\|\mathbf{d}\| = 1$. Then,

$$\mathbf{d}^T\boldsymbol{\mathcal{W}}_o\left(t, t+\delta\right)\mathbf{d} = \int_t^{t+\delta} \left\|\mathbf{f}\left(\tau, t\right)\right\|^2 d\tau,$$

with

$$\mathbf{f}\left(\tau, t\right) = \mathbf{C}\left(\tau\right)\boldsymbol{\phi}\left(\tau, t\right)\mathbf{d}.$$

Notice that

$$\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t) = \mathbf{L}_j(\tau)\,\boldsymbol{\phi}(\tau,t)\,\mathbf{d} \tag{6}$$

for all integers $0 \le j \le q$.

As a preliminary remark, some bounds need to be derived. From the assumptions of the theorem, define $A > 0$ such that

$$\|\mathbf{A}(t)\| \le A$$

for all $t \ge t_0$. Notice that there exists a positive constant $C_\phi > 0$ such that

$$\|\boldsymbol{\phi}(\tau,t)\,\mathbf{d}\| \le C_\phi$$

and

$$\left\|\frac{\partial}{\partial \tau}\boldsymbol{\phi}(\tau,t)\right\| \le AC_\phi$$

for all $\tau \in [t, t+\delta]$, $t \ge t_0$. In addition, by assumption, there exist positive constants $L_j > 0$ such that

$$\|\mathbf{L}_j(t)\| \le L_j$$

for all $t \ge t_0$, $j = 0, \ldots, q$. Therefore, for $j = 0, \ldots, q-1$, $\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t)$ is Lipchitz, with Lipchitz constant $C_\phi L_{j+1}$. Also, as both $\mathbf{L}_q(\tau)$ and $\boldsymbol{\phi}(\tau,t)$ are bounded and Lipchitz for $\tau \in [t, t+\delta]$, $t \ge t_0$, it follows that $\frac{\partial^q}{\partial \tau^q}\mathbf{f}(\tau,t)$ is also Lipchitz for $\tau \in [t, t+\delta]$, $t \ge t_0$, with Lipchitz constant $C_\phi(AL_q + C_q)$. Define

$$C := \max\left(C_\phi L_1, \ldots, C_\phi L_{q-1}, C_\phi(AL_q + C_q)\right).$$

Under the conditions of the theorem, the system matrix $\mathbf{A}(t)$ is bounded. Hence, it follows from Lemma 2 that there exists a positive constant $c'_\phi > 0$ such that

$$\|\boldsymbol{\phi}(\tau,t)\,\mathbf{d}\| \ge e^{-c'_\phi|\tau-t|}\,\|\mathbf{d}\| \ge e^{-c'_\phi|\tau-t|}$$

for all $\tau \ge t \ge t_0$. In particular, it follows that

$$\|\boldsymbol{\phi}(\tau,t)\,\mathbf{d}\| \ge c_\phi \tag{7}$$

for all $\tau \in [t, t+\delta]$, $t \ge t_0$, with $c_\phi := e^{-c'_\phi \delta}$.

By assumption, there exist $\alpha > 0$ and $\delta > 0$ such that, for all $t \ge t_0$, it is possible to choose $t_i \in [t, t+\delta]$ such that, for all unit vectors $\mathbf{d}' \in \mathbb{R}^n$, it is true that $\|\boldsymbol{\mathcal{L}}(t_i)\,\mathbf{d}'\| \ge \sqrt{\alpha}$, i.e.,

$$\sqrt{\alpha} \le \left\|\begin{bmatrix}\mathbf{L}_0(t_i)\,\mathbf{d}' \\ \vdots \\ \mathbf{L}_q(t_i)\,\mathbf{d}'\end{bmatrix}\right\|.$$

Using the equivalence of norms (2) one may write

$$\sqrt{\alpha} \le \sqrt{p(q+1)}\left\|\begin{bmatrix}\mathbf{L}_0(t_i)\,\mathbf{d}' \\ \vdots \\ \mathbf{L}_q(t_i)\,\mathbf{d}'\end{bmatrix}\right\|_\infty.$$

