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Run-to-run control with nonlinearity and delay uncertainty

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Abstract: In this paper, we study a simple single-input single output nonlinear system controlled by a Run-to-run algorithm. Besides the usually considered model uncertainty, the particularity of the system under consideration is that measurements available to the control algorithm suffer from large and varying measurement delays. The control algorithm is a nonlinear sampled model-based controller with successive model inversion and bias correction. The main contribution of this article is its proof of global convergence. In particular, the model error and the varying delays are treated using monotonicity of the system and a detailed analysis of the closed-loop behavior of the sampled dynamics, in an appropriate norm.

Keywords: Run-to-run control, data-sampled system, varying time delays, stability analysis

1. INTRODUCTION

In this article, we consider the effects of delay variability on the Run-to-run control of a nonlinear process. Run-to-run is a popular and efficient class of techniques, originally proposed in Sachs et al. [1991], specifically tailored for processes lacking in situ measurement for the quality of the production (see Wang et al. [2009]). Numerous examples of implementations have been reported in the semiconductor, and materials industry, in particular, see e.g. Wang et al. [2009], Moyne et al. [2000] and references therein. However, in applications, two practical problems often arise: model uncertainty and delay uncertainty.

First, the interactions between the input and the system states can be rather complex, which, in turn, causes some non-negligible uncertainty on the quantitative effects of the input. These can be addressed as model mismatch.

Second, the measurements are available after a long time lag covering the various tasks of sample collection, receipt, preparation, analysis and transfer of data through an information technology (IT) system to the control system. Therefore, measurements are impacted by large delays, which can be varying to a large extent, and in some cases be state- or input-dependant. This variability of the delay builds up with the intrinsic information technology (IT) dating uncertainty, because, usually, no reliable timestamp can be associated to the measurements. The delay variability cannot be easily represented by Gaussian models, nor can it be fully described as deterministic input or state dependant delay, nor known varying delays that could be compensated for by predictor techniques (as done in e.g. Bresch-Pietri et al. [2012, 2014], Bekiaris-Liberis and Krstic [2013b,c,a]). As is well known, the uncertainty and the variability of delay may jeopardize closed-loop stability Krstic [2009] and references therein. In the particular context of Run-to-run control, it is known, see Wang et al. [2005], that such metrology delay coupled with inaccurate process model could lead to closed-loop instability. For these reasons, the problem under consideration in this article can be considered as challenging and of importance for applications.

In this paper, we consider a simple single-input single-output problem of Run-to-run control. As is well known, such control problem can also be seen as an adaptive control scheme or a simple nonlinear implementation of an internal model controller (IMC, see e.g Morari and Zafiriou [1989]). Besides the usual model mismatch (both model and true system behavior are assumed to be monotonic), we address the effects of the discussed delay uncertainty.

The nonlinearity does not cause too much difficulty. In the absence of delay, robust stability in the presence of model mismatch can be readily established, using the monotonicity of the system and model. The study of delay effects is more involved. Once expressed in the sampled time-scale, the control scheme exhibits a variable delay discrete-time dynamics. Hence, a simple Nyquist criterion analysis cannot be used to infer stability and some more specific investigations are required. In details, the control scheme involves an uncertain positive bounded delay. From there, a complete stability analysis in a space of sufficiently large dimension, with a well chosen norm, yields a proof of robust stability under a small gain condition. Interestingly, the small-gain bound is reasonably sharp, so that it can serve as guideline for practical implementation. The novelty of the approach presented in this article lies in the proof considered. It does not treat the uncertainty of the delay using the Padé approximate approach considered in Zhang et al. [2009], but directly uses an extended dimension of the discrete time dynamics. In future works, these arguments of proof could be extended to address more general problems, in particular to higher dimensional forms (lifted forms) usually considered to recast general iterative learning control into Run-to-run as clearly explained in Wang et al. [2009].

The paper is organized as follows. In Section 2 notations are given. In Section 3 the process under consideration is exposed. In Section 4 robust stability results are established. In Section 5 simulations results are reported. Conclusions and future directions are given in Section 6.

2. NOTATIONS AND PRELIMINARY RESULTS

2.1 Notations

Given $\mathcal{I}$ an interval of $\mathbb{R}$, and $f : \mathcal{I} \to \mathbb{R}$ a smooth function, we define

$$\|f\|_\infty = \sup_{x\in\mathcal{I}} |f(x)|$$

For any vector X, we note $\|X\|_1$, $\|X\|_2$ and $\|X\|_\infty$ its 1-norm, its Euclidean norm and its infinity norm, respectively. Note $\|.\|_*$ any of the vector norms above. For any square matrix A, we note $\|A\|_*$ the norm of A, subordinate to $\|.\|_*$. Classically (e.g. Higham [2008]), for all A, B

$$\|AB\|_* \leq \|A\|_* \|B\|_*$$

We note $\lfloor x \rfloor$ the floor value of x, mapping x to the largest previous integer.

