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Predictor-Feedback Control of a Model of Microfluidic Process With Hydraulic Input-Dependent Input Delay

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***Abstract*— We consider a model of a microfluidic process under Zweifach-Fung effect, which gives rise to a second-order nonlinear, non-affine system with control input that affects the plant both without delay and with an input-dependent delay defined implicitly through an integral of the past input values (that arises from a transport process with transport speed being the control input itself). We construct a predictor-feedback control law that exponentially stabilizes the output to a desired reference point. This is the first time that a predictor-feedback design is constructed that achieves *complete* input delay compensation for such a type of input delay and despite that control input affects the plant also without delay. This is attributed to the particular structure of the nonlinear system considered, which allows to deriving an implementable formula for the predictor state at the proper prediction horizon.**

I. Introduction

Microfluidic processes are ubiquitous in lab-on-a-chip applications, see, for example, [13], [18]. An important phenomenon evident in such processes is the so-called Zweifach-Fung effect, which appears in microfluidic systems that involve separation of particles within a fluid at a bifurcation point, with a separation volume ratio that depends on the flow rates at the two daughter branches of the main channel. Fig. 1 illustrates such a setup example. This phenomenon

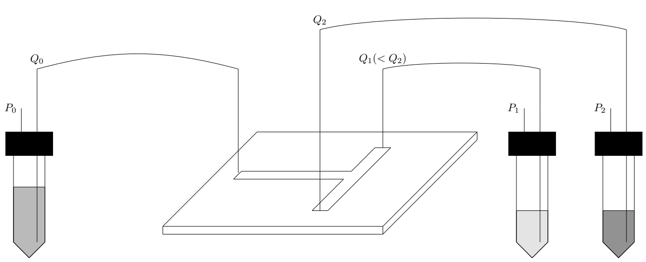

Fig. 1. An example of a microfluidic process. Particles within a fluid are separated in the bifurcation point at a volume ratio that depends on the flow rates Q_1 and Q_2 at each daughter branch, which in turn can be manipulated via the respective pressures P_1 and P_2 in the reservoirs. The flow rate and pressure in the main channel are denoted by Q_0 and P_0, respectively.

can be utilized for applications, such as, for example, blood purification [20], while it is studied within the framework of analysis of microcirculation dynamics, see, for example, [7], [10], [11]. Regulating the volume fraction of particles in one of the daughter channels is crucial for applications that involve, e.g., filtering or enrichment of particles in a fluid.

A control-oriented model of such a phenomenon is presented in [14]. The main features of this model are the following. The control input is the flow ratio (with respect to total flow) at the reservoir of the first channel, while the output is the volume fraction of particles in the first reservoir. Owing to the transport of particles from the bifurcation point to the first reservoir there is a delay of hydraulic type (i.e., defined implicitly through an integral of past values of flow ratio), because the transport speed depends explicitly on the flow ratio itself. In addition, the Zweifach-Fung effect at the bifurcation point, results in a nonlinear term in the dynamic equation for the volume ratio, which depends on the flow ratio at the delay time. Moreover, the flow ratio also affects directly the volume ratio of particles in the first reservoir, which gives rise to a term that depends on an undelayed form of the flow ratio. Despite the practical importance of control of such processes and existence of a control-oriented model there is no attempt to design a delay-compensating feedback law. As a result, the related literature for this problem can be categorized into results dealing with modeling and analysis of such processes; see, for example, [7], [10], [11], [14], [20], and into results dealing with predictor-based control of systems with input-dependent input delays; see, for example, [1], [2], [4], [5], [6], [8], [15], [17], and of systems with distributed input delay; see, e.g., [1], [12], [16], [19], [21].

In this paper, we develop a predictor-feedback control law for a nonlinear model of a microfluidic process under the Zweifach-Fung effect, which achieves exponential stabilization of a desired reference point. The design relies on two ingredients-the construction of an *exact* predictor state and the design of a nominal feedback law. Despite that the delay is defined implicitly through an integral of the control input (over an interval from the delay time to the current time) and despite that the control input enters the plant both in delayed and undelayed form, the construction of the predictor state is made possible owing to the particular structure of the nonlinear system considered and its specific dependence on the input variable (in fact, the predictor state is given in explicit formulae). The nominal feedback law is designed based on a particular delay-free system, which is not obtained in an obvious manner (for example, considering that the input only appears in undelayed form in the right-hand side of the respective system's dynamic equations). It is rather derived constructing a stabilizing feedback law for the system in a new time variable, which allows, in fact, to recasting the problem of design of the nominal feedback law as a problem

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of design of a feedback law for a delay-free, time-varying, nonlinear non-affine system.