Thus, for every t_i and $\mathbf{d}'$, there exists an integer $0 \le j \le q$ such that

$$\|\mathbf{L}_j(t_i)\,\mathbf{d}'\|_\infty \ge \sqrt{\frac{\alpha}{p(q+1)}},$$

and, using again the equivalence of norms (2),

$$\|\mathbf{L}_j(t_i)\,\mathbf{d}'\| \ge \sqrt{\frac{\alpha}{p(q+1)}}. \tag{8}$$

For any unit vector $\mathbf{d}$, choose

$$\mathbf{d}' = \frac{\boldsymbol{\phi}(t_i,t)\,\mathbf{d}}{\|\boldsymbol{\phi}(t_i,t)\,\mathbf{d}\|}$$

and rewrite (6), for $\tau = t_i$, as

$$\left.\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t)\right|_{\tau=t_i} = \|\boldsymbol{\phi}(t_i,t)\,\mathbf{d}\|\,\mathbf{L}_j(t_i)\,\mathbf{d}'.$$

Using (7) and (8) allows to write

$$\left\|\left.\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t)\right|_{\tau=t_i}\right\| \ge c_\phi\sqrt{\frac{\alpha}{p(q+1)}}. \tag{9}$$

If $j > 0$ one may conclude, resorting to Lemma 1, that there exists a positive constant $\beta_{j,0} > 0$ and a time instant $t'_j \in [t, t+\delta]$ such that

$$\left\|\int_t^{t'_j}\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t)\,d\tau\right\| \ge \beta_{j,0}. \tag{10}$$

Now, write

$$\left\|\left.\frac{\partial^{j-1}}{\partial \tau^{j-1}}\mathbf{f}(\tau,t)\right|_{\tau=t'_j} - \left.\frac{\partial^{j-1}}{\partial \tau^{j-1}}\mathbf{f}(\tau,t)\right|_{\tau=t}\right\| = \left\|\int_t^{t'_j}\frac{\partial^j}{\partial \tau^j}\mathbf{f}(\tau,t)\,d\tau\right\|. \tag{11}$$

As the right side of (11) satisfies (10), one gets that one of the two vectors $\left.\frac{\partial^{j-1}}{\partial \tau^{j-1}}\mathbf{f}(\tau,t)\right|_{\tau=t'_j}$ or $\left.\frac{\partial^{j-1}}{\partial \tau^{j-1}}\mathbf{f}(\tau,t)\right|_{\tau=t}$ must be, in norm, greater than $\beta_{j,0}/2$. Therefore, one has obtained a time instant $t_{i,1} \in [t, t+\delta]$ such that

$$\left\|\left.\frac{\partial^{j-1}}{\partial \tau^{j-1}}\mathbf{f}(\tau,t)\right|_{\tau=t_{i,1}}\right\| \ge \alpha_{j,1},$$

with

$$\alpha_{j,1} := \beta_{j,0}/2.$$

Following the same train of thought and proceeding with j iterations, one concludes that there exists a positive constant $\alpha_{j,j} > 0$ and a time instant $t_{i,j} \in [t, t+\delta]$ such that

$$\|\mathbf{f}(t_{i,j},t)\| \ge \alpha_{j,j}. \tag{12}$$

From (9) and (12), one may conclude that there exists a positive constant α' and a time instant $t_1 \in [t, t+\delta]$ such that

$$\|\mathbf{f}(t_1,t)\| \ge \alpha'.$$

The construction of α' proceeds as follows. It has been shown that, under the assumptions of the theorem, there exists $\alpha' > 0$ and $\delta > 0$ such that, for all $t \ge t_0$ and all unit vectors $\mathbf{d}$, one can construct $t_1 \in [t, t+\delta]$ such that $\|\mathbf{f}(t_1,t)\| \ge \alpha'$. According to Lemma 1, define

$$l(\delta, C, \alpha) := \frac{1}{4}\min\left(\delta, \frac{\alpha}{\sqrt{p}C}\right)\frac{\alpha}{\sqrt{p}} > 0$$

for $\alpha > 0$. From

$$\beta_0 := l\left(\delta, C, c_\phi\sqrt{\frac{\alpha}{p(q+1)}}\right) > 0$$

iterate

$$\beta_k := l\left(\delta, C, \frac{\beta_{k-1}}{2}\right)$$

q times, and take

$$\alpha' = \min_{j=0,\ldots,q}\beta_k > 0. \tag{13}$$

By applying Lemma 1 again, it follows that there exists a time instant $t_2 \in [t, t+\delta]$ and a positive constant

$\alpha'' > 0$, $\alpha'' = l\left(\delta, C, (\alpha')^2\right)$, which, according to (13) solely depends on δ, C, α and the bounds on the system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$, such that

$$\boldsymbol{\mathcal{W}}_o(t, t+t_2) \succeq \alpha''\mathbf{I}.$$

Finally, this implies the desired result, i.e., for all $t \geq t_0$,

$$\boldsymbol{\mathcal{W}}_o(t, t+\delta) \succeq \alpha''\mathbf{I},$$

which means that the pair $(\mathbf{A}(t), \mathbf{C}(t))$ is UCO.

