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Optimization of dynamical systems with time-varying or input-varying delays

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Abstract— **We study the optimization of dynamical systems exhibiting variable time delays. We consider time-varying delays, and delays implicitly defined by input variables as they appear in systems involving fluid transport phenomena. We establish the necessary optimality conditions. Simulations results are presented.**

Key words: optimal control, variable time delays, fluid transport processes, process control

I. INTRODUCTION

Because delays have a well-established potential to decrease the performance of a closed-loop system and even deter its stability, they have been the subject of a rich literature in control theory. For fixed time delays, extensive work has been carried out to establish robustness criteria and to design adequate compensation strategies. For time-varying delays, robust stabilization techniques have been designed and studied. The reader can refer to [14], [17], [5], [4], [3], [9], [8] and references therein for numerous contributions covering cases of unknown or modeled variabilities.

In the field of dynamic optimization, which is the scope of our paper, delay systems are systems of importance, especially in process control applications [19]. For this reason, most commercial Model Predictive Control (MPC) tools routinely take into account fixed time delays, and implementations are common place in industrial applications. Practically, delays are usually treated directly in the time-discretization schemes. Formally, the numerical reformulations rely on the optimality conditions that have been investigated from early on by the control system community, see [13], [16], [10], [2], [22]. These works cover cases of multiple input and state delays in Pontryagin's maximum principle. Besides MPC techniques, other approaches have focused on optimal synthesis, for fixed delays, resulting in feedback control laws. A detailed panorama can be found in [11], [12] which also propose numerical methods for implementation.

Interestingly enough, it appears that only little attention has been given to dynamic optimization problems under varying delays. Since the seminal work of [1], most research efforts have focused on closed-form solutions to LQR problems for dynamics impacted by time-varying delays, see [7]. In facts, in most applications where delays are *a priori* known to be variable, this information is simply ignored. The delays are assumed to be fixed (e.g. set to an expected value) while the controller is left to deal with some level of unstructured uncertainty and is tuned so that it features sufficient robustness with regard to delay errors. Some adaptive control schemes, including those mentioned earlier in this discussion, can be implemented to improve this robustness or to compensate for the variability of the delay. Nevertheless, even when these strategies are successful at providing asymptotic stability (with zero asymptotic error), the performance is sub-optimal because an undesirable transient error is resulting from the purposely created lack of accuracy of the delay model. This sub-optimality can be particularly costly in certain applications where transients represent a dominant part of operating conditions. A typical class of situations where such variable input delays exist encompasses systems where the delay is related to fluid transport as discussed in [15], [20] or [25]. These examples feature "hydraulic delays". In many cases, these delays can be modelled relatively effortlessly.

This paper addresses optimization problems for systems with time-varying delays or with hydraulic delays. We take this variability into account, be it explicit as a time function or implicit in terms of the control variable, and we establish necessary (first-order) optimality conditions. These conditions are the main contribution of the paper.

The paper is organized as follows. In Section II, we define hydraulic delays. In Section III, two illustrative practical examples are presented. We treat problems of calculus of variations in Section IV, and optimal control problems in Section V. We detail the complexities created by time-varying delays and hydraulic delays in the stationarity conditions. As the reader will note it, the calculations performed in these two sections have similarities with those of [10] or [24] for fixed time delays. Simulation results illustrating our theoretical results are given in Section VI. Some conclusions and perspectives are given in Section VII.

II. NOTATIONS AND PRELIMINARY RESULT

In this article, a delay $t \mapsto D(t)$ is a smooth (scalar valued) positive function. For any delay law $t \mapsto D(t)$, we note

$$r : t \mapsto t - D(t).$$

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It is assumed that $\dot{D}(t) < 1$, so that r is increasing and bijective.

Given a control input $\mathbb{R} \ni t \mapsto u(t) \in \mathbb{R}^n$, we call "hydraulic delay" any delay law $(t, u(.)) \mapsto D(t, u(.))$ defined by a relation of the type

$$\int_{t-D(t,u(.))}^{t} \phi(u(\tau))\, \mathrm{d}\tau = 1, \qquad (1)$$

where ϕ is any given positive, bounded away from zero, smooth function. For the sake of convenience, we will simply use the notation $D(t, u)$ instead of $D(t, u(.))$. Similarly, we note

$$r_u : t \mapsto t - D(t, u).$$

Under the preceding assumptions on ϕ in (1), r_u is necessarily increasing and bijective [6]. This class of delays is practically important in describing hydraulic transport phenomenons, as outlined in examples of Section III.