For guaranteeing the delay properties required for design of a predictor state and for well-posedness of the system a feasibility condition that the control input is lower and upper bounded by positive constants needs to be satisfied. This imposes derivation of a local stability result in the supremum norm of the delayed, actuator state. The proof of exponential stability of the closed-loop system under the constructed predictor-feedback law relies on deriving estimates on solutions and on relating the norm of the overall infinite-dimensional system to the norm of the predictor state. An alternative formulation of the plant and the respective predictor-feedback law using transport Partial Differential Equation (PDE) representation of the delayed actuator state is also provided. We also present simulation results of a microfluidic process with a sinusoidal nonlinearity, describing the Zweifach-Fung effect, see, for example, [9], [14], which confirms the performance improvement of the closed-loop system under predictor feedback, as compared, for instance, to employment of an open-loop control strategy.

II. Model of the Process and Open-Loop Behavior

A. Model of the Process

We consider the system

$$\dot{Y}(t) = \frac{f\left(U\left(t-D(t)\right)\right)-Y(t)}{X(t)}U(t) \tag{1}$$

$$\dot{X}(t) = U(t) \tag{2}$$

$$\int_{t-D(t)}^{t} U(s)ds = L, \tag{3}$$

where $Y>0$ denotes the ratio of particles volume in the first channel with respect to total volume, $X>0$ is total volume, $U>0$ is flow ratio between flow in the first channel and total flow, which is the manipulated variable, $L>0$ is the ratio between total volume and total flow, and $t \geq 0$ is time variable. The goal is to regulate Y to a desired reference value. To guarantee well-posedness of system (1)–(3) and for system (1)–(3) to be a realistic model of the process the following feasibility condition has to be satisfied

$$0<c_1 \leq U(\theta) \leq c_2 < 1, \quad \text{for all } \theta \geq -D(0), \tag{4}$$

for some positive constants c_1, c_2. In fact, condition (4) guarantees that the delay D, defined implicitly via (3), satisfies all requirements of time-varying input delays that imply a uniquely defined delay that is positive and upper bounded, as well as that its rate is less than one and lower bounded, see, for example, [4]. These requirements also allow to guarantee well-posedness of a predictor state design, see, for example, [3], [4]. We impose the following realistic (see, for example, [14]) assumption on f.

Assumption 1: The function $f:[c_1,c_2]\to[d_1,d_2]$, with $0<d_1<d_2<1$, is Lipschitz with constant L_1, strictly increasing, and its inverse $f^{-1}:[d_1,d_2]\to[c_1,c_2]$ is Lipschitz with constant L_2.

B. Open-Loop Behavior

Lemma 1: Consider system (1), (2) with initial conditions $Y(0)=Y_0>0$, $X(0)=X_0>0$, and $U_0 \in \text{Lip}\left([-D(0),0],(c_1,c_2)\right)$ with $U_0(0)=c$ for some $c_1<c<c_2$, under a reference input $U(t)=c$, $t\geq 0$. Then the following holds[1]

$$Y(t)=\frac{Y_0X_0}{ct+X_0}+\frac{c\int_0^t f\left(U_0\left(s-D(s)\right)\right)ds}{ct+X_0}, \quad 0\leq t\leq \frac{L}{c} \tag{5}$$

$$Y(t)-f(c)=\left(Y\left(\frac{L}{c}\right)-f(c)\right)\frac{L+X_0}{ct+X_0}, \quad t>\frac{L}{c}. \tag{6}$$

Proof: The proof can be found in Appendix A. ∎

Lemma 1 implies that the equilibrium point $\bar{Y}=f(c)$ is asymptotically stable for a constant, reference input $U(t)=\bar{U}=c$, $t\geq 0$ (in fact, the statement of Lemma 1 holds for f that is only continuous on $[c_1,c_2]$). However, to improve performance (such as, for example, the convergence rate) and robustness of the open-loop system, we design next a predictor-feedback control law.