The second result of the paper is the dual result for uniform complete controllability. First, the controllability equivalent to Theorem 1 is introduced.

Theorem 4. (Rugh (1995)). Suppose that q is a positive integer such that, for all $t \geq t_0$, $\mathbf{B}(t)$ is a q times continuously differentiable matrix and $\boldsymbol{A}(t)$ is a $(q-1)$ times continuously differentiable matrix. Define

$$\boldsymbol{\mathcal{M}}(t) = [\,\mathbf{M}_0(t)\ \ldots\ \mathbf{M}_q(t)\,],$$

where

$$\begin{cases} \mathbf{M}_0(t) = \mathbf{B}(t) \\ \mathbf{M}_i(t) = -\mathbf{A}(t)\mathbf{M}_{i-1}(t) + \dot{\mathbf{M}}_{i-1}(t),\ i = 1, \ldots, q. \end{cases}$$

Then, the linear system (1) is controllable on $[t_0, t_f]$ if, for some $t_a \in [t_0, t_f]$, $\operatorname{rank}(\boldsymbol{\mathcal{M}}(t)) = \mathrm{n}$.

The following theorem is the controllability equivalent to Theorem 3.

Theorem 5. (extension). Suppose that $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are bounded and sufficiently smooth matrices such that it is possible to compute $\boldsymbol{\mathcal{M}}(t)$, for some $q \geq 0$. Further suppose that $\boldsymbol{\mathcal{M}}(t)$ is bounded and that $\mathbf{M}_q(t)$ satisfies the Lipchitz condition

$$\|\mathbf{M}_q(t_1) - \mathbf{M}_q(t_2)\| \leq C_L\,|t_1 - t_2|\,,\ C_L > 0,$$

for all $t_1 \geq t_0$ and $t_2 \geq t_0$. If there exist positive constants $\alpha > 0$ and $\delta > 0$ such that, for all $t \geq t_0$, it is possible to choose $t_i \in [t, t+\delta]$ such that

$$\boldsymbol{\mathcal{M}}(t_i)\,\boldsymbol{\mathcal{M}}^T(t_i) \succeq \alpha\mathbf{I},$$

with $\boldsymbol{\mathcal{M}}(t) \in \mathbb{R}^{w\times n}$, $w \geq n$, then the pair $(\mathbf{A}(t), \mathbf{B}(t))$ is UCC.

Proof. The proof is the dual of Theorem 3 for controllability and therefore it is omitted.

4. EXAMPLES OF APPLICATION

This section presents some examples of application of the previous results.

4.1 Improvement on previous results

Consider the linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{14}$$

with

$$\mathbf{C}(t) = \begin{cases} \mathbf{0}, 2kT \leq t - t_0 < (2k+1)\,T \\ \dfrac{t - (2k+1)\,T}{T/2}\mathbf{I}, (2k+1)\,T \leq t - t_0 < (2k+1)\,T + T/2 \\ \dfrac{(2k+2)\,T - t}{T/2}\mathbf{I}, (2k+1)\,T + T/2 \leq t - t_0 < (2k+2)\,T \end{cases},$$