In the optimal control problems under consideration, we note $t_0, T > 0$, $x_0, a, b \in \mathbb{R}^n$, some fixed parameters, and L and ψ are smooth scalar valued functions.

$\mathbb{1}_{[a,b]} \rightarrow \{0, 1\}$ denotes the indicator function of the interval $[a, b]$.

For any $f : \mathbb{R}^{n_1} \times ... \times R^{n_m} \rightarrow \mathbb{R}$ a smooth function, we note

$$\partial_i f = (\frac{\partial f}{\partial x_{\sum_{j=1}^{i-1} n_j + 1}}, ..., \frac{\partial f}{\partial x_{\sum_{j=1}^{i} n_j}}),$$

the partial derivation of f with respect to the n_i variables of the i^{th} subset of its arguments.

For $(u, h) \in C^2(\Omega, \mathbb{R}^n)^2$ where Ω is an open subinterval of $\mathbb{R}$ and $J : C^2(\Omega, \mathbb{R}^n) \rightarrow \mathbb{R}$ a functional, we consider the Gâteaux derivative [21] of J in the direction h at u as

$$D_h J(u) \triangleq \lim_{\delta \to 0} \frac{J(u + \delta h) - J(u)}{\delta}.$$

With these notations, one has

Proposition 1 (Sensitivity of hydraulic delay with respect to input variation) For all $t \in r_u^{-1}(\Omega) \cap \Omega$, the Gâteaux derivative w.r.t. u of the hydraulic delay is

$$\begin{aligned} D_h D(t, u) &\triangleq \lim_{\delta \to 0} \frac{D(t, u + \delta h) - D(t, u)}{\delta} \\ &= -\frac{1}{\phi(u(r_u(t)))} \int_{r_u(t)}^{t} \frac{\partial \phi}{\partial u}(u(\tau)) h(\tau)\, \mathrm{d}\tau. \end{aligned}$$

Proof: see Appendix. ■

III. Examples of systems with hydraulic delays

A. First example: a mixing unit with pre-blend, the "paint" problem

A very simple system illustrating the effects of hydraulic delays is described in Figure 1 that represents a mixing unit with pre-blend. This example can be found in many process applications, e.g. [23]. One can describe the control objective as follows, see [18]. Three batches of paint of different colours (inputs) are to be mixed to provide a product of a desired colour (output). To minimize pipe lengths, the outlet of the batch 1 and 2 are first blended, then go through a transport pipe having a dead volume V before being finally mixed with the product 3. This color of the final product is the output of the unit. The control variables are the flow-rates out of each batch $(u_i)_{i \in [\![1;3]\!]}$. It is desired to control the instantaneous recipe of the output product, y_{out}. Balance equations give

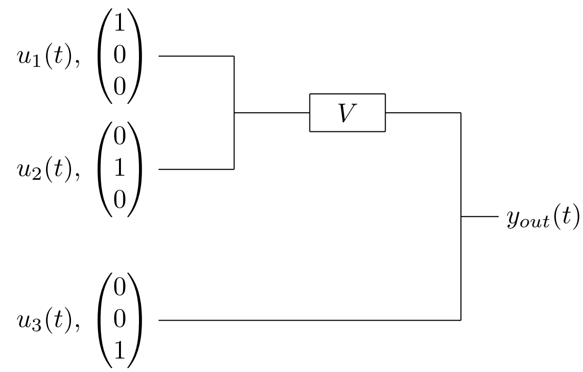


Fig. 1: Three batch mixing unit with a dead volume V: the "paint" problem

$$y_{out}(t) = \begin{pmatrix} \frac{u_1(t-D(t,u))}{u_1(t-D(t,u))+u_2(t-D(t,u))} \frac{u_1(t)+u_2(t)}{u_1(t)+u_2(t)+u_3(t)} \\ \frac{u_2(t-D(t,u))}{u_1(t-D(t,u))+u_2(t-D(t,u))} \frac{u_1(t)+u_2(t)}{u_1(t)+u_2(t)+u_3(t)} \\ \frac{u_3(t)}{u_1(t)+u_2(t)+u_3(t)} \end{pmatrix}, \qquad (2)$$

where

$$\frac{1}{V} \int_{t-D(t,u)}^{t} u_1(\tau) + u_2(\tau)\, \mathrm{d}\tau = 1. \qquad (3)$$

Despite its simplicity, the implicit integral-type relation between D and the control makes both open-loop motion planning and tracking problems relatively difficult.