For any matrix dimension, we define E_{ij} the matrix of general term $e_{k,l}$

$$\forall (k,l), \quad e_{k,l} = \delta_{k,i}\delta_{l,j} \tag{1}$$

where δ is the Kronecker delta $\delta_{k,i} = 1$ if $k = i$ and 0 otherwise.

2.2 Preliminary results on discrete linear time-varying systems

The event-driven nature of the control scheme leads us to consider discrete time dynamics. Below, we formulate a simple technical result, instrumental in the rest of the paper.

Consider a discrete linear time-varying system (2) of dimension s, and $\mathcal{A}$ a bounded set of possible transition matrices in $\mathcal{M}_s(\mathbb{R})$ and initial condition X_0

$$\forall n \geq 0, X_{n+1} = A_n X_n, \quad A_n \in \mathcal{A} \tag{2}$$

For any vector norm $\|.\|_*$ and any $N \in \mathbb{N}$, we define

$$M_{N,*} \triangleq \sup_{A_{N-i}\in\mathcal{A}} \| \prod_{i=1}^{N} A_{N-i} \|_* = \sup_{A_i\in\mathcal{A}} \| \prod_{i=0}^{N-1} A_i \|_* \tag{3}$$

Proposition 1. (Suff. cond. for exp. stab.). Consider the system (2). If there exists $N_0 \in \mathbb{N}^*$ such that $M_{N_0,*} < 1$, then the system (2) (globally) exponentially converges to 0. One has, for some $K > 0$,

$$\forall n \in \mathbb{N}^*, \quad \|X_n\|_* \leq K \|X_0\|_* \left(M_{N_0,*}\right)^{\left\lfloor \frac{n}{N_0} \right\rfloor}$$

Proof. see appendix

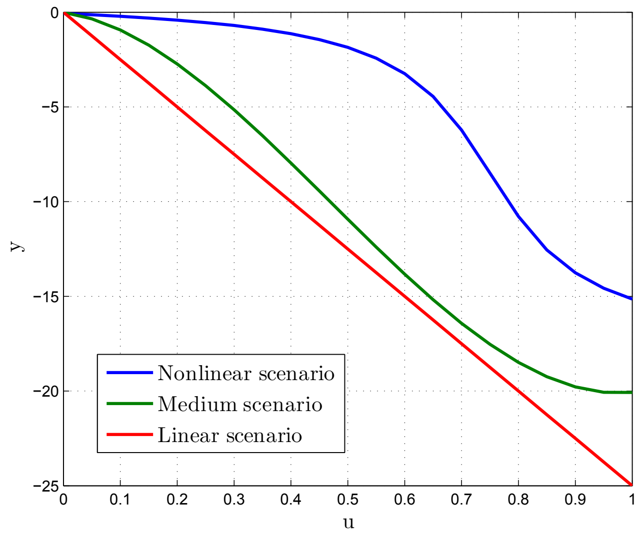


Fig. 1. Examples of possible input-output mappings f

3. PROBLEM STATEMENT

3.1 Model

We note y the controlled variable (output) and u the control variable (input). It is assumed that there exists f a strictly monotonous smooth function such that

$$y = f(u)$$

Although f is unknown, we can use a model of it, f_p, which is also smooth and monotonous [1], such that $f_p(0) = f(0)$. Usually, f_p is a rough estimate of f. Typical models are represented in Figure 1. For the simulations considered in this article, the model error can be as large as 20-40%, which is representative of needs for industrial applications.

The target value c for the controlled variable is assumed to be reachable by both the system and the model, *i.e.* there exists u_c and u_p verifying

$$f(u_c) = c, \quad f_p(u_p) = c$$

3.2 Measurements

A measurement system sporadically provides measurements of y. Once a value is available, a new measurement process is initiated.

In many cases, the measurement time is varying, and the measurement delay directly depends on the value of the measured variable. Besides this state-dependent delay, another source of lag is related to the industrial IT. In many cases, no track is kept of the time the specimen was

[1] In practice, it can result from the analysis of sensitivity look-up tables obtained from experiments and derivation of interpolating models.

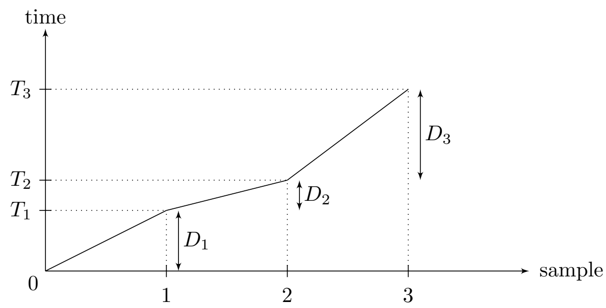


Fig. 2. Representation of times when measures become available (T_1, ... T_n) and measurement delays (D_1, ... D_n).

taken on the plant. This causes an additional uncertainty on the delay.