$T > 0$, $k \in \mathbb{N}_0$. Computing $\boldsymbol{\mathcal{L}}(t)$ for this system simply gives

$$\boldsymbol{\mathcal{L}}(t) = \mathbf{C}(t)$$

as $\mathbf{C}(t)$ is not differentiable for all $t \geq t_0$. In this case, Theorem 2 cannot be invoked to show that the system is uniformly completely observable. Indeed, for any unit vector $\mathbf{d} \in \mathbb{R}^n$,

$$\boldsymbol{\mathcal{L}}(t)\mathbf{d} = \mathbf{0}$$

for all $2kT \leq t < (2k+1)\,T$, which implies that (3) does not hold for all $t \geq t_0$. Yet, (14) is uniformly completely observable, which can be shown invoking Theorem 3, with $\alpha = 1$, $\delta = 2T$. Indeed, for all $t \geq t_0$, it is possible to choose $t_i \in [t, t+\delta]$, $t_i = t_0 + (2k+1)\,T + T/2$ such that

$$\boldsymbol{\mathcal{L}}(t_i) = \mathbf{I},$$

meaning that (5) is verified, while the Lipchitz condition (4) is also verified, with $C_L = 2/T$. This example shows the improvement obtained with the relaxed condition derived in this paper over previous existing results.

4.2 Example of conservativeness

Consider the linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{15}$$

where

$$\mathbf{C}^T(t) = \begin{cases} \begin{bmatrix} \dfrac{t - 2kT}{T/2} \\ 0 \end{bmatrix}, 2kT \leq t - t_0 < 2kT + T/2 \\ \begin{bmatrix} \dfrac{(2k+1)\,T - t}{T/2} \\ 0 \end{bmatrix}, 2kT + T/2 \leq t - t_0 < (2k+1)\,T \\ \begin{bmatrix} 0 \\ \dfrac{t - (2k+1)\,T}{T/2} \end{bmatrix}, (2k+1)\,T \leq t - t_0 < (2k+1)\,T + T/2 \\ \begin{bmatrix} 0 \\ \dfrac{(2k+2)\,T - t}{T/2} \end{bmatrix}, (2k+1)\,T + T/2 \leq t - t_0 < (2k+2)\,T \end{cases},$$

$T > 0$, $k \in \mathbb{N}_0$. Computing $\boldsymbol{\mathcal{L}}(t)$ for this system simply gives

$$\boldsymbol{\mathcal{L}}(t) = \mathbf{C}(t)$$

as $\mathbf{C}(t)$ is not differentiable for all $t \geq t_0$. In this case, Theorem 3 cannot be invoked to show that the system is uniformly completely observable. Yet, the system is uniformly completely observable. Indeed, for this system, the transition matrix is a simple identity and hence it can be written, for all unit vectors

$$\mathbf{d} = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \in \mathbb{R}^2$$

and all $t \geq t_0$

$$\mathbf{d}^T\boldsymbol{\mathcal{W}}_o(t, t+2T)\,\mathbf{d} = \int_t^{t+2T} \|\mathbf{C}(\sigma)\mathbf{d}\|^2\,d\sigma.$$

As $\mathbf{C}(t)$ is periodic, with period $2T$, it is possible to write

$$\begin{aligned}
\mathbf{d}^T \boldsymbol{\mathcal{W}}_o\left(t, t+2T\right)\mathbf{d} &= \int_{t-t_0}^{t-t_0+2T} \left\|\mathbf{C}(\sigma)\mathbf{d}\right\|^2 d\sigma \\
&= \int_{2kT}^{2kT+T/2} \left(\frac{\sigma - 2kT}{T/2} d_x\right)^2 d\sigma \\
&+ \int_{2kT+T/2}^{(2k+1)T} \left(\frac{(2k+1)\,T - \sigma}{T/2} d_x\right)^2 d\sigma \\
&+ \int_{(2k+1)T}^{(2k+1)T+T/2} \left(\frac{\sigma - (2k+1)\,T}{T/2} d_y\right)^2 d\sigma \\
&+ \int_{(2k+1)T+T/2}^{(2k+2)T} \left(\frac{(2k+2)\,T - \sigma}{T/2} d_y\right)^2 d\sigma \\
&= \frac{T}{3} d_x^2 + \frac{T}{3} d_y^2 = \frac{T}{3}
\end{aligned}$$

for all $t \geq t_0$ and all unit vectors $\mathbf{d}$, which allows to conclude that

$$\boldsymbol{\mathcal{W}}_o\left(t, t+2T\right) \succeq \frac{T}{3}\mathbf{I}$$

and hence the system (15) is uniformly completely observable. This example shows that there is still conservativeness in the relaxed condition derived in this paper.

