Motion planning for two classes of nonlinear systems with delays depending on the control

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Abstract

Two classes of nonlinear delayed controlled systems are considered: nonlinear hyperbolic equations (such as the Burgers equation without diffusion) and models of mixing processes with non negligible pipes holdups. Both can be seen as systems with delays depending on the control. As for flat systems, the trajectories of such systems can be explicitly parameterized. This is achieved by enlarging the set of allowed manipulations (classical algebraic computations and time derivations) by using compositions and inversions of functions. This provides an easy motion planning algorithm.

Keywords: nonlinear delay systems, hyperbolic partial differential equation, flatness, motion planning.

1 Introduction

Motion planning is an easy problem when the system is flat (see Fliess et al. [3]): the corresponding parameterization of all the trajectories of the system yields the solution. Such systems are numerous among nonlinear systems.

For linear systems with constant delays, module theory provides systematic tools that are closely related to flatness [7, 4], namely δ-freeness and π-freeness, for solving the motion planning problem. Several physical examples, such as torsion beams [5], antenna systems [8] and classical process control models [10] (see also [6]) show that this approach is relevant. In [9] the motion planning of a class of nonlinear chemical reactors with time-delays is solved by combining the classical algebraic manipulations and derivation (used in the flatness approach [3]) and advance (specific to the δ-freeness approach).

The goal of this paper is to indicate that adding composition and inversion of functions to these computation rules can be very useful for solving the motion planning problems. This assertion relies on two classes of systems admitting a clear physical meaning and engineering interest. These systems are shown to be controllable in the sense of Willems [12, 11]: the explicit parameterization of their trajectories provides here a simple and systematic way for constructing steering trajectories on $[0,T]$ for some $T>0$, from a prescribed past trajectory on $]-\infty,0]$ to another prescribed future trajectory on $[T,+\infty[$ in order to obtain, after concatenation, system trajectories defined on $]-\infty,+\infty[$.

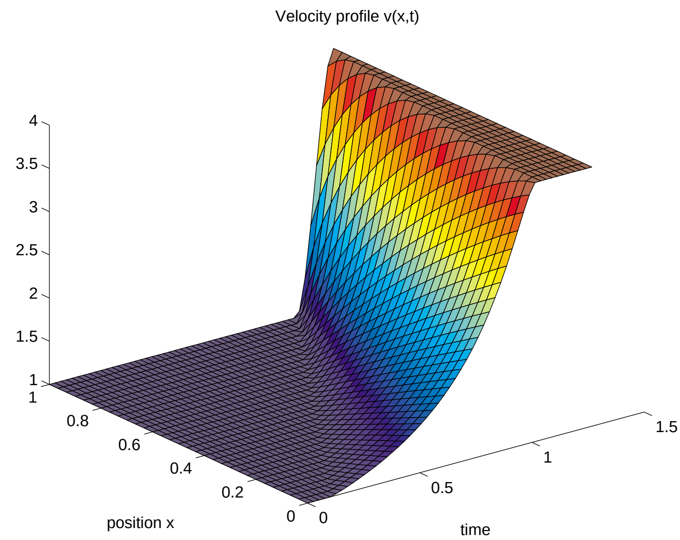

Figure 1: steering the Burgers equation from a low velocity $v\equiv 1$ to a high velocity $v\equiv 4$ without a compression shock wave.

2 Burgers equation

We consider here the classical Burgers equation without diffusion admitting the control $u>0$ on the boundary $x=0$:

$$\begin{aligned}\frac{dv}{dt} = v_t + v v_x = 0 \quad x\in[0,1]\\ v(0,t)=u(t).\end{aligned} \tag{1}$$

The field $x \mapsto v(x,t)$ describes the velocity field of a gas of particles without interaction (inertial motion) in a one dimensional tube $0 \leq x \leq 1$. Here the input velocity is the control u.

We are interested in the output velocity $y(t) = v(1,t)$ and its relationship with respect to the input u. Burgers equation says that the acceleration of each particle is zero: its velocity is constant. The particle which is in $x = 1$ at time t admits $y(t)$ as velocity. Thus at time $t - 1/y(t)$ the same particle was at $x = 0$ with velocity $u(t - 1/y(t))$. Since its velocity remains constant we have the nonlinear delay relation between u and y:

$$y(t) = u(t - 1/y(t)).$$

Symmetrically, we have also;

$$u(t) = y(t + 1/u(t)).$$

More generally the velocity field is related to the output velocity via

$$y(t) = v(t - (1-x)/y(t), x) \quad x \in [0,1]$$

and to the input velocity by

$$u(t) = v(t + x/u(t), x) \quad x \in [0,1].$$

Formally, we have a one to one correspondence between the trajectory profile $t \mapsto v(\blacksquare, t)$, solution of (1), and $t \mapsto y(t)$. This correspondence is effective as soon as $y > 0$ is continuously differentiable and $t \mapsto t - (1-x)/y(t)$ is increasing for each x, that is, for each t, $\dot{y}(t) > -y^2(t)$. This condition corresponds to smooth solutions of Burgers equation and avoids solution with shock waves. Notice that the transition from high to low velocity can be easily and smoothly achieved via any decreasing input $t \mapsto u(t)$ (see e.g., [1]).