In the system considered in this article, the measurements available for feedback in a control loop have thus two specificities. They are sporadic, and each value y_n, $n = 1$,... becomes available after a delay $D(y_n)$ which depends on the value measured, and uncertain. Exact dating of the data is impossible because each measurement y_n is corrupted with noise, and because the specimen date itself is uncertain.

3.3 Control problem

The above description leads us to consider an event triggered discretization of the process in which a new sampling time n is created at every time T_n when a measurement y_n is received. By definition,

$$T_n - T_{n-1} = D(y_n) \triangleq D_n$$

These variables are represented in Figure 2.

A closed-loop controller can be designed for the system. Every time a measurement is received, the control is updated and the value of the control remains constant until the next measurement is received, creating piece-wise constant control signals (with varying step-lengths). To account for delay variability and estimate the date of each measurement, it is necessary to use a model of it, $D_p(y)$, providing one with an estimation of the measurement delay associated with a given measured value y.

The control design should aim at solving the following problem.

Control problem *Create a sequence* (u_n) *using the approximate model* f_p *and the delayed measurements* $(f(u_n))$ *of* y_n *such that* $\lim_{n\to+\infty} f(u_n) = c$

At this stage, we can propose a simple nonlinear IMC algorithm to address the problem. This algorithm adapts a bias term used in a model inversion. Ignoring the measurement delay effects, the implementation of such an algorithm would be

$$\begin{cases} u_0 = 0, \quad \delta_0 = 0, \quad \alpha \in]0;1] \\ n \geqslant 0, \quad u_{n+1} = f_p^{-1}(c - \delta_n) \\ \quad \delta_{n+1} = \delta_n + \alpha(y_n - f_p(u_n) - \delta_n) \end{cases} \tag{4}$$

which can be wrapped up in the following familiar block diagram of Figure 3.

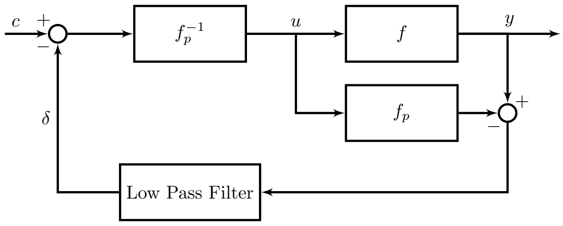


Fig. 3. Idealized closed-loop control scheme.

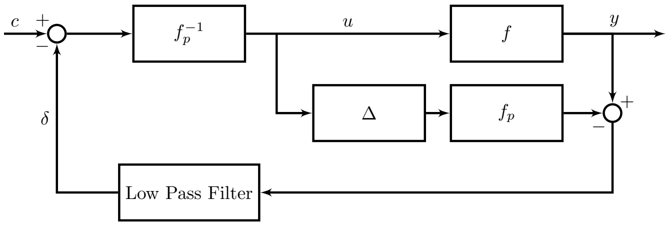


Fig. 4. Mis-synchronization due to delay measurement creates a varying delay in the IMC scheme.

However, the measurement delays have an impact on the controller dynamics. Instead of (4), one is able to implement the following

$$\begin{cases} u_0 = 0, \quad \delta_0 = 0, \quad T_0 = 0 \\ n \geqslant 0, \quad u_{n+1} = f_p^{-1}(c - \delta_n) \\ \delta_{n+1} = \delta_n + \alpha(y_n - f_p(u_{\mathrm{ind}(T_{n+1}-D_p(y_n))}) - \delta_n) \end{cases} \tag{5}$$

where the ind function is defined as

$$\mathrm{ind}(t) = \begin{cases} n \text{ such as } t \in [T_n; T_{n+1}[\\ 0 \quad \text{if } t < 0 \end{cases}$$

Besides,

$$\mathrm{ind}(T_{n+1} - D_p(y_n)) = \mathrm{ind}(T_n + D(y_n) - D_p(y_n))$$

Note

$$\mathrm{ind}(T_n + D(y_n) - D_p(y_n)) = n - \Delta_n$$

then $\Delta_n \in \mathbb{N}$. It can be interpreted as an explicit mis-synchronization term.

Equivalently, equations (5) can be rewritten as

$$\begin{cases} u_0 = 0, \quad \delta_0 = 0, \quad T_0 = 0 \\ n \geqslant 0, \quad u_{n+1} = f_p^{-1}(c - \delta_n) \\ \delta_{n+1} = \delta_n + \alpha(y_n - f_p(u_{n-\Delta_n}) - \delta_n) \end{cases} \tag{6}$$

Interestingly, if the delay model is perfect i.e. $D \equiv D_p$, it is straightforward to see that (6) simplifies to (4). Otherwise, some mis-synchronization appears between the measurement and the associated prediction in the calculation of the bias. The situation is pictured in Figure 4. It is necessary to investigate the stability of the controller in this case.

4. STABILITY ANALYSIS

4.1 Convergence with model mismatch, without delays

In the analysis, two problems must be treated: model mismatch and mis-synchronisation.