4.3 Order of quantifiers

The main condition of Theorem 3 may be written as

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists}\ \underset{t\geq t_0}{\forall}\ \underset{t_i\in[t,t+\delta]}{\exists}\ \underset{\substack{\mathbf{d}\in\mathbb{R}^n\\ \|\mathbf{d}\|=1}}{\forall}\ \left\|\boldsymbol{\mathcal{L}}\left(t_i\right)\mathbf{d}\right\|^2 \geq \alpha. \tag{16}$$

If Theorem 3 remained true if one changed the order of the last two quantifies in (16), then it would be possible to show that the system presented in the previous example was uniformly completely observable invoking the theorem presented in this paper, which would reduce even further the conservativeness of the result. However, it is not possible to change the order of the two quantifiers, i.e., a system can satisfy

$$\underset{\substack{\alpha>0\\ \delta>0}}{\exists}\ \underset{t\geq t_0}{\forall}\ \underset{\substack{\mathbf{d}\in\mathbb{R}^n\\ \|\mathbf{d}\|=1}}{\forall}\ \underset{t_i\in[t,t+\delta]}{\exists}\ \left\|\boldsymbol{\mathcal{L}}\left(t_i\right)\mathbf{d}\right\|^2 \geq \alpha. \tag{17}$$

and yet not be uniformly completely observable, as the next example demonstrates.

Consider the linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -W \\ W & 0 \end{bmatrix} \in \mathbb{R}^2, \ \ W > 0,$$

and

$$\mathbf{C}(t) = \left[\,1\ \ 0\,\right]\mathbf{R}(t),$$

where $\mathbf{R}(t)$ is the rotation matrix that satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{A}.$$

In this case,

$$\boldsymbol{\mathcal{L}}(t) = \begin{bmatrix} \left[\,1\ \ 0\,\right]\mathbf{R}(t) \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

and it is not possible to invoke Theorem 3 as

$$\boldsymbol{\mathcal{L}}(t)\mathbf{R}^T(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

for all $t \geq t_0$. However, condition (17) is verified, with $\alpha = 1$ and $\delta = 2\pi/W$. Notice that W corresponds to the angular velocity, which is constant, and therefore, for all $t \geq t_0$ and all unit vectors $\mathbf{d}$, it is possible to choose $t_i \in [t, t+\delta]$ such that

$$\mathbf{R}\left(t_i\right)\mathbf{d} = \left[\,1\ \ 0\,\right]$$

and hence

$$\left\|\boldsymbol{\mathcal{L}}\left(t_i\right)\mathbf{d}\right\|^2 = 1.$$

Yet, the system (18) is not uniformly completely observable. In fact it is not even observable. Indeed, the transition matrix associated with this system is

$$\boldsymbol{\phi}\left(\tau, t\right) = \mathbf{R}^T\left(\tau\right)\mathbf{R}(t)$$

and the observability Gramian is given by

$$\boldsymbol{\mathcal{W}}_o\left(t, t+\delta\right) = \delta\mathbf{R}^T(t)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{R}(t),$$

which is not full rank.

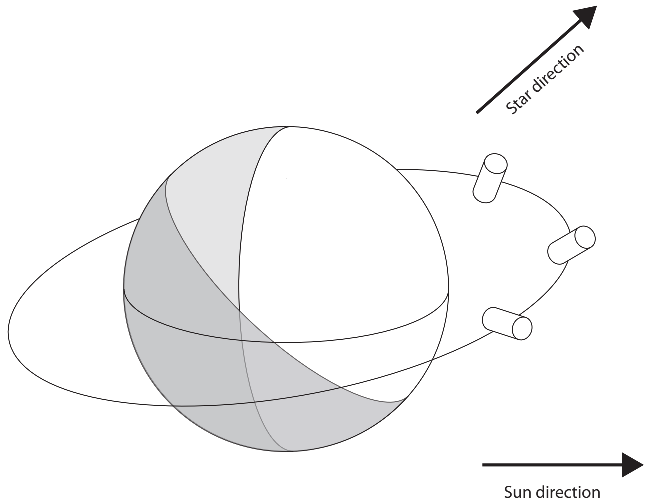

Fig. 1. A spun satellite in orbit, using two vector measurements to determine its angular velocity.

4.4 Rigid body in rotation

Consider a rigid body in rotation equipped with two direction sensors. One such system is a spinning satellite equipped with Sun sensors, a star tracker or magnetometers (among other sensing technologies). Estimating the angular velocity of the rigid body is a question of practical importance, especially for orientation control, because most methods employed for controlling the second order rotation dynamics (e.g. Lyapunov control design, feedback linearization, or computed torque) require angular velocity information Magnis and Petit (2016).