B. Second example: control of a water heating process

Another example has been presented in [15] and used as an experimental test case for the study of a non-linear MPC-based approach to the control of systems with variable time delays. The process is pictured in Figure 2. It consists of a heated tank, the level of which is controlled so that the water hold-up remains constant. A submerged electrical heater delivers a constant heat flux to warm up the liquid. The control objective is the temperature at the outlet T_{out} and the input is the flow-rate of water through the tank, q.

As shown in [15], the temperature in the tank $T_{tank}(t)$ satisfies the following balance equation

$$\rho c_p V \frac{dT_{tank}(t)}{dt} = Q + \rho c_p q(t)(T_{in}(t) - T_{tank}(t)),$$

where ρ, c_p and Q are the density of water, its specific heat and the power of supplied heat, respectively. Neglecting the heat losses in the pipe and the mixing time, we have

$$T_{out}(t) = T_{tank}(t - \delta(t, q)),$$

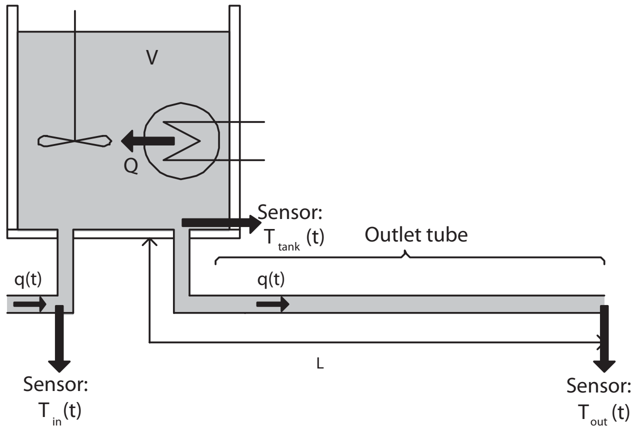


Fig. 2: Schematic of the water heating process from [15]

with

$$\int_{t-\delta(t,q)}^{t} q(\tau)\,\mathrm{d}\tau = LS.$$

where L and S are the length and cross-section of the pipe. After normalization and changes of state variables, this system is equivalent to

$$\dot{T}_i(t) = 1 - u(t)T_i(t), \quad x(t) = T_i(t - D(t,u)),$$

and

$$\int_{t-D(t,u)}^{t} u(\tau)\,\mathrm{d}\tau = 1,$$

where x is the variable to control, T_i the internal state of the system and u the control. This can equivalently be rewritten as

$$\begin{aligned}\dot{x}(t) &= \frac{u(t)}{u(t-D(t))}(1 - u(t-D(t,u))x(t)) \\ &\triangleq f(x(t), u(t), u(t-D(t,u))).\end{aligned}$$

IV. Calculus of Variations

In this section, we study the calculus of variations for time-delayed systems where the time delay either is a (known) time-varying function or is a hydraulic delay. These studies are closely connected to the optimal control problems addressed in Section V.

A. Calculus of variations with time-varying delay

Here, D is a fixed function of time (it does not depend on u in any way). As will appear, this variation creates new terms in the calculus of variations. For conciseness of notations, we define

$$[u,\dot{u}]_D(t) \triangleq (t, u(t), u(t-D(t)), \dot{u}(t), \dot{u}(t-D(t))).$$

Consider the following optimization problem
Problem 1:

$$\min_{\substack{u\in C^2([r(t_0);t_0+T],\mathbb{R})\\ u(t_0)=a, u(t_0+T)=b}} J(u) = \int_{t_0}^{t_0+T} L([u,\dot{u}]_D(t))\,\mathrm{d}t. \quad (4)$$

We wish to establish necessary stationarity conditions for Problem 1. Since the control is fixed in the past $t < t_0$, any admissible variation h is such that

$$\forall t \le t_0, \quad h(t) = 0, \quad \text{and} \quad h(t_0+T) = 0.$$

As a consequence,

$$\begin{aligned}D_h J(u) = \int_{t_0}^{t_0+T} &\partial_2 L([u,\dot{u}]_D(t))h(t) \\ &+ \partial_4 L([u,\dot{u}]_D(t))\dot{h}(t) \\ &+ \partial_3 L([u,\dot{u}]_D(t))h(t-D(t)) \\ &+ \partial_5 L([u,\dot{u}]_D(t))\dot{h}(t-D(t))\,\mathrm{d}t.\end{aligned}$$