This correspondence can be used, as for flat systems [3, 6] to generate control trajectories $t \mapsto u(t)$ steering the system from a low velocity profile $v(\blacksquare, 0) \equiv v_1 > 0$ to a high profile $v(\blacksquare, T) \equiv v_2 > v_1$ avoiding discontinuity in the transition profile due to compression shock waves [1].

Figure 1 shows the obtained profile trajectory for $v_1 = 1$, $v_2 = 4$, $y(t) = v_1 + (v_2 - v_1)s^2(2-3s)$ with $s = (t - 1.25)/0.25$ for $t \in [1.25, 1.5]$, $y(t) \equiv v_1$ for $t < 1.25$ and $y(t) \equiv v_2$ for $t > 1.5$.

Finally such a correspondence can be extended to any hyperbolic equation of the form

$$\begin{gathered} v_t + \lambda(v) v_x = 0 \quad x \in [0,1] \\ v(0,t) = u(t) \end{gathered}$$

since the relation between $y(t) = v(1,t)$ and u, v are (see, e.g., [2, page 41]):

$$y(t) = u[t - 1/\lambda(y(t))], \quad y(t) = v[t - (1-x)/\lambda(y(t)), x].$$

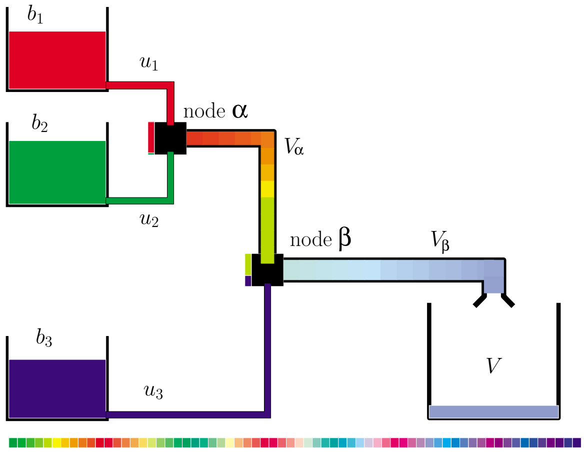

Figure 2: The color batch process: input-varying delays are due to non-negligible pipe holdups V_α and V_β.

3 Color batch process

Another basic example of a system where input-varying delays are to be taken into account can be described by figure 2. Using the outcome flows from three input tanks where pure colors are available, a desired color is to be achieved in output tank. Due to the non-negligible pipe holdups, delays appear which are dependent on the velocity in the pipes. Under the assumption of plug-flow in pipes α and β, we show that the trajectories of the system can be explicitly parameterized by the output tank holdups $t \mapsto \boldsymbol{Y} = (Y_1, Y_2, Y_3)$: the three controls $\boldsymbol{u} = (u_1, u_2, u_3)$ and the color profile in pipe α and β can be expressed via $\boldsymbol{Y}$ and its time derivatives $\dot{\boldsymbol{Y}}$.

3.1 Notations

Notations are partly explained on figure 2.

- a color (a composition) is a triplet (c_1, c_2, c_3) with $\forall i = 1,2,3$ $0 \leq c_i \leq 1$ and $c_1 + c_2 + c_3 = 1$.
- $\boldsymbol{b_i} = (\delta_{ij})^T_{j=1,2,3}$ corresponds to the fundamental color contained in tank i, $i = 1,2,3$.
- $\boldsymbol{u} = (u_1, u_2, u_3)^T$: outcome flows from input tanks (the inputs).
- $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0)^T$: color at node α.
- $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T$: color at node β.
- V: level of output tank.
- $\boldsymbol{X}$: color of the output tank.
- $\boldsymbol{Y} = (Y_1, Y_2, Y_3)^T = (V.X_1, V.X_2, V.X_3)^T$ the holdups in output tank. Y_1, Y_2, Y_3 are strictly increasing functions of time.
- V_α: holdup of pipe α.

- V_β: holdup of pipe β.