We first consider the system without measurement delays.

Error dynamics Used in closed loop, (4) gives

$$\begin{cases} u_0 = 0, \quad \delta_0 = 0, \quad \delta_1 = \alpha_0(f(0) - f_p(0)) \\ n \geqslant 0, \quad u_{n+1} = f_p^{-1}(c - \delta_n) \\ \quad \delta_{n+2} = (1-\alpha)\delta_{n+1} + \alpha(\delta_n - c + f \circ {f_p}^{-1}(c - \delta_n)) \end{cases} \tag{7}$$

The asymptotic behaviour of (7) is determined by the second order dynamics of (δ_n). If (u_n) and (δ_n) converge toward the limits u and δ respectively, then, necessarily,

$$u = u_c \quad \text{and} \quad \delta = c - f_p(u_c)$$

Define the sequence $(d_n \triangleq \delta_n - \delta, n \geq 0)$. The error dynamics is equivalently represented by the second order equation

$$d_{n+2} = (1-\alpha)d_{n+1} + \alpha(d_n + f \circ {f_p}^{-1}(f_p(u_c) - d_n)) - \alpha c$$

Applying the mean value theorem to the function $x \mapsto x + f \circ f_p^{-1}(f_p(u_c) - x)$, one deduces that there exists

$$a_n \in [\min(0, d_n); \max(0, d_n)]$$

such that

$$d_{n+2} = (1-\alpha)d_{n+1} + \alpha\left(1 - \frac{f' \circ {f_p}^{-1}(f_p(u_c) - a_n)}{f'_p \circ {f_p}^{-1}(f_p(u_c) - a_n)}\right) d_n$$

This can be rewritten as a two-dimensional linear time-varying (LTV) system

$$X_{n+1} = A_n X_n \tag{8}$$

with

$$X_n = \begin{pmatrix} d_n \\ d_{n+1} \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} 0 & 1 \\ \alpha h(a_n) & 1-\alpha \end{pmatrix}$$

where

$$h(a_n) = 1 - \frac{f' \circ {f_p}^{-1}(f_p(u_c) - a_n)}{f'_p \circ {f_p}^{-1}(f_p(u_c) - a_n)}$$

Interestingly, h can be interpreted as a metric of the model error: if $f \equiv f_p$, we indeed get $h \equiv 0$. Then, (8) becomes a simple linear time invariant system (LTI) which is trivially exponentially stable. Otherwise, one needs further investigations to establish the following result, showing that the control problem is solved, in the absence of delay variations.

Theorem 2. (Global exponential convergence). Given any $\alpha \in]0; 1]$, if there exists η such that $\|h\|_\infty \leq \eta < 1$, then the closed loop error (7) converges exponentially and $\lim_{n \to +\infty} f(u_n) = c$.

Remark 1. In particular, one can notice that f' and f'_p must have the same sign so that the condition can be verified. In this case, if

$$0 < \|\frac{f'}{f'_p}\|_\infty < 2$$

then the sufficient condition is satisfied.

Proof. Establishing the asymptotic (not to say exponential) convergence of a general LTV discrete time system is a difficult task. In particular, it is not sufficient to study its eigenvalues (see Rugh [1996]). Some results have long been available for slowly varying systems and have recently been refined in Hill and Ilchmann [2010], in particular. However, in our present case, it is not necessary to use them. The particular structure of the varying term allows more straightforward investigations.

We define the (infinite) set

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 \\ \alpha h(x) & 1-\alpha \end{pmatrix}, \quad x \in \mathbb{R} \right\}$$

Under the assumption $\|h\|_\infty \leq \eta < 1$, $\mathcal{A}$ is bounded. Consider any $(A_1, A_2) \in \mathcal{A}^2$

$$A_1 = \begin{pmatrix} 0 & 1 \\ \alpha h_1 & 1-\alpha \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ \alpha h_2 & 1-\alpha \end{pmatrix}$$

Then,

$$A_2 A_1 = \begin{pmatrix} \alpha h_1 & 1-\alpha \\ (1-\alpha)\alpha h_1 & \alpha h_2 + (1-\alpha)^2 \end{pmatrix} \triangleq \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

Since

$$\|L_1\|_1 \leq 1 - \alpha(1-\eta) \triangleq l < 1$$

and

$$\|L_2\|_1 \leq (1-\alpha)^2 + \alpha(2-\alpha)\eta \triangleq l' < 1$$

then, we have for all $(A_1, A_2) \in \mathcal{A}^2$

$$\|A_2 A_1\|_\infty \leq \max(l, l') < 1$$

As a consequence, using the notation (3)

$$M_{2,\infty} = \sup_{(A_1, A_2) \in \mathcal{A}^2} \|A_2 A_1\|_\infty < 1$$

which, according to Proposition 1, yields the conclusion.