Consider $t_k = t_{kick-in}$ the uniquely defined time instant such that

$$r(t_0+t_k) = t_0 + t_k - D(t_0+t_k) \triangleq t_0,$$

then, the expression of the Gâteaux derivative $D_h J(u)$ becomes

$$\begin{aligned}&\int_{t_0}^{t_0+T} \partial_2 L([u,\dot{u}]_D(t))h(t) + \partial_4 L([u,\dot{u}]_D(t))\dot{h}(t)\,\mathrm{d}t \\ &+ \int_{t_0+t_k}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))h(t-D(t)) \\ &\qquad + \partial_5 L([u,\dot{u}]_D(t))\dot{h}(t-D(t))\,\mathrm{d}t.\end{aligned}$$

The first integral is well-known and will bring the usual terms of the Euler-Lagrange equations. To deal with the second one, we use the change of variable $t = r^{-1}(\tau)$. The new expression of $D_h J(u)$ is

$$\begin{aligned}&\int_{t_0}^{t_0+T} (\partial_2 L([u,\dot{u}]_D(t)) - \frac{d}{dt}\partial_4 L([u,\dot{u}]_D(t)))h(t)\,\mathrm{d}t \\ &\quad + \int_{t_0}^{r(t_0+T)} \partial_3 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t)h(t) \\ &\qquad + \partial_5 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t)\dot{h}(t)\,\mathrm{d}t,\end{aligned}$$

and, after integrations by parts

$$\begin{aligned}&\int_{t_0}^{t_0+T} (\partial_2 L([u,\dot{u}]_D(t)) - \frac{d}{dt}\partial_4 L([u,\dot{u}]_D(t)))h(t)\,\mathrm{d}t \\ &+ \int_{t_0}^{r(t_0+T)} \partial_3 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t)h(t) \\ &- \frac{d}{dt}\partial_5 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t)^2 h(t) \\ &- \partial_5 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})''(t)h(t)\,\mathrm{d}t \\ &+ \partial_5 L([u,\dot{u}]_D(t_0+T))(r^{-1})'(r(t_0+T))h(r(t_0+T)).\end{aligned}$$

Invoking Du Bois-Reymond lemma, we deduce the optimality conditions (5)-(6) of Problem 1:

$$\begin{aligned}&\partial_2 L([u,\dot{u}]_D(t)) - \frac{d}{dt}\partial_4 L([u,\dot{u}]_D(t)) \\ &\quad + \mathbb{1}_{[t_0;r(t_0+T)]}(t)\cdot\Big(\partial_3 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t) \\ &\qquad - \frac{d}{dt}\partial_5 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})'(t)^2 \\ &\qquad - \partial_5 L([u,\dot{u}]_D(r^{-1}(t)))(r^{-1})''(t)\Big) = 0, \quad (5)\end{aligned}$$

and, because $r^{-1}(t) \neq 0$ by assumption

$$\partial_5 L([u,\dot{u}]_D(t_0+T)) = 0. \quad (6)$$

B. Calculus of variations with hydraulic delay

Let us now consider that the system has a hydraulic delay as defined by (1). For ease of notation, we define a similar vector

$$[u,\dot{u}]_{D_u}(t) = (t, u(t), u(t-D(t,u)), \dot{u}(t), \dot{u}(t-D(t,u))).$$

Problem 2: Consider functions $u \in C^2(\Omega_u)$, where Ω_u is an open set characterised by $[r_u(t_0); t_0+T] \subset \Omega_u$, solve the following problem

$$\min_{\substack{u\in C^2(\Omega_u,\mathbb{R})\\ u(t_0)=a, u(t_0+T)=b}} J(u) = \int_{t_0}^{t_0+T} L([u,\dot{u}]_{D_u}(t))\,\mathrm{d}t. \quad (7)$$

For conciseness, in the following development we will simply use the notation $[u,\dot{u}]_D$ instead of $[u,\dot{u}]_{D_u}$.

Since u is fixed in the past $t < t_0$ and noting as before $t_k(u) = t_{kick-in}(u)$, with

$$r_u(t_0+t_k(u)) = t_0 + t_k(u) - D(t_0+t_k(u)) \triangleq t_0,$$

for any admissible variation $h \in C^2(\Omega_u, \mathbb{R})$

$$\begin{aligned} &D_h J(u) = \\ &\int_{t_0}^{t_0+T} \partial_2 L([u,\dot{u}]_D(t))h(t) + \partial_4 L([u,\dot{u}]_D(t))\dot{h}(t)\,\mathrm{d}t \\ &+ \int_{t_0+t_k}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))h(t-D(t)) \\ &\qquad + \partial_5 L([u,\dot{u}]_D(t))\dot{h}(t-D(t))\,\mathrm{d}t \\ &- \int_{t_0}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))\dot{u}(t-D(t,u)) \cdot D_h D(t,u)\,\mathrm{d}t \\ &- \int_{t_0}^{t_0+T} \partial_5 L([u,\dot{u}]_D(t))\ddot{u}(t-D(t,u)) \cdot D_h D(t,u)\,\mathrm{d}t. \end{aligned}$$