Notice that

$$\alpha_1+\alpha_2=1,\quad \beta_1+\beta_2+\beta_3=1,\quad X_1+X_2+X_3=1.$$

3.2 Balance equations

Plug-flow assumption in pipe α tells us that the color of the infinitesimal amount of fluid that reaches node β at time t is $\boldsymbol{\alpha}(\Gamma_\alpha)$ where the time $\Gamma_\alpha < t$ depends on t and is defined implicitly by:

$$V_\alpha = \int_{\Gamma_\alpha}^{t} (u_1(s)+u_2(s))\ ds. \tag{2}$$

Seemingly, the color of the infinitesimal amount of fluid that reaches the output tank at time t is $\boldsymbol{\beta}(\Gamma_\beta)$, where time $\Gamma_\beta < t$ is defined by:

$$V_\beta = \int_{\Gamma_\beta}^{t} (u_1+u_2+u_3)(s)\ ds. \tag{3}$$

Mixing is assumed instantaneous and linear in node α and β:

$$\boldsymbol{\alpha}(t) = \frac{u_1(t)\ \boldsymbol{b_1} + u_2(t)\ \boldsymbol{b_2}}{u_1(t)+u_2(t)}, \tag{4}$$

$$\boldsymbol{\beta}(t) = \frac{(u_1(t)+u_2(t))\ \ \boldsymbol{\alpha}(\Gamma_\alpha(t)) + u_3(t)\ \boldsymbol{b_3}}{u_1(t)+u_2(t)+u_3(t)}, \tag{5}$$

The dilution dynamics in the output tank yields:

$$\frac{d\boldsymbol{Y}}{dt} = \frac{d}{dt}(V\boldsymbol{X}) = (u_1+u_2+u_3)(t)\ \boldsymbol{\beta}(\Gamma_\beta(t)). \tag{6}$$

Since $\beta_1+\beta_2+\beta_3=1$, we deduce from (6) that

$$u_1(t)+u_2(t)+u_3(t) = \dot V(t).$$

Equations (2,3,4,5,6) describe the system dynamics: the relations between $\boldsymbol{u}$ and $\boldsymbol{Y}$ include delays depending nonlinearly on $\boldsymbol{u}$ via the implicit equations defining Γ_α and Γ_β. It also involves a differential equation (6), in opposition to the Burgers system.

3.3 Parameterization of the trajectories

In the following we show that one can write every quantity of the system in terms of $\boldsymbol{Y}$ and a finite number of its derivatives. More precisely, there is a one to one correspondence between $\boldsymbol{Y} = V\boldsymbol{X}$ and the set $(V, \boldsymbol{X}, \boldsymbol{\alpha}, \boldsymbol{\beta}, u_1, u_2, u_3)$ solution of (2,3,4,5,6). Each component of $\boldsymbol{Y}$ being a known increasing differentiable time function, let us compute the quantities $(V, \boldsymbol{X}, \boldsymbol{\alpha}, \boldsymbol{\beta}, u_1, u_2, u_3)$.

We have

$$V = Y_1+Y_2+Y_3. \tag{7}$$

Since $u_1(t)+u_2(t)+u_3(t)=\dot V(t)$, equation (3) reads:

$$V_\beta = V(t) - V(\Gamma_\beta(t)).$$

Since V is a strictly increasing function, we can invert it:

$$\Gamma_\beta = V^{-1}\circ(V-V_\beta). \tag{8}$$

Thus Γ_β is also a strictly increasing function and its inverse is given by:

$$\Gamma_\beta^{-1} = V^{-1}\circ(V+V_\beta).$$

Equation (6) writes:

$$\dot{\boldsymbol{Y}}(t) \quad = \quad \boldsymbol{\beta}(\Gamma_\beta(t))\ \dot V(t)$$

which gives $\boldsymbol{\beta}$ via:

$$\boldsymbol{\beta} = \left[\frac{\dot{\boldsymbol{Y}}}{\dot V}\right]\circ\Gamma_\beta^{-1}. \tag{9}$$

Since $\boldsymbol{\alpha}$ and $\boldsymbol{b_3}$ are orthogonal vectors, equation (5) implies that:

$$\beta_3 \quad = \quad \frac{u_3(t)}{u_1(t)+u_2(t)+u_3(t)} = \frac{u_3}{\dot V}.$$