Consider the case when the satellite is travelling along an orbit and can only obtain the two vector measurements (e.g. Sun direction and another star direction) when it is in direct visibility of it. The Earth is a major obstructing object, see Figure 1. Considering that the orbit of the satellite is T-periodic (according to Newton's law of universal gravitation in the simplest gravity model), then the measurements can be performed on kT-long regularly spaced periods (with $0 \leq k \leq 1/2$) , say $[0, kT]$, $[T, (k+1)T]$,... This scenario can be addressed by Theorem 3 as shown below.

Let $\mathring{\mathbf{a}}$ and $\mathring{\mathbf{b}}$ denote the two reference (three dimensional) unit vectors expressed in an inertial frame. Denote by

$\boldsymbol{\omega}$ the (three dimensional) angular velocity vector corresponding to the rotation matrix $\boldsymbol{R}$ from this inertial body frame to a body frame attached to the rigid body, expressed in this frame. The two reference unit vectors are measured with sensors arranged on the rigid body, producing a vector of measurements $\boldsymbol{y}(t) = \boldsymbol{R}(t)\begin{bmatrix}\mathring{\mathbf{a}}\\ \mathring{\mathbf{b}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{a}(t)\\ \boldsymbol{b}(t)\end{bmatrix}$ at each instant. Then, one has

$$\dot{\boldsymbol{a}}(t) = \boldsymbol{a}(t) \times \boldsymbol{\omega} = [\boldsymbol{a}(t)_\times]\boldsymbol{\omega} \tag{19}$$

where $\times$ denotes the cross-product, and $[\boldsymbol{a}_\times]$ is the skew-symmetric cross-product matrix associated to $\boldsymbol{a}$.

The right-handside of (19) is bilinear in the measurement $\boldsymbol{a}$ and $\boldsymbol{\omega}$. In Magnis and Petit (2016), it was proposed to model the unknown $\boldsymbol{\omega}$ by introducing its governing equation (the Euler equation of free rotation). Here, we follow a simpler method, commonly employed in Kalman filtering, which is to model $\boldsymbol{\omega}$ as an unknown constant (or slowly varying) variable driven by noise. With this assumption, the nominal (noise-less) equation governing $\boldsymbol{\omega}$ is simply

$$\dot{\boldsymbol{\omega}}(t) = 0$$

Note the state vector $\boldsymbol{x} = \begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\\ \boldsymbol{\omega}\end{pmatrix}$. One has

$$\dot{\boldsymbol{x}} = \begin{pmatrix}\boldsymbol{a}\times\boldsymbol{\omega}\\ \boldsymbol{b}\times\boldsymbol{\omega}\\ 0\end{pmatrix} = \boldsymbol{A}(t)\boldsymbol{x}, \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

with

$$\boldsymbol{A}(t) = \begin{pmatrix}0 & 0 & [\boldsymbol{a}(t)_\times]\\ 0 & 0 & [\boldsymbol{b}(t)_\times]\\ 0 & 0 & 0\end{pmatrix}$$

and

$$\mathbf{C} = \begin{pmatrix}I & 0 & 0\\ 0 & I & 0\end{pmatrix}$$

With the notations of Theorem 3, consider $q = 1$. Then,

$$\boldsymbol{\mathcal{L}}^T(t)\boldsymbol{\mathcal{L}}(t) = \begin{pmatrix}I\,0 & 0\\ 0\,I & 0\\ 0\,0 & [\boldsymbol{a}(t)_\times]^T[\boldsymbol{a}(t)_\times] + [\boldsymbol{b}(t)_\times]^T[\boldsymbol{b}(t)_\times]\end{pmatrix}.$$