With Proposition 1, it follows that $D_h J(u)$ can be expressed as

$$\begin{aligned} &\int_{t_0}^{t_0+T} \partial_2 L([u,\dot{u}]_D(t))h(t) + \partial_4 L([u,\dot{u}]_D(t))\dot{h}(t)\,\mathrm{d}t \\ &+ \int_{t_0+t_k}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))h(t-D(t)) \\ &\qquad + \partial_5 L([u,\dot{u}]_D(t))\dot{h}(t-D(t))\,\mathrm{d}t \\ &+ \int_{t_0}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))\dot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &+ \int_{t_0}^{t_0+T} \partial_5 L([u,\dot{u}]_D(t))\ddot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t. \end{aligned}$$

Finally, denoting

$$\begin{aligned} A = &\int_{t_0}^{t_0+t_k(u)} \partial_3 L([u,\dot{u}]_D(t))\dot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t_0}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} B = &\int_{t_0+t_k(u)}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))\dot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} C = &\int_{t_0}^{t_0+t_k(u)} \partial_5 L([u,\dot{u}]_D(t))\ddot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t_0}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} D = &\int_{t_0+t_k(u)}^{t_0+T} \partial_5 L([u,\dot{u}]_D(t))\ddot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))} \int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\,\mathrm{d}t, \end{aligned}$$

we get

$$\begin{aligned} &D_h J(u) = A + B + C + D \\ &+ \int_{t_0}^{t_0+T} \partial_2 L([u,\dot{u}]_D(t))h(t) + \partial_4 L([u,\dot{u}]_D(t))\dot{h}(t)\,\mathrm{d}t \\ &+ \int_{t_0+t_k}^{t_0+T} \partial_3 L([u,\dot{u}]_D(t))h(t-D(t)) \\ &\qquad + \partial_5 L([u,\dot{u}]_D(t))\dot{h}(t-D(t))\,\mathrm{d}t. \end{aligned}$$

Using Fubini's theorem to reorder the striped integration domains, we get

$$\begin{aligned} A = &\int_{t_0}^{t_0+t_k(u)} \int_{\tau}^{t_0+t_k(u)} \partial_3 L([u,\dot{u}]_D(t))\dot{u}(t-D(t,u)) \\ &\qquad \frac{1}{\phi(u(t-D(t,u)))}\,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau, \end{aligned}$$

and

$$\begin{aligned} B = &\int_{t_0}^{t_0+t_k(u)} \int_{t_0+t_k(u)}^{\min(r_u^{-1}(\tau), t_0+T)} \partial_3 L([u,\dot{u}]_D(t)) \\ &\dot{u}(t-D(t,u)) \frac{1}{\phi(u(t-D(t,u)))}\,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \\ &+ \int_{t_0+t_k(u)}^{t_0+T} \int_{\tau}^{\min(r_u^{-1}(\tau), t_0+T)} \partial_3 L([u,\dot{u}]_D(t)) \\ &\frac{\partial u}{\partial t}(t-D(t,u)) \frac{1}{\phi(u(t-D(t,u)))}\,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau, \end{aligned}$$

As a consequence, after some grouping of terms

$$\begin{aligned} A + B = &\int_{t_0}^{t_0+T} \int_{\tau}^{\min(r_u^{-1}(\tau), t_0+T)} \partial_3 L([u,\dot{u}]_D(t)) \\ &\dot{u}(t-D(t,u)) \frac{1}{\phi(u(t-D(t,u)))}\,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau, \end{aligned}$$

and, similarly

$$C + D = \int_{t_0}^{t_0+T} \int_{\tau}^{\min(r_u^{-1}(\tau), t_0+T)} \partial_5 L([u, \dot{u}]_D(t))$$
$$\ddot{u}(t - D(t,u)) \frac{1}{\phi(u(t - D(t,u)))} \,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau)) h(\tau) \,\mathrm{d}\tau,$$