4.2 Convergence with measurement delays

We now consider the implementation of the same controller on the more realistic system with variable measurement delays causing the discussed mis-synchronization.

Error dynamics Using the same transformation as in § 4.1, we establish the closed-loop error

$$\begin{aligned} d_{n+2} =& (1-\alpha)d_{n+1} + \alpha(f(f_p^{-1}(f_p(u_c) - d_n)) \\ & - f_p(u_c) + d_{n-\Delta_{n+1}}) - \alpha(c - f_p(u_c)) \end{aligned}$$

and, applying the mean value theorem, we get that

$$d_{n+2} = (1-\alpha)d_{n+1} - \alpha\rho(a_n)d_n + \alpha d_{n-\Delta_{n+1}}$$

where

$$\rho(a_n) = \frac{f'(f_p^{-1}(f_p(u_c) - a_n))}{f'_p(f_p^{-1}(f_p(u_c) - a_n))}$$

and

$$a_n \in [\min(0, d_n); \max(0, d_n)]$$

We will now assume that the desynchronization is bounded in terms of sampling times, *i.e.* we assume the following

Assumption 1. There exits Δ_{max} such that $\forall n \in \mathbb{N}$ one has $\Delta_n \leq \Delta_{max}$

If this reasonable assumption holds, the system can be written as a LTV system of dimension $\Delta_{max} + 2$

$$X_{n+1} = A_n X_n \tag{9}$$

where

$$X_n = (d_{n-\Delta_{max}} \cdots d_{n+1})^T$$

with

$$A_n = \begin{pmatrix} 0 & 1 & \cdots\cdots\cdots & & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots \quad \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots\cdots\cdots & 0 & & -\alpha\rho(a_n) & 1-\alpha \end{pmatrix} + \quad \alpha F_n$$

and, with the notation (1),

$$F_n = E_{\Delta_{max}+2,\Delta_{max}+1-\Delta_{n+1}}$$

Convergence analysis without model error Let us first assume that there is no model error. Under this assumption

$$\rho = 1$$

and the transition matrices A_n of the dynamic (9) all belong to the finite set

$$\mathcal{A} = \{C + \alpha E_{D_{max}+2,k}, \quad k \in [\![1; D_{max}+1]\!]\}$$

where

$$C = \begin{pmatrix} 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & -\alpha & 1-\alpha \end{pmatrix}$$

Theorem 3. (Global exponential convergence). If $\rho = 1$ and if Assumption 1 holds, then for $\alpha \leq \frac{3-\sqrt{5}}{2}$ the controller (5) guarantees $\lim\limits_{n\to+\infty} f(u_n) = c$.

Proof. The proof is built, recursively, on the fact that for all $n \in \mathbb{N}^*$

$$M_{n,\infty} = \| \prod_{i=1}^{n} A_{n-i} \|_\infty \leq (1+\alpha)(1-\alpha^2)^{\lfloor \frac{n-1}{p+1} \rfloor} \quad (10)$$

Consider a sequence of n transition matrices $(A_i)_{i\in[\![0;n-1]\!]} \in \mathcal{A}^n$. Define

$$\forall k \in [\![1;n]\!], \quad \Pi_k = \prod_{i=1}^{k} A_{k-i} = \begin{pmatrix} L_1^k \\ \vdots \\ L_p^k \end{pmatrix}$$

where L_i^k designates the i^{th} row of the product of the k matrices and $p = \Delta_{max} + 2$ is the dimension of the transition matrices.

With these notations, one has

$$\|\Pi_k\|_\infty = \max_{i\in[\![1;p]\!]} \|L_i^k\|_1$$

For all $n \geq 2$, we wish to prove that the following relations (11), (12), (13) hold.

$$\forall (j,k) \in [\![1;p]\!] \times \{n-1;n\}, \quad \|L_j^k\|_1 \leq (1+\alpha)(1-\alpha^2)^{\lfloor \frac{k+j-2}{p+1} \rfloor} \quad (11)$$

$$\forall j \in [\![1;p-1]\!], \quad L_j^n = L_{j+1}^{n-1} \quad (12)$$

$$\exists l \in [\![1;p-1]\!], \quad L_p^n = (1-\alpha)L_p^{n-1} - \alpha L_{p-1}^{n-1} + \alpha L_l^{n-1} \quad (13)$$

If $\alpha \leq \frac{3-\sqrt{5}}{2}$, the property can be initialized by a straightforward computation for $n = 2$.