Interestingly, the time-varying 3×3 matrix

$$[\boldsymbol{a}(t)_\times]^T[\boldsymbol{a}(t)_\times] + [\boldsymbol{b}(t)_\times]^T[\boldsymbol{b}(t)_\times]$$

has 3 *constant* eigenvalues $\left\{1 \pm \mathring{\mathbf{a}}^T\mathring{\mathbf{b}}, \quad 2\right\}$ This fact can be easily proven, after some lines of calculus, by considering the orthogonal basis $\left\{\mathring{\mathbf{a}} - r\mathring{\mathbf{b}}, \mathring{\mathbf{b}} - r\mathring{\mathbf{a}}, \mathring{\mathbf{b}}\times\mathring{\mathbf{a}}\right\}$ where $\mathring{\mathbf{b}}\times\mathring{\mathbf{a}}$ is an eigenvector with 2 as eigenvalue, and r is a root of the polynomial $\mathring{\mathbf{a}}^T\mathring{\mathbf{b}}x^2 - 2r + \mathring{\mathbf{a}}^T\mathring{\mathbf{b}} = 0$.

It follows that (5) holds provided that the two unit vectors $\mathring{\mathbf{a}}$ and $\mathring{\mathbf{b}}$ are not aligned, with $\delta = T$, $\alpha = 1 - |\mathring{\mathbf{a}}^T\mathring{\mathbf{b}}|$. Besides, $\boldsymbol{A}(t)$ is bounded and Lipschitz because $\boldsymbol{\omega}$ is bounded [2], while $\boldsymbol{C}$ is constant. This guarantees the Lipschitz condition (4).

In conclusion, the system is UCO if $\mathring{\mathbf{a}}$ and $\mathring{\mathbf{b}}$ are not aligned. This assumption, and the result, will seem familiar to the reader aware of the classic results on Wahba's problem Wahba (1965). Here we deduce that the angular velocity can be estimated even in the case of periodic obstruction from the Earth using two distinct vector measurements.

[2] The interested reader will note that the angular velocity of any rigid body in free rotation (i.e. without any external torque) is bounded at all times Magnis and Petit (2016).

5. CONCLUSIONS

Uniform completely controllability and observability are key concepts that often play roles not restricted to the design of controllers and observers for LTV systems. While simple, it often proves cumbersome to show that a particular system is UCO or UCC. This paper presented novel sufficient conditions that allow, in some cases, to show UCO and UCC. These are relaxed conditions of a previous recent result and examples of application show the improvement over existing results, as well as their applicability.

Appendix A. TWO TECHNICAL RESULTS

Lemma 1. Let $\mathbf{f} : [t_0, t_f] \subset \mathbb{R} \to \mathbb{R}^n$ be a continuous function on $\mathcal{I} := [t_0, t_f]$, $t_f > t_0$. Let

$$C := \max_{\substack{\tau_1,\tau_2\in\mathcal{I}\\ \tau_1\neq\tau_2}} \frac{\|\mathbf{f}(\tau_2) - \mathbf{f}(\tau_1)\|}{|\tau_2 - \tau_1|}.$$

If there exists $\alpha > 0$ and $t_1 \in \mathcal{I}$ such that $\|\mathbf{f}(t_1)\| \geq \alpha$, then there exists $t_2 \in \mathcal{I}$ such that $\left\|\int_{t_0}^{t_2}\mathbf{f}(\tau)\,d\tau\right\| \geq \beta$, with

$$\beta := \frac{1}{4}\min\left(T, \frac{\alpha}{\sqrt{n}C}\right)\frac{\alpha}{\sqrt{n}},$$

where $T := t_f - t_0$.

Proof. Suppose that the hypothesis of the theorem holds, i.e., suppose that there exists $\alpha > 0$ and $t_1 \in \mathcal{I}$ such that $\|\mathbf{f}(t_1)\| \geq \alpha$. Using the norm inequality (2) one gets

$$\|\mathbf{f}(t_1)\|_\infty \geq \frac{\alpha}{\sqrt{n}}.$$

Set $1 \leq k \leq n$ such that

$$f_k(t_1) = \frac{\alpha}{\sqrt{n}},$$

where

$$\mathbf{f}(t_1) = \begin{bmatrix}f_1(t_1)\\ f_2(t_1)\\ \vdots\\ f_n(t_1)\end{bmatrix}.$$