III. Predictor-Feedback Control Design

A. Nominal Control Design

Under Assumption 1 as long as the feasibility condition (4) is satisfied we can construct a nominal feedback law, which stabilizes a particular system. This is a typical requirement of predictor feedback in order to guarantee availability of a nominal, delay-free stabilizing feedback law. We choose the following nominal feedback law function

$$\kappa(\tau,H)=f^{-1}\left(H-k\tau\left(H-f(c)\right)\right), \tag{7}$$

with some $k>0$ and $c_1<c<c_2$, which renders the equilibrium $\bar{H}=f(c)$ of system

$$\frac{dH(\tau)}{d\tau}=\frac{1}{\tau}\left(f\left(\kappa\left(\tau,H(\tau)\right)\right)-H(\tau)\right), \tag{8}$$

asymptotically stable. The requirement that the nominal feedback law is chosen such that it stabilizes system (8) it may not be obvious and is explained as follows. With the change of variables

$$\tau=X(t), \tag{9}$$

(with $X(0)=X_0>0$) for the time variable t, under (4) (implying that the change of variables is invertible) we get from (1)–(3), using the fact that $t-D(t)=X^{-1}\left(X(t)-L\right)$ for $t-D(t)\geq 0$

$$\frac{dH(\tau)}{d\tau} = \frac{1}{\tau}\left(f\left(W(\tau-L)\right)-H(\tau)\right) \tag{10}$$

$$\frac{dZ(\tau)}{d\tau} = 1, \tag{11}$$

[1]Throughout the paper it is assumed that f satisfies Assumption 1. We do not, however, explicitly state this in Lemma 1 because its statement holds for f that is only continuous on $[c_1,c_2]$ (and takes values in $[d_1,d_2]$ to guarantee positivity of Y).

for $\tau \geq L$, where

$$H(\tau) = Y\left(X^{-1}(\tau)\right) \tag{12}$$
$$Z(\tau) = \tau \tag{13}$$
$$W(\tau) = U\left(X^{-1}(\tau)\right). \tag{14}$$

System (10) is a time-varying nonlinear system with a constant input delay. If there is a feedback law $\kappa(\tau, H(\tau))$ that stabilizes H to a desired reference point, say $\bar{H} = f(c)$, then we can employ a predictor-feedback control law to stabilize (10).

Even though in order to achieve the desired convergence rate in the original time variable t one should attentively choose a feedback law $\kappa(\tau, H(\tau))$ in τ, the alternative representation (10), (11) reveals that X could be viewed more as time variable (rather than as state), and thus, as regards a nominal, delay-free design, one could seek a feedback law of the form $\kappa(X, Y)$ that stabilizes (10), (11), which is simpler than (1). Moreover, the construction of the predictor state could be even performed for a system with a constant, rather than an input-dependent, delay.

B. Predictor-Feedback Design

Given a nominal, stabilizing feedback law κ, we construct the following predictor-feedback law

$$U(t) = \kappa(X(t) + L, P(t)) \tag{15}$$
$$P(t) = \frac{Y(t)X(t)}{X(t) + L} + \frac{\int_{\phi(t)}^{t} f(U(s))\, U(s) ds}{X(t) + L}. \tag{16}$$

Note that P is the predictor state of Y at the proper, for complete input delay compensation, prediction horizon, whereas $X + L$ is the predictor state of X. Both of these facts are explained as follows. Denoting the delay time as $\phi(t) = t - D(t)$ and the prediction time as $\sigma(t) = \phi^{-1}(t)$ (that exists as long as (4) is satisfied) we get that for $t \geq 0$

$$\int_{t}^{\sigma(t)} U(s) ds = L. \tag{17}$$

Therefore, using (2) we get that

$$X(\sigma(t)) = X(t) + L, \tag{18}$$

which shows that the predictor state of X, i.e., $X(\sigma)$ is $X + L$. Moreover, the prediction horizon needed is given by

$$\sigma(t) = X^{-1}(X(t) + L). \tag{19}$$

To find the predictor state of Y we substitute $t = \sigma(\theta)$, for $\phi(t) \leq \theta \leq t$, in (1) to obtain

$$\frac{dY(\sigma(\theta))}{d\theta} = \frac{d\sigma(\theta)}{d\theta} \frac{f(U(\theta)) - Y(\sigma(\theta))}{X(\sigma(\theta))} U(\sigma(\theta)), \tag{20}$$

and thus, defining $Y(\sigma(\theta)) = P(\theta)$ and using the fact that $U(\sigma(\theta)) \frac{d\sigma(\theta)}{d\theta} = U(\theta)$ (that follows differentiating (17) with respect to the time variable) we get

$$\frac{d(P(\theta)X(\sigma(\theta)))}{d\theta} = f(U(\theta))\, U(\theta). \tag{21}$$