Given $n \geq 2$, let us assume that the property is true for this rank. One has

$$\Pi_{n+1} = \prod_{i=1}^{n+1} A_{n+1-i} = \begin{pmatrix} L_1^{n+1} \\ \vdots \\ L_p^{n+1} \end{pmatrix}$$

Computing Π_{n+1} gives

$$\forall j \in [\![1;p-1]\!], \quad L_j^{n+1} = L_{j+1}^n$$

and

$$\exists l \in [\![1;p-1]\!], \quad L_p^{n+1} = (1-\alpha)L_p^n - \alpha L_{p-1}^n + \alpha L_l^n$$

This proves (12) and (13) at the rank $n+1$. Furthermore, according to (13) at rank n

$$\exists l' \in [\![1;p-1]\!], \quad L_p^n = (1-\alpha)L_p^{n-1} - \alpha L_{p-1}^{n-1} + \alpha L_{l'}^{n-1}$$

Hence, according to (12) at rank n

$$L_p^n = (1-\alpha)L_{p-1}^n - \alpha L_{p-1}^{n-1} + \alpha L_{l'}^{n-1}$$

As a consequence,

$$L_p^{n+1} = [(1-\alpha)^2 - \alpha]L_{p-1}^n - \alpha(1-\alpha)L_{p-1}^{n-1} + \alpha(1-\alpha)L_{l'}^{n-1} + \alpha L_l^n$$

Leading to

$$\|L_p^{n+1}\|_1 \leq [|(1-\alpha)^2 - \alpha| + \alpha(1-\alpha) + \alpha + \alpha(1-\alpha)] \max_{\substack{j\in[\![1;p-1]\!] \\ k\in\{n-1;n\}}} \|L_j^k\|_1$$

If $(1-\alpha)^2 - \alpha \geq 0$, *i.e.* $\alpha \leq \frac{3-\sqrt{5}}{2}$, (11) implies

$$\begin{aligned} \|L_p^{n+1}\|_1 &\leq (1-\alpha^2) \max_{\substack{j\in[\![1;p-1]\!] \\ k\in\{n-1;n\}}} \|L_j^k\|_1 \\ &\leq (1+\alpha)(1-\alpha^2)^{\lfloor \frac{n-2}{p+1} \rfloor + 1} \\ &\leq (1+\alpha)(1-\alpha^2)^{\lfloor \frac{n+1+p-2}{p+1} \rfloor} \end{aligned}$$

Hence proving (11) at rank $n+1$. As a result we deduce for all $n \geq 2$, (11), (12) and (13) hold.

The proof directly follows using

$$\forall n \in \mathbb{N}, \quad \|\Pi_n\|_\infty = \max_{i\in[\![1;p]\!]} \|L_i^n\|_1$$

Remark 2. In particular, one sees from (10) that the larger D_{max} is, the slower the guaranteed convergence is.

General case Based on this first result, we extend it to the case with small model error, by continuity. This last result shows that the proposed controller solves the control problem at stake, in the presence of both model mismatch and varying delay.

Corollary 4. (Small model errors). Under Assumption 1, consider the controller (5). For any $\alpha \leq \frac{3-\sqrt{5}}{2}$ there exists $\epsilon \in \mathbb{R}_+$ such that if $\|\rho - 1\|_\infty \leq \epsilon$, then the controller converges and $\lim\limits_{n\to+\infty} f(u_n) = c$.

Proof. According to Theorem 3, there exists $N_0 \in \mathbb{N}$ such that if there is no model error

$$M_{N_0,\infty} \leq \frac{1}{2}$$

With model error, every transition matrix of the dynamics A_n can be written under the additive form

$$A_n = A_n^0 + P_n$$

where A_n^0 is a matrix of the dynamics for $\rho = 1$

$$A_n^0 \in \{C + \alpha E_{D_{max}+2,k}, \quad k \in [\![1; D_{max}+1]\!]\}$$

and P_n is a perturbation matrix of general term p_{ij}^n

$$p_{ij}^n = -\alpha(\rho(x_n) - 1)\delta_{i,\Delta_{max}+2}\delta_{j,\Delta_{max}+1}$$

with x_n a given real number. Consider any N_0 sized collection of such matrices $(A_i)_{i\in[\![0;N_0-1]\!]}$, then

$$\| \prod_{i=0}^{N_0-1} A_i \|_\infty \leq \| \prod_{i=0}^{N_0-1} A_i^0 \|_\infty + \sum_{i=0}^{N_0-1} C_{N_0}^i (1+\alpha)^i \epsilon^{N_0-i}$$
$$\leq \frac{1}{2} + \sum_{i=0}^{N_0-1} C_{N_0}^i (1+\alpha)^i \epsilon^{N_0-i}$$

By upper-bounding the (finite) sum appearing in the right-hand side, it follows that there exists a sufficiently small value of ϵ such that for any $(A_i)_{i\in[\![0;N_0-1]\!]}$

$$\| \prod_{i=0}^{N_0-1} A_i \|_\infty \leq \frac{3}{4} < 1$$

Then, Proposition 1 yields the conclusion.