This gives the control u_3:

$$u_3 = \dot V\ \beta_3, \tag{10}$$

thus

$$u_1+u_2 = \dot V\ (1-\beta_3) = \dot V\ (\beta_1+\beta_2).$$

Equation (2) defining implicitly Γ_α involves an integral that can be integrated explicitly. Since

$$(u_1+u_2)(t) = \dot V(t)\ (\beta_1+\beta_2)(t)$$

where

$$\begin{aligned}\boldsymbol{\beta}(t) &= \left[\frac{\dot{\boldsymbol{Y}}}{\dot V}\right]\circ\Gamma_\beta^{-1}(t)\\ &= \left[\frac{\dot{\boldsymbol{Y}}}{\dot V}\right]\circ V^{-1}\circ(V+V_\beta)(t).\end{aligned}$$

We have

$$(u_1+u_2)(t) = \dot V(t)\left(\left[\frac{\dot Y_1+\dot Y_2}{\dot V}\right]\circ V^{-1}\circ(V+V_\beta)(t)\right).$$

Thus the integrand in (2) can be expressed as (d denotes here the total differentiation operator)

$$\begin{aligned}(u_1(s)+u_2(s))\ ds &= \left(\left[\frac{\dot Y_1+\dot Y_2}{\dot V}\right]\circ V^{-1}\circ(V+V_\beta)\right)\ dV\\ &= \left(\left[\frac{\dot Y_1+\dot Y_2}{\dot V}\right]\circ V^{-1}\circ(V+V_\beta)\right)\ d(V+V_\beta).\end{aligned}$$

The right-hand side is of the form

$$\left(\left[\frac{df}{dg}\right] \circ g^{-1} \circ h\right) dh$$

since $df = \dot{f}\ dt$ and $dg = \dot{g}\ dt$. But

$$\left(\left[\frac{df}{dg}\right] \circ g^{-1} \circ h\right)\ dh = d(f \circ g^{-1} \circ h).$$

We finally obtain

$$(u_1 + u_2)\ ds = d\left((Y_1 + Y_2) \circ V^{-1} \circ (V + V_\beta)\right).$$

This allows to write (2) as:

$$\left[(Y_1 + Y_2) \circ V^{-1} \circ (V + V_\beta)\right]_{\Gamma_\alpha(t)}^{t} = V_\alpha$$
$$(Y_1 + Y_2) \circ \Gamma_\beta^{-1} = (Y_1 + Y_2) \circ \Gamma_\beta^{-1} \circ \Gamma_\alpha + V_\alpha,$$

and deduce that

$$\Gamma_\alpha = \Gamma_\beta \circ (Y_1 + Y_2)^{-1} \circ \Big((Y_1 + Y_2) \circ \Gamma_\beta^{-1} - V_\alpha\Big), \tag{11}$$

$$\Gamma_\alpha^{-1} = \Gamma_\beta \circ (Y_1 + Y_2)^{-1} \circ \Big((Y_1 + Y_2) \circ \Gamma_\beta^{-1} + V_\alpha\Big). \tag{12}$$

Projection of (5) onto b_1 gives:

$$\alpha_1 \circ \Gamma_\alpha = \frac{\dot{V}\ \beta_1}{\dot{V}(1 - \beta_3)} = \frac{\beta_1}{\beta_1 + \beta_2}.$$

Using (9), we get:

$$\alpha_1 \circ \Gamma_\alpha = \left[\frac{\dot{Y}_1}{\dot{Y}_1 + \dot{Y}_2}\right] \circ \Gamma_\beta^{-1}.$$

In the end:

$$\alpha_1 = \left[\frac{\dot{Y}_1}{\dot{Y}_1 + \dot{Y}_2}\right] \circ \Gamma_\beta^{-1} \circ \Gamma_\alpha^{-1} \tag{13}$$

and

$$\alpha_2 = 1 - \alpha_1. \tag{14}$$

We obtain the remaining controls u_1 and u_2 from

$$\alpha_1 = \frac{u_1}{u_1 + u_2}.$$

Then

$$u_1 = \alpha_1\ \dot{V}\ (1 - \beta_3). \tag{15}$$

Next, the projection of (4) onto b_2 gives:

$$u_2 = \alpha_2\ \dot{V}\ (1 - \beta_3). \tag{16}$$

At last we already know u_3 from (10).

Gathering all the formulae (7,8,9,10,12,13,14,15,16) we can write all the quantities of the system in terms of $\boldsymbol{Y}$.

3.4 Motion planning: scheduling several color batches.

Let us summarize, using time scaling, the relation between $\boldsymbol{Y}$ and $\boldsymbol{u}$. The function $t \mapsto \sigma(t) \mapsto \boldsymbol{Y}(\sigma(t))$ is given with $t \mapsto \sigma(t)$ an increasing differentiable function and $\sigma \mapsto Y_i(\sigma)$ positive, differentiable and increasing for each $i = 1, 2, 3$. The calculations of previous section leads to the following simple algorithm allowing to compute $\boldsymbol{u}(t)$ from $\boldsymbol{Y}$ ($'$ denotes $d/d\sigma$):

1. solve (via, e.g., Newton-like method) the scalar equation

   $$\sum_{i=1}^{3} Y_i(\sigma_\beta) = \sum_{i=1}^{3} Y_i(\sigma(t)) + V_\beta$$

   with σ_β corresponding to $\Gamma_\beta^{-1}(t)$ as unknown.