Integrating (21) from $\theta = \phi(t)$ to $\theta = t$ and using (18) we get (16). Note that, according to (20), the Ordinary Differential Equation (ODE) satisfied by the predictor state is

$$\dot{P}(t) = \frac{f(U(t)) - P(t)}{X(t) + L} U(t). \tag{22}$$

IV. Stability Analysis

Theorem 1: Consider the closed-loop system consisting of the plant (1), (2) and the control law (15), (16) with (7). Under Assumption 1 there exists a strictly decreasing function $\epsilon \in C((0, +\infty), (0, +\infty))$ such that for all initial conditions $Y(0) = Y_0 > 0$, $X(0) = X_0 > 0$, $U_0 \in \mathrm{Lip}([-D(0), 0], (c_1, c_2))$, which satisfy

$$\Omega_0 < \epsilon(X_0) \tag{23}$$
$$\Omega_0 = |Y_0 - f(c)| + \sup_{-D(0) \leq \theta \leq 0} |U_0(\theta) - c|, \tag{24}$$

and $U_0(0) = \kappa\left(X_0 + L, \frac{Y_0 X_0}{X_0 + L} + \frac{\int_{-D(0)}^{0} f(U_0(s)) U_0(s) ds}{X_0 + L}\right)$, there exists a unique solution such that $Y(t) \in C^1[0, +\infty)$, $X(t) \in C^1[0, +\infty)$, $U(t)$ being locally Lipschitz on $[0, +\infty)$, and the following hold for $t \geq 0$

$$|Y(t) - f(c)| \leq \Omega_0 e^{kL}(1 + 2L_1) e^{-kc_1 t} \tag{25}$$
$$\sup_{t - D(t) \leq \theta \leq t} |U(\theta) - c| \leq \Omega_0 (1 + L_1)(L_2 + 1) e^{kL} e^{-kc_1 t} \times (1 + kL + kX_0 + kc_2 t). \tag{26}$$

Moreover, the feasibility condition (4) is satisfied.

Proof: The proof can be found in Appendix B. ■

V. Transport PDE Alternative

A. Model of the Process

An alternative representation of the process using transport PDE actuator state instead of delayed actuator state is given by

$$\dot{Y}(t) = \frac{f(u(0, t)) - Y(t)}{X(t)} U(t) \tag{27}$$
$$\dot{X}(t) = U(t) \tag{28}$$
$$u_t(x, t) = U(t) u_x(x, t) \tag{29}$$
$$u(L, t) = U(t), \tag{30}$$

where $x \in [0, L]$ is spatial variable and $u > 0$ denotes transport PDE state due to transportation of particles. To guarantee well-posedness of system (27)–(30) in the sense of guaranteeing a transport speed that is positive and lower/upper bounded (and for system (27)–(30) to be a realistic model of the process) one has to establish that (4) is satisfied.

B. Predictor-Feedback Control Design

The predictor state aims at compensating the delay due to the transport effect of the transport process given in (29). We keep here the PDE formulation as the original process is a transport process. Given a nominal, stabilizing feedback law, we construct the following predictor-feedback law

$$U(t)=\kappa\left(X(t)+L,p\left(L,t\right)\right) \tag{31}$$

$$p\left(x,t\right)=Y(t)\frac{X(t)}{X(t)+x}+\frac{1}{X(t)+x}\int_0^x f\left(u(y,t)\right)dy,\quad x\in[0,L]. \tag{32}$$

The fact that p is the predictor state could be explained as follows. In view of (29) the predictor state should satisfy

$$p(0,t) = Y(t) \tag{33}$$

$$p_t(x,t) = U(t)p_x(x,t), \tag{34}$$

which implies that relation $p(x,t)=Y\left(X^{-1}\left(X(t)+x\right)\right)$ holds (provided that U remains positive, and thus, that X is strictly increasing). It can be shown taking time and spatial derivatives of (32) and using (27)–(30) that (34) holds.