5. SIMULATION

Destabilization may arise without any model error on the function f, simply because of mis-synchronization between prediction and measurement. For this purpose, we consider a situation where $f_p \equiv f$ with small measurement errors to excite the system and

$$D(y) = D_p(y) + \delta D$$

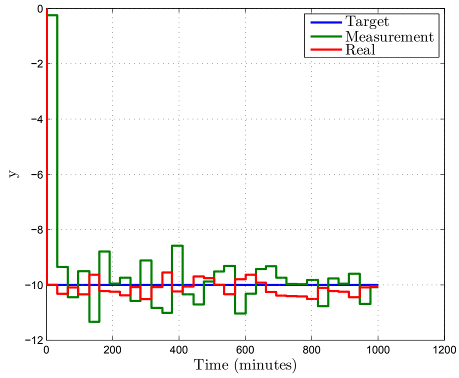
where δD is a stochastic term governed by a uniform law (D_p is simply given by an affine law with values ranging between 15 and 50 units of time for the values of y considered here). We simulate the system for different values of the filtering parameter α. The results of these simulations are given on Figure 5.

We also give an illustration of Corollary 4, by simulating the same system without measurement error but a model error (f_p and f being given respectively by the medium and non-linear scenarios of Figure 1, *i.e.* $\epsilon = 2$). The results of this second simulation batch are presented on Figure 6.

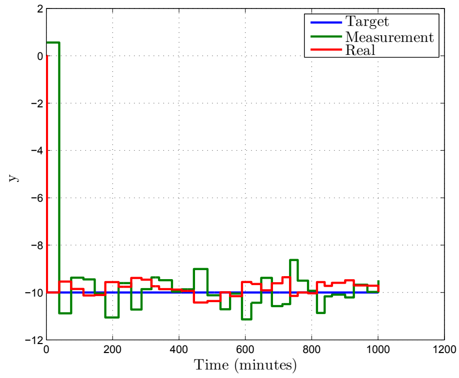
These simulations illustrate the merits of the theoretical results established in this article. A tuning of the controller gain following the (conservative) estimate provided by the small-gain condition gives satisfactory closed-loop responses even when the delay variability is not negligible and not perfectly known. If the gain is chosen above the threshold, some divergence (or strong oscillations) can be observed.

6. CONCLUSIONS

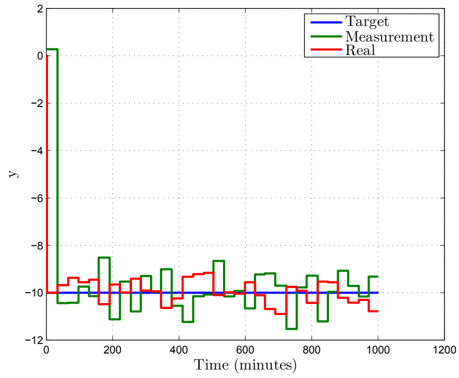
As a static SISO control problem, the core problem tackled in this paper appears, at first sight, as simple as it could be. However, the variability of the delay makes the problem particularly tricky. We have provided explicit robustness margins in regard of model error and asymptotic analysis on the consequences of imperfect timestamping. Indeed, while the situation of timestamping error is relatively frequent in real closed-loop control systems (see Petit [2015]), to the best of our knowledge, it as received limited theoretical attention since timestamping is usually implicitly assumed to be exact (especially in contributions studying the control of delayed systems such as Krstic



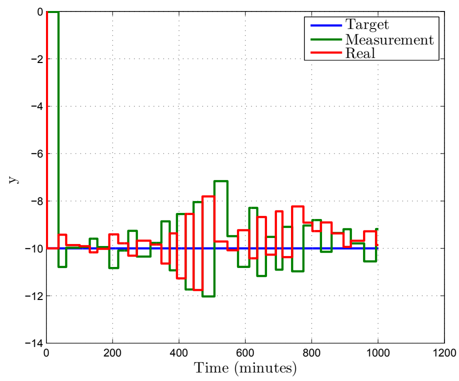
(a) Perfectly known delay, $\alpha \simeq 0.48$