2. solve a second scalar equation

   $$Y_1(\sigma_\alpha) + Y_2(\sigma_\alpha) = Y_1(\sigma_\beta) + Y_2(\sigma_\beta) + V_\alpha$$

   with σ_α corresponding to $\Gamma_\alpha^{-1} \circ \Gamma_\beta^{-1}(t)$ as unknown.

3. Set

   $$\alpha_1(t) = \frac{Y_1'(\sigma_\alpha)}{Y_1'(\sigma_\alpha) + Y_2'(\sigma_\alpha)}, \quad \alpha_2(t) = 1 - \alpha_1(t)$$

   and

   $$\boldsymbol{\beta}(t) = \left[\frac{\boldsymbol{Y}'}{V'}\right](s_\beta)$$

   with $V = Y_1 + Y_2 + Y_3$.

4. Set

   $$u_i(t) = \alpha_i(t)\ (Y_1'(\sigma_\beta) + Y_2'(\sigma_\beta))\ V'(\sigma(t))\ \dot{\sigma}(t) \quad i = 1, 2$$

   and

   $$u_3(t) = V'(\sigma(t))\ \dot{\sigma}(t) - u_1(t) - u_2(t).$$

Assume that we have to produce $(a, b, c \dots)$, a series of colored mixtures in the output tank, defined by $Q^a = (Q_1^a, Q_2^a, Q_3^a)$, Q^b, Q^c, ... without flushing pipes α and β between each batch. We only need to define an increasing curve for each component of $\boldsymbol{Y}$, $t \mapsto \sigma \mapsto Y_i$ as displayed on figure 3 and where the part of the curve under Q_i^{init} (the initial quantity in pipes α and β of color i) is deduced from the initial profiles in pipes α and β. Notice also the design of $t \mapsto \sigma(t)$ with horizontal tangent at time t^a, t^b, t^c, ... in order to smoothly stop and start between each batch change where the content of output tank is flushed.

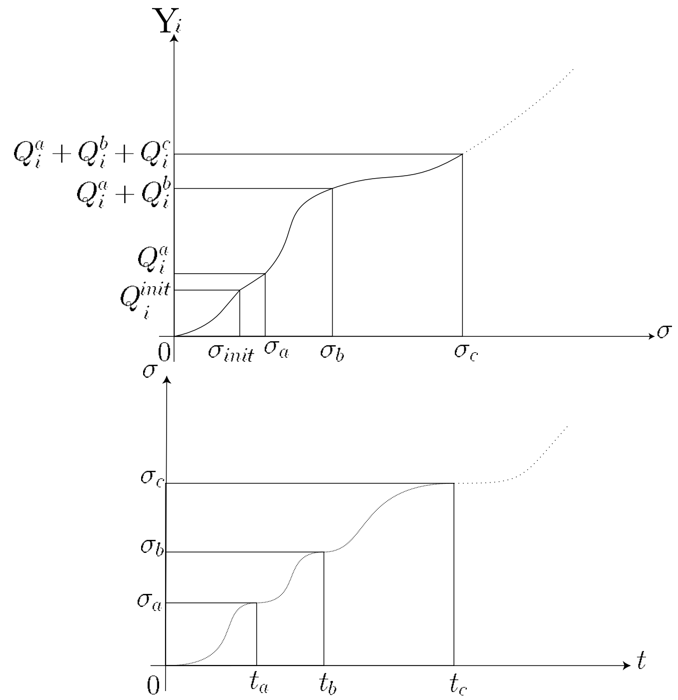

Figure 3: Color batch scheduling by defining a suitable "flat output" trajectory $t \mapsto Y(t)$.

4 Conclusion

Thanks to two physical examples, we have shown that composition and inversion of functions are very useful for solving the motion planning problem.

These examples belong to more general classes of systems for which the preceding methods give a similar answer. In facts the extension to more complex flowsheets of alike mixture processes is rather straightforward. The extension for hyperbolic equation in one dimensional space that can be solved by the classical method of characteristics is also easy. On the other hand, hyperbolic equations with several unknowns, such as the dynamics of a compressible polytropic gas (see [1]), may be much more difficult to handle.

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