VI. Simulation Example

We consider the example from [14] in which $f:\left(\frac{1}{4},\frac{3}{4}\right)\rightarrow\left(\frac{1}{4},\frac{3}{4}\right)$ with $f(U)=\frac{1}{2}-\frac{1}{4}\sin\left(2\pi U\right)$ and $f^{-1}(U)=\frac{\pi-\arcsin(2-4U)}{2\pi}$. We show in Fig. 2 both functions f and f^{-1}. We choose the desired reference point as $\bar{Y}=f(c)=\frac{3}{5}$ with

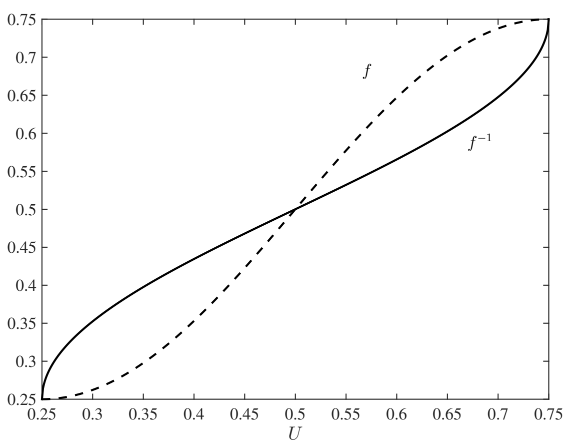

Fig. 2. The function f modeling the Zweifach-Fung effect and its inverse f^{-1} employed in the nominal feedback law (7).

$\bar{U}=c=0.566$ and a control gain $k=1.5$. In Fig. 3 we compare the responses in the cases of the open-loop system and for the closed-loop system under the proposed predictor-feedback law. One can observe that the predictor-feedback law stabilizes the desired equilibrium faster than the open-loop controller. In Fig. 4 we show the respective control efforts. Note that because the initial conditions for Y and u are at an equilibrium (although not at the desired one), there is a time interval in which Y remains constant. (This is consistent with equation (B.13); see also Lemma 1.)

VII. Conclusions

We constructed a predictor-feedback law for a second-order, nonlinear non-affine system with input-dependent input delay of hydraulic type arising in control of microfluidic

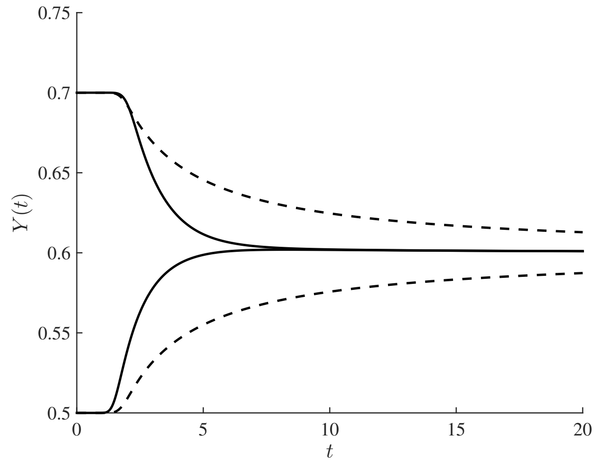

Fig. 3. Solid: Output $Y(t)$ of system (27) for two different initial conditions, namely, $u_0\equiv\frac{1}{2}$, $Y_0=f\left(\frac{1}{2}\right)=\frac{1}{2}$ and $u_0\equiv 0.65$, $Y_0=f\left(0.65\right)=0.7$, with $X_0=\frac{1}{2}$, under the predictor-feedback control law (31), (32) with (7). Dashed: Output $Y(t)$ of system (27) for two different initial conditions, namely, $u_0\equiv\frac{1}{2}$, $Y_0=f\left(\frac{1}{2}\right)=\frac{1}{2}$ and $u_0\equiv 0.65$, $Y_0=f\left(0.65\right)=0.7$, with $X_0=\frac{1}{2}$, under the open-loop control law $U(t)=\bar{U}$, for all $t\geq 0$.