Invoking Du Bois-Reymond lemma, the stationarity conditions (8)-(9) of Problem 2 can finally be derived as

$$\begin{aligned}
&\partial_2 L([u,\dot{u}]_D(t)) - \frac{d}{dt}\partial_4 L([u,\dot{u}]_D(t)) \\
&\quad + \mathbb{1}_{[t_0; r_u(t_0+T)]}(t) \cdot [\partial_3 L([u,\dot{u}]_D(r_u^{-1}(t)))(r_u^{-1})'(t) \\
&\qquad - \frac{d}{dt}\partial_5 L([u,\dot{u}]_D(r_u^{-1}(t)))(r_u^{-1})'(t)^2 \\
&\qquad - \partial_5 L([u,\dot{u}]_D(r_u^{-1}(t)))(r_u^{-1})''(t)] \\
&\quad + \int_t^{\min(r_u^{-1}(t), t_0+T)} \partial_3 L([u,\dot{u}]_D(\tau)) \dot{u}(\tau - D(\tau,u)) \\
&\qquad \frac{1}{\phi(u(\tau - D(\tau,u)))} \,\mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)) \\
&\quad + \int_t^{\min(r_u^{-1}(t), t_0+T)} \partial_5 L([u,\dot{u}]_D(\tau)) \ddot{u}(\tau - D(\tau,u)) \\
&\qquad \frac{1}{\phi(u(\tau - D(\tau,u)))} \,\mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)) = 0, \qquad (8)
\end{aligned}$$

and, because $r_u^{-1}(t) \neq 0$

$$\partial_5 L([u,\dot{u}]_D(t_0+T)) = 0. \qquad (9)$$

V. Optimal control

Having introduced new terms in the stationarity conditions of the calculus of variations of the preceding systems, we are now ready to address the problems of optimal control. Again, we successively treat the case of time-varying delays and the case of hydraulic delay appearing in the right-hand side of differential equations, by order of complexity.

A. *Optimal control with time-varying delay*

For now, D is a fixed function of time. Very generally, we define the objective function

$$J(x,u) = \psi(x(t_0+T)) + \int_{t_0}^{t_0+T} L(t, x(t), u(t)) \,\mathrm{d}t. \qquad (10)$$

We now consider the following optimal control problem

Problem 3:

$$\begin{aligned}
&\min_{x,u} J(x,u) \\
&\quad s.t. \quad \dot{x} = f(t, x(t), u(t - D(t))) \cdot \\
&\qquad\quad x(t_0) = x_0
\end{aligned} \qquad (11)$$

After adjoining the constraints, we obtain

$$\begin{aligned}
\min_{\substack{x,u,\lambda \\ x(t_0)=x_0}} \bar{J}(x,u) = \psi(x(t_0+T)) + \int_{t_0}^{t_0+T} & L(t, x(t), u(t)) \\
& + \lambda(t)^T [f(t, x(t), u(t - D(t))) - \dot{x}] \,\mathrm{d}t.
\end{aligned}$$

The Gâteaux derivative w.r.t. the control variable u is

$$\begin{aligned}
D_h \bar{J}(x,u) = \int_{t_0}^{t_0+T} & \frac{\partial L}{\partial u}(t, x(t), u(t)) h(t) \\
& + \lambda(t)^T \cdot \frac{\partial f}{\partial u}(t, x(t), u(t - D(t))) h(t - D(t)) \,\mathrm{d}t,
\end{aligned}$$

hence

$$\begin{aligned}
D_h \bar{J}(x,u) = &\int_{t_0}^{t_0+T} \frac{\partial L}{\partial u}(t, x(t), u(t)) h(t) \,\mathrm{d}t \\
&+ \int_{t_0}^{r(t_0+T)} \lambda(r^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r^{-1}(t), x(r^{-1}(t)), u(t)) \\
&\qquad (r^{-1})'(t) h(t) \,\mathrm{d}t,
\end{aligned}$$

finally, Du Bois-Reymond lemma gives

$$\begin{aligned}
&\frac{\partial L}{\partial u}(t, x(t), u(t)) \,\mathrm{d}t + \mathbb{1}_{[t_0; r(t_0+T)]}(t) \cdot \\
&\lambda(r^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r^{-1}(t), x(r^{-1}(t)), u(t))(r^{-1})'(t) = 0.
\end{aligned}$$

This condition is analogous to the one of the classical two-point boundary problem $\frac{\partial H}{\partial u} \triangleq L + \lambda^T f = 0$, where H is the Hamiltonian of the system. Calculating the Gâteaux derivatives with respect to x and λ, we classically get the following two-point boundary value problem (TPBVP) which represents stationarity conditions for Problem 3.