Using the assumptions of the lemma and the equivalence of norms (2), one may conclude that

$$|f_k(t) - f_k(t_1)| \leq C\,|t - t_1|$$

for all $t \in \mathcal{I}$, which in turn allows to write

$$f_k(t) \geq f_k(t_1) - C\,|t - t_1|$$

for all $t \in \mathcal{I}$. Now, without loss of generality, suppose that $f_k(t_1) > 0$. Then, there exists an interval $\mathcal{I}_1 := [t_3, t_4] \subset \mathcal{I}$, $t_3 < t_4$, of length

$$T_1 = t_4 - t_3 = \min\left(T, \frac{\alpha}{\sqrt{n}C}\right)$$

such that $f_k(t) > 0$ for all $t \in \mathcal{I}_1$ and consequently

$$\int_{\mathcal{I}_1} f_k(t)dt \geq \beta', \tag{A.1}$$

with

$$\beta' := \frac{T_1}{2}\frac{\alpha}{\sqrt{n}}.$$

Notice that the construction of the interval $\mathcal{I}_1$ is conservative to cover all possible cases when the intersection of straight lines of slope $\pm C$ from $\left(t_1, \frac{\alpha}{\sqrt{n}}\right)$ crosses the abscissa axis outside or strictly within the interval $\mathcal{I}$. Now, write

$$\left|\int_{t_0}^{t_4} f_k(t)dt - \int_{t_0}^{t_3} f_k(t)dt\right| = \left|\int_{\mathcal{I}_1} f_k(t)dt\right|. \tag{A.2}$$

As the right side of (A.2) satisfies (A.1), one gets that one of the two terms $\int_{t_0}^{t_4} f_k(t)dt$ or $\int_{t_0}^{t_3} f_k(t)dt$ must be greater, in modulus, than $\frac{\beta'}{2}$. As such, there always exists $t_2 \in \mathcal{I}$ such that $\left|\int_{t_0}^{t_2} f_k(t)dt\right| \geq \beta$, with

$$\beta := \beta'/2 = \frac{1}{4}\min\left(T, \frac{\alpha}{\sqrt{n}C}\right)\frac{\alpha}{\sqrt{n}},$$

which concludes the proof.

Lemma 2. Let $\boldsymbol{\phi}(t,t_0)$ denote the transition matrix from t_0 to t, $t \geq t_0$, associated with the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$

where $\mathbf{A}(t)$ is continuous. Suppose that $\mathbf{A}(t)$ is bounded and define

$$A := \max_{\tau\in[t_0,t]} \|\mathbf{A}(\tau)\|.$$

Then,

$$\|\boldsymbol{\phi}(t,t_0)\| \geq e^{-A(t-t_0)}.$$

Proof. Define

$$r(t) := \|\mathbf{x}(t)\|^2.$$

Then,

$$\dot{r}(t) = \mathbf{x}^T(t)\left[\mathbf{A}(t) + \mathbf{A}^T(t)\right]\mathbf{x}(t)$$

and, using

$$\mathbf{x}^T(t)\left[\mathbf{A}(t) + \mathbf{A}^T(t)\right]\mathbf{x}(t) \geq -\left\|\mathbf{A}(t) + \mathbf{A}^T(t)\right\|\|\mathbf{x}(t)\|^2$$

and the definition of $r(t)$, one may conclude that

$$\dot{r}(t) \geq -\left\|\mathbf{A}(t) + \mathbf{A}^T(t)\right\| r(t).$$

Due to the boundedness of $\mathbf{A}(t)$, it is possible to further write

$$\dot{r}(t) \geq -2Ar(t). \tag{A.3}$$

Now, define

$$s(t) := r(T-t)$$

for some $T > 0$. Using the chain rule and (A.3), one concludes that

$$\dot{s}(t) \leq 2As(t),$$

from which follows, using the Gronwall-Bellman inequality, that

$$s(t) \leq s(t_0)\, e^{2A(t-t_0)}.$$

Now, using the definition of $s(t)$, further write

$$r(T-t) \leq r(T-t_0)\, e^{2A(t-t_0)}$$

and, with $T = t_0 + t > 0$, one gets

$$r(t_0) \leq r(t)e^{2A(t-t_0)}$$

or, equivalently,

$$\|\mathbf{x}(t)\| \geq \|\mathbf{x}(t_0)\|\, e^{-A(t-t_0)}.$$

Finally, as

$$\|\mathbf{x}(t)\| = \|\boldsymbol{\phi}(t,t_0)\,\mathbf{x}(t_0)\| \leq \|\boldsymbol{\phi}(t,t_0)\|\,\|\mathbf{x}(t_0)\|,$$

one gets the desired result, thus concluding the proof.

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