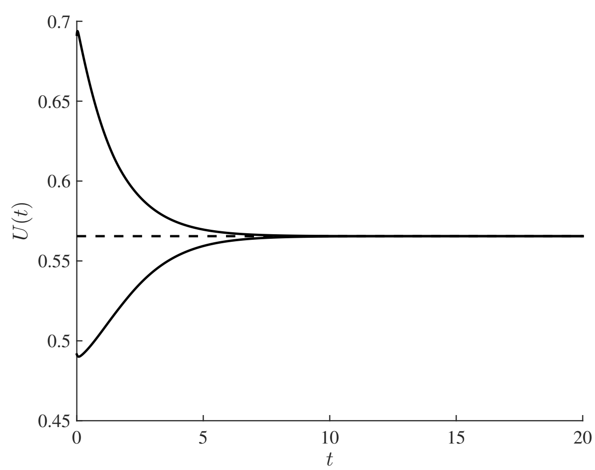

Fig. 4. Solid: Control input $U(t)$ given by (31), (32) with (7), for two different initial conditions, namely, $u_0\equiv\frac{1}{2}$, $Y_0=f\left(\frac{1}{2}\right)=\frac{1}{2}$ and $u_0\equiv 0.65$, $Y_0=f\left(0.65\right)=0.7$, with $X_0=\frac{1}{2}$. Dashed: Control input $U(t)=\bar{U}$, for all $t\geq 0$, for two different initial conditions, namely, $u_0\equiv\frac{1}{2}$, $Y_0=f\left(\frac{1}{2}\right)=\frac{1}{2}$ and $u_0\equiv 0.65$, $Y_0=f\left(0.65\right)=0.7$, with $X_0=\frac{1}{2}$.

processes under the Zweifach-Fung effect. We proved exponential stability of the reference point in closed loop utilizing estimates on solutions. The simulation results provided confirm the performance improvement of the closed-loop system under the developed control design.

Even though we impose the assumption on invertibility of the nonlinearity due to the Zweifach-Fung effect, this is not restrictive. The reason is that, in certain applications, the operation region of interest lies in medium flow ratios. However, to operate over the whole spectrum of potential flow ratios, where f may not be increasing, one has to remove such an assumption. This is an issue that we currently investigate.

Appendix A

Proof of Lemma 1

Denote as t_1 the time instant at which $\phi(t_1) = 0$. This time instant always exists since ϕ is increasing and for $0 \le t \le t_1$ it holds from (3) that $\int_{\phi(t)}^{0} U_0(s)ds = L - ct$. Hence, it should be the case that $\int_{\phi(t_1)}^{0} U_0(s)ds = 0$, where $t_1 = \frac{L}{c}$, since U_0 is continuous and strictly positive (by assumption). Thus, for $t \ge \frac{L}{c}$ system (1), (2) evolves according to

$$\dot{Y}(t) = \frac{f(c) - Y(t)}{X(t)} c \tag{A.1}$$

$$\dot{X}(t) = c. \tag{A.2}$$

Solving (A.1) we get that $Y(t)X(t) - Y\left(\frac{L}{c}\right) X\left(\frac{L}{c}\right) = cf(c)\left(t - \frac{L}{c}\right)$ and $X(t) = ct + X_0$, for $t \ge \frac{L}{c}$, from which we obtain (6) for $t \ge \frac{L}{c}$. For $0 \le t \le \frac{L}{c}$ we get from (1), (2) that

$$\dot{Y}(t) = \frac{f(U_0(t - D(t))) - Y(t)}{ct + X_0} c. \tag{A.3}$$

Solving (A.3) we obtain (6) for $0 \le t \le \frac{L}{c}$.

Appendix B

Proof of Theorem 1

The feasibility condition (4) is satisfied for $-D(0) \le \theta \le 0$ by the assumption on the initial condition for U. In order to guarantee that the feasibility condition is satisfied for $t \ge 0$ we need to establish that the following holds for $t \ge 0$

$$f(c_1) \le P(t) - k(X(t) + L)(P(t) - f(c)) \le f(c_2), \tag{B.1}$$

which can be satisfied provided that the following holds

$$\left|\tilde{P}(t)\right| |1 - k(X(t) + L)| < \delta, \quad t \ge 0, \tag{B.2}$$

where $\tilde{P} = P - f(c)$ and $\delta = \min\{f(c_2) - f(c), f(c) - f(c_1)\}$. As long as U satisfies inequality (4), from (7), (15), (22) it follows that the predictor state P satisfies $\dot{\tilde{P}}(t) = -k\tilde{P}(t)\dot{X}(t)$, and thus,

$$\tilde{P}(t) = \tilde{P}(0)\mathrm{e}^{-k(X(t) - X_0)}. \tag{B.3}$$

Furthermore, as long as U satisfies inequality (4), it holds that $c_1 t + X_0 \le X(t) \le c_2 t + X_0$. Therefore,

$$\left|\tilde{P}(t)\right| |1 - k(X(t) + L)| \le \left|\tilde{P}(0)\right| \mathrm{e}^{-kc_1 t}(1 + kL + kX_0 + kc_2 t). \tag{B.4}$$