$$\begin{cases}
\dot{x}(t) = f(x, u(t - D(t))) \\
\dot{\lambda}(t)^T = -\frac{\partial L}{\partial x}(t, x(t), u(t)) \\
\qquad + \lambda(t)^T \frac{\partial f}{\partial x}(t, x(t), u(t - D(t))) \\
\lambda(t_0+T)^T = \frac{\partial}{\partial x(t_0+T)} \psi(x(t_0+T)) \\
\frac{\partial L}{\partial u}(t, x(t), u(t)) \,\mathrm{d}t + \mathbb{1}_{[t_0; r(t_0+T)]}(t) \cdot \\
\quad \lambda(r^{-1}(t))^T \frac{\partial f}{\partial u}(r^{-1}(t), x(r^{-1}(t)), u(t))(r^{-1})'(t) = 0
\end{cases} \qquad (12)$$

B. *Optimal control with hydraulic delay*

Let us now consider that the system has a hydraulic delay as defined by (1) impacting the differential equation. The cost function (10) is unchanged. Then, the optimal control problem under consideration is

Problem 4:

$$\begin{aligned}
&\min_{x,u} J(x,u) \\
&\quad s.t. \quad \dot{x} = f(t, x(t), u(t - D(t,u))) \cdot \\
&\qquad\quad x(t_0) = x_0
\end{aligned} \qquad (13)$$

Adjoining the constraints gives

$$\begin{aligned}
\min_{\substack{x,u,\lambda \\ x(t_0)=x_0}} \bar{J}(x,u) = \psi(x(t_0+T)) + \int_{t_0}^{t_0+T} & L(t, x(t), u(t)) \\
& + \lambda(t)^T [f(x(t), u(t - D(t,u))) - \dot{x}] \,\mathrm{d}t.
\end{aligned}$$

With respect to the control variable u, we have

$$D_h\bar{J}(u) = \int_{t_0}^{t_0+T} \frac{\partial L}{\partial u}(t, x(t), u(t))h(t)\,\mathrm{d}t + \int_{t_0}^{r_u(t_0+T)} \lambda(r_u^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r_u^{-1}(t), x(r_u^{-1}(t)), u(t)) (r_u^{-1})'(t)h(t)\,\mathrm{d}t + \int_{t_0}^{t_0+T} \lambda(t)^T \cdot \frac{\partial f}{\partial u}(t, x(t), u(t-D(t,u))\dot{u}(t-D(t,u)) \frac{1}{\phi(u(t-D(t,u)))}\int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\ \mathrm{d}t.$$

Again, using Fubini's theorem as in the previous section, we get

$$D_h\bar{J}(u) = \int_{t_0}^{t_0+T} \frac{\partial L}{\partial u}(t, x(t), u(t))h(t)\,\mathrm{d}t + \int_{t_0}^{r_u(t_0+T)} \lambda(r_u^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r_u^{-1}(t), x(r_u^{-1}(t)), u(t)) (r_u^{-1})'(t)h(t)\,\mathrm{d}t + \int_{t_0}^{t_0+T} \int_{\tau}^{\min(r_u^{-1}(\tau), t_0+T)} \lambda(t)^T \cdot \frac{\partial f}{\partial u}(t, x(t), u(t-D(t,u)) \dot{u}(t-D(t,u))\frac{1}{\phi(u(t-D(t,u)))}\,\mathrm{d}t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau.$$

Finally, Dubois-Reymond lemma gives

$$\frac{\partial L}{\partial u}(t, x(t), u(t))\,\mathrm{d}t + \mathbb{1}_{[t_0; r(t_0+T)]}(t). \lambda(r^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r^{-1}(t), x(r^{-1}(t)), u(t))(r^{-1})'(t) + \int_{t}^{\min(r_u^{-1}(t), t_0+T)} \lambda(\tau)^T \cdot \frac{\partial f}{\partial u}(\tau, x(\tau), u(\tau - D(\tau,u)) \dot{u}(\tau - D(\tau,u))\frac{1}{\phi(u(\tau - D(\tau,u)))}\,\mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)) = 0.$$

Calculating the Gâteaux derivatives with respect to x and λ, we classically get the (TPBVP) which formulates stationarity conditions for Problem 4

$$\begin{cases} \dot{x}(t) = f(x, u(t-D(t,u))) \\ \dot{\lambda}(t)^T = -\frac{\partial L}{\partial x}(t, x(t), u(t)) \\ \qquad + \lambda(t)^T \frac{\partial f}{\partial x}(t, x(t), u(t-D(t,u))) \\ \lambda(t_0+T)^T = \frac{\partial}{\partial x(t_0+T)}\psi(x(t_0+T)) \\ \frac{\partial L}{\partial u}(t, x(t), u(t))\,\mathrm{d}t + \mathbb{1}_{[t_0; r_u(t_0+T)]}(t). \\ \quad \lambda(r_u^{-1}(t))^T \cdot \frac{\partial f}{\partial u}(r_u^{-1}(t), x(r_u^{-1}(t)), u(t))(r_u^{-1})'(t) \\ \quad + \int_{t}^{\min(r_u^{-1}(t), t_0+T)} \lambda(\tau)^T \cdot \frac{\partial f}{\partial u}(\tau, x(\tau), u(\tau - D(\tau,u)) \\ \quad \dot{u}(\tau - D(\tau,u))\frac{1}{\phi(u(\tau - D(\tau,u)))}\,\mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)) = 0 \end{cases} \tag{14}$$