(b) Delay known with error ($\pm$ 10 min), $\alpha \simeq 0.48$

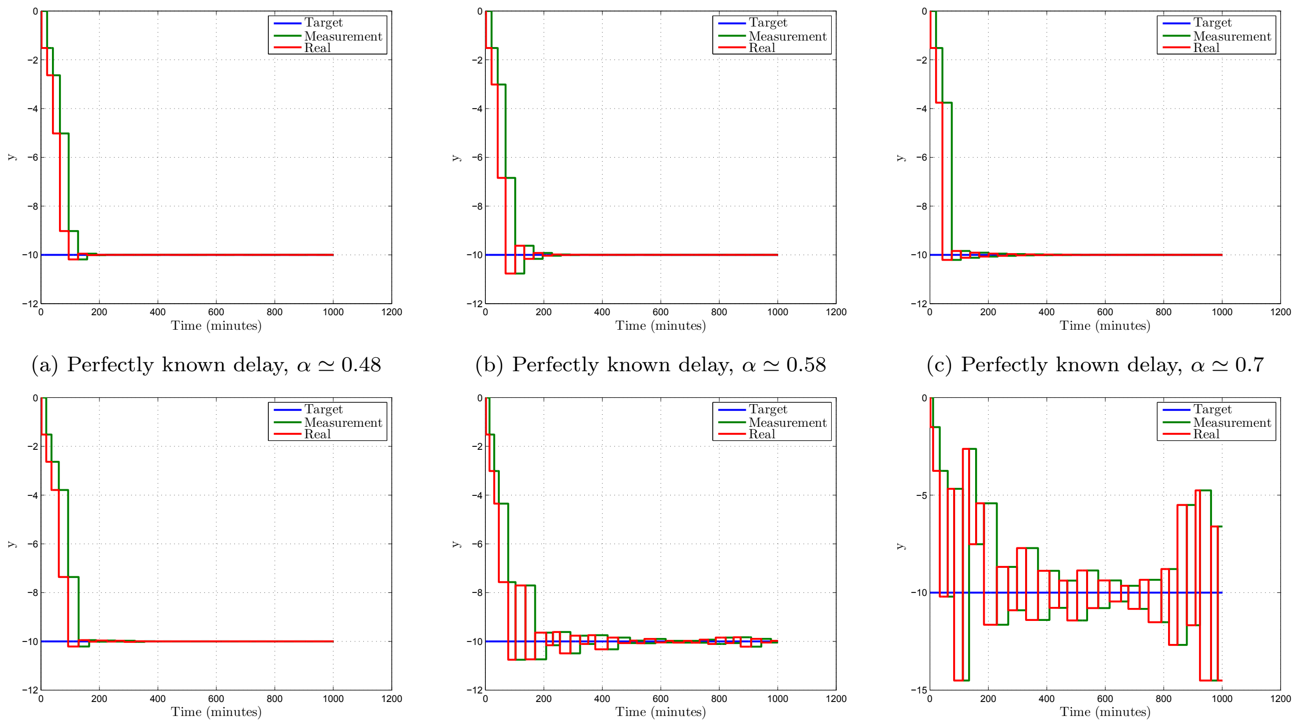


(c) Perfectly known delay, $\alpha \simeq 0.7$



(d) Delay known with error ($\pm$ 10 min), $\alpha \simeq 0.7$

Fig. 5. System behaviour without model error, with measurement error and with and without delay mismatch under different filtering parameters



(a) Perfectly known delay, $\alpha \simeq 0.48$ (b) Perfectly known delay, $\alpha \simeq 0.58$ (c) Perfectly known delay, $\alpha \simeq 0.7$

(d) Delay known with error ($\pm$ 10 units of time), $\alpha \simeq 0.48$ (e) Delay known with error ($\pm$ 10 units of time), $\alpha \simeq 0.58$ (f) Delay known with error ($\pm$ 10 units of time), $\alpha \simeq 0.7$

Fig. 6. System behaviour with model error, with and without delay mismatch under different filtering parameters

[2009], Niculescu [2001]). In the case where an underlying dynamical system should be considered to model the system, the preceding approach should be updated, significantly. Because the measurement will remain sampled by nature, the closed loop system will naturally become a sampled-data ordinary differential equation as considered in e.g. Fridman et al. [2004]. Also, it is known, see e.g. Cacace et al. [2014] that the introduction of time-varying gains may improve the exponential convergence, when measurements are subjected to (known) delays. If estimates of the delay are available, such tuning rules could bring some performance improvement. While the problem becomes significantly harder due to the time-varying nature of the discretized system transition matrices, it would be interesting to investigate whether, in a more general context of multi-input multi-output (MIMO) dynamical systems, an event-triggered discretization approach such as the one developed in this paper could be used to obtain results on the influence of timestamping uncertainty.

Appendix A. PROOF OF PROPOSITION 1

The proof is relatively straightforward

$$\forall n \in \mathbb{N}, \quad X_n = \prod_{i=1}^{n} A_{n-i} X_0$$

Hence, grouping terms in N_0-size bundles starting from the right

$$\|X_n\|_* \leq \| \prod_{i=1}^{n-\lfloor \frac{n}{N_0} \rfloor N_0} A_{n-i} \|_* \times \prod_{i=1}^{\lfloor \frac{n}{N_0} \rfloor} \| \prod_{j=1}^{N_0} A_{\lfloor \frac{n}{N_0} \rfloor N_0 - (i-1)N_0 - j} \|_* \|X_0\|_*$$

and

$$\|X_n\|_* \leq M_{n-\lfloor \frac{n}{N_0} \rfloor N_0, *} M_{N_0,*}^{\lfloor \frac{n}{N_0} \rfloor} \|X_0\|_*$$

Besides,

$$\forall n \in \mathbb{N}, \quad 0 \leq n - \left\lfloor \frac{n}{N_0} \right\rfloor < N_0$$

Hence, we get the desired result by defining

$$K \triangleq \max_{k \in [\![0; N_0 - 1]\!]} M_{k,*}$$

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