From (16) for $t = 0$ it follows using (3) that

$$\left|\tilde{P}(0)\right| \le \frac{|Y_0 - f(c)| X_0}{X_0 + L} + \frac{\int_{\phi(0)}^{0} |f(U_0(s)) - f(c)| U_0(s)ds}{X_0 + L}. \tag{B.5}$$

Under Assumption 1 (f being Lipschitz) we get from (B.5) using (3) that

$$\left|\tilde{P}(0)\right| \le \frac{|Y_0 - f(c)| X_0}{X_0 + L} + \frac{LL_1 \sup_{-D(0) \le s \le 0} |U_0(s) - c|}{X_0 + L}, \tag{B.6}$$

and thus (since $X_0 > 0$ by assumption),

$$\left|\tilde{P}(0)\right| \le |Y_0 - f(c)| + L_1 \sup_{-D(0) \le s \le 0} |U_0(s) - c|. \tag{B.7}$$

Using (B.4), it follows that (B.2) is satisfied provided that

$$\left|\tilde{P}(0)\right| (1 + kL + kX_0 + kc_2 t) \mathrm{e}^{-kc_1 t} < \delta, \quad t \ge 0, \tag{B.8}$$

which is satisfied whenever

$$\left|\tilde{P}(0)\right| < \frac{\delta}{M(X_0)} \tag{B.9}$$

$$M(X_0) = \max\left\{1 + kL + kX_0, \frac{c_2}{c_1}\mathrm{e}^{-1 + \frac{c_1}{c_2}(kL + 1 + kX_0)}\right\}. \tag{B.10}$$

Using (B.7) we obtain that condition (B.9), and hence, also (B.2), is satisfied whenever (23) holds with

$$\epsilon(X_0) = \frac{\delta}{\max\{1, L_1\} M(X_0)}. \tag{B.11}$$

In order to derive stability estimates (25) and (26) we start noting that, since under (4) the state X remains an increasing function of time, there exists a finite time instant $\sigma(0) \ge 0$ such that $X(\sigma(0)) = X_0 + L$ (in fact, from (15), (17) it follows that $\sigma(0) \le \frac{L}{c_1}$), and hence, $\phi(\sigma(0)) = 0$. For all $0 \le t \le \sigma(0)$ we then obtain from (1), (2) that

$$\frac{d(Y(t)X(t))}{dt} = f(U_0(t - D(t)))U(t), \tag{B.12}$$

and hence,

$$Y(t) - f(c) = \frac{\int_0^t (f(U_0(s - D(s))) - f(c)) U(s)ds}{X(t)} + \frac{(Y_0 - f(c)) X_0}{X(t)}, \quad 0 \le t \le \sigma(0). \tag{B.13}$$

Under Assumption 1 (f being Lipschitz) and the assumption on U_0 we get from (B.13) that

$$|Y(t) - f(c)| \le \frac{L_1 \sup_{0 \le s \le t} |U_0(s - D(s)) - c| \int_0^t U(s)ds}{X(t)} + \frac{|Y_0 - f(c)| X_0}{X(t)}. \tag{B.14}$$

Since $X(t) \ge c_1 t + X_0$ (under (4)), using (2) we obtain from (B.14) that for $0 \le t \le \sigma(0)$ it holds that

$$|Y(t) - f(c)| \le |Y_0 - f(c)| + 2L_1 \sup_{-D(0) \le s \le 0} |U_0(s) - c|. \tag{B.15}$$

For $t \ge \sigma(0)$, which implies that $t - D(t) \ge 0$, as $X(\sigma(t)) = X(t) + L$ and $P(t) = Y(\sigma(t))$, we obtain from (15) that

$$\dot{Y}(t) = \frac{f(\kappa(X(t), Y(t))) - Y(t)}{X(t)} \dot{X}(t), \tag{B.16}$$

and hence, from (7) we get that

$$\dot{Y}(t) = -k(Y(t) - f(c)) \dot{X}(t). \tag{B.17}$$

Thus, we get by explicitly solving (B.17) and using (18) that

$$\begin{aligned} Y(t) &= f(c) + \mathrm{e}^{-k(X(t)-X_0-L)} \\ &\quad \times \left(Y\left(\sigma(0)\right) - f(c)\right), \end{aligned} \tag{B.18}$$

and hence,

$$|Y(t) - f(c)| \leq \mathrm{e}^{-kc_1t}\mathrm{e}^{kL}\left|Y\left(\sigma(0)\right) - f(c)\right|. \tag{B.19}$$

Using (B.15) and the fact that for $t \leq \sigma(0)$ it holds that $X(t) \leq X_0 + L$ (since X is increasing), we obtain (25). Since (B.1) holds and since f^{-1} is Lipschitz (by assumption), it follows from (7), (15) that for $t \geq 0$

$$|U(t) - c| \leq L_2\left|\tilde{P}(t)\right|\left|\left(1 - k\left(X(t) + L\right)\right)\right|, \tag{B.20}$$

and hence, using (B.4) it follows that

$$\begin{aligned} |U(t) - c| &\leq L_2\left|\tilde{P}(0)\right|\left(1 + kL + kX_0 + kc_2t\right) \\ &\quad \times \mathrm{e}^{-kc_1t}, \quad t \geq 0. \end{aligned} \tag{B.21}$$

Using (B.7) we obtain from (B.21) that

$$\begin{aligned} |U(t) - c| &\leq (1 + L_1)L_2\Omega_0\left(1 + kL + kX_0 + kc_2t\right) \\ &\quad \times \mathrm{e}^{-kc_1t}, \quad t \geq 0. \end{aligned} \tag{B.22}$$

Thus, for $t \geq \sigma(0)$ we obtain from (B.22) that

$$\begin{aligned} \sup_{t-D(t)\leq\theta\leq t} |U(\theta) - c| &\leq \Omega_0(1 + L_1)L_2\mathrm{e}^{kL}\mathrm{e}^{-kc_1t} \\ &\quad \times \left(1 + kL + kX_0 + kc_2t\right), \end{aligned} \tag{B.23}$$

where we used the fact that $D(t) \leq \frac{L}{c_1}$, $t \geq 0$, which follows from (3), (4). Using the fact that $\sup_{t-D(t)\leq\theta\leq t}|U(\theta) - c| \leq \left(\sup_{-D(0)\leq\theta\leq 0}|U_0(\theta) - c| + \sup_{0\leq\theta\leq t}|U(\theta) - c|\right) \times \mathrm{e}^{-kc_1(t-\sigma(0))}$, $0 \leq t \leq \sigma(0)$, we obtain using (B.22) that

$$\begin{aligned} \sup_{t-D(t)\leq\theta\leq t} |U(\theta) - c| &\leq \Omega_0(1 + L_1)(L_2 + 1)\mathrm{e}^{kc_1\sigma(0)}\mathrm{e}^{-kc_1t} \\ &\quad \times \left(1 + kL + kX_0 + kc_2t\right), \end{aligned} \tag{B.24}$$

for $0 \leq t \leq \sigma(0)$. Using (4), (17) it follows that $\sigma(0) \leq \frac{L}{c_1}$, and hence, using (B.23), (B.24) we obtain (26).

To study the regularity properties of the closed-loop system we first note that from (2), (15), (22) it follows that

$$\dot{P}(t) = -k\left(P(t) - f(c)\right)\kappa\left(X(t) + L, P(t)\right) \tag{B.25}$$

$$\dot{X}(t) = \kappa\left(X(t) + L, P(t)\right), \tag{B.26}$$

and thus, since the right-hand side of the above ODE in (P, X) is locally Lipschitz in (P, X) we get (with (B.3) and $c_1t + X_0 \leq X(t) \leq c_2t + X_0$) existence and uniqueness of a solution $(P(t), X(t)) \in C^1[0, +\infty)$. Therefore, from (7), (15), it follows from Assumption 1 (and the assumption on U_0) that $U(t)$ is locally Lipschitz on $[0, +\infty)$. Moreover, since $\phi'(t) = \frac{U(t)}{U(\phi(t))}$ with U being locally Lipschitz, we obtain (with $t - \frac{L}{c_1} \leq \phi(t) \leq t - \frac{L}{c_2}$, which follows from (3), (4)) that there exists a unique solution $\phi(t) \in C^1[0, +\infty)$. Thus, from (B.12) it follows (with (25)) that there exists a unique solution $Y(t) \in C^1[0, \sigma(0))$. Similarly, from (B.17) it follows (with (25)) that there exists a unique solution $Y(t) \in C^1(\sigma(0), +\infty)$. Compatibility of U_0 with the feedback law guarantees that Y is continuously differentiable also at $t = \sigma(0)$.

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