VI. SIMULATIONS

In this section, we present simulation results for the paint mixing problem presented in § III-A. We assume that the total output flow-rate is fixed equal to 1 $\mathrm{m}^3.\mathrm{s}^{-1}$, $V = 50\ \mathrm{m}^3$ and that our control variables are the output flow-rates of the three batches $(u_i)_{i\in[\![1;3]\!]}$ where $u_1+u_2+u_3 = 1\ \mathrm{m}^3.\mathrm{s}^{-1}$ and $0 < u_{i\in[\![1;3]\!]}$. Given a reference trajectory yielding a smooth change of set-point for the output recipe of the product $y_{out_{ref}}$, we seek to achieve its optimal tracking in the sense of a quadratic norm. We assume that

$$\forall t < 0, \quad u(t) = u(0).$$

Invoking (2), we deduce

$$y_{out}(t) = \Gamma(u(t - D(t,u)))u(t),$$

where

$$\Gamma(u(t-D(t,u))) = \begin{pmatrix} \frac{u_1(t-D(t,u))}{u_1(t-D(t,u))+u_2(t-D(t,u))} & \frac{u_1(t-D(t,u))}{u_1(t-D(t,u)+u_2(t-D(t,u))} & 0 \\ \frac{u_2(t-D(t,u))}{u_1(t-D(t,u))+u_2(t-D(t,u))} & \frac{u_2(t-D(t,u))}{u_1(t-D(t,u))+u_2(t-D(t,u))} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with D the hydraulic delay defined by (3) and

$$u(t) = \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \end{pmatrix}^T.$$

For comparisons, we first define a "naive" control law as the input trajectory defined, straightforwardly but erroneously, by

$$u_{naive}(t) \triangleq y_{out_{ref}}(t).$$

This control input would be appropriate in the case $V = 0$, and in any case yields an error going asymptotically to zero if the reference signal itself is asymptotically converging. We simulate the output obtained using this control law. As expected, it generates a significant discrepancy with the reference, *cf.* Figures 3a and 3d.

We then solve the following simplified problem

$$\min_u J(u) = \int_0^T \|\Gamma(u(t-\tau))u(t) - y_{out_{ref}}(t)\|^2\,\mathrm{d}t,$$

where the delay is supposed to be a constant τ defined as

$$\tau = \frac{V}{u_1(0)+u_2(0)},$$

and T is set to 450s. The Figures 3b and 3e illustrate that while this approach improves the transient behaviour of the system, synchronization errors remain, creating a significant bump during the transient.

Finally, using the results of this paper, we solve the exact tracking problem

$$\min_u J(u) = \int_0^T \|\Gamma(u(t-D(t,u)))u(t) - y_{out_{ref}}(t)\|^2\,\mathrm{d}t. \tag{15}$$

To solve this problem, we design an iterative numerical scheme.

In the case of (15), the stationarity conditions (8) become

$$\partial_2 L([u,\dot{u}]_D(t)) + \mathbb{1}_{[t_0;r_u(t_0+T)]}(t)\cdot\partial_3 L([u,\dot{u}]_D(r_u^{-1}(t)))(r_u^{-1})'(t) + \int_t^{\min(r_u^{-1}(t),t_0+T)} \partial_3 L([u,\dot{u}]_D(\tau))\dot{u}(\tau - D(\tau,u)) \frac{1}{\phi(u(\tau-D(\tau,u)))}\,\mathrm{d}\tau \frac{\partial\phi}{\partial u}(u(t)) = 0, \quad (16)$$

where

$$\phi = u_1 + u_2,$$

and

$$L(t,[u,\dot{u}]_D(t)) = \|\Gamma(u(t-D(t,u)))u(t) - y_{out_{ref}}(t)\|^2.$$

Then, we define a sequence of functions $(u_k(.))_{k\in\mathbb{N}}$ as

$$u_0 = u_{naive},$$

and $\forall k \geq 0$, u_{k+1} is such as $\forall t \in [t_0; t_0+T]$

$$\partial_2 L([u_{k+1},\dot{u}_{k+1}]_{D_k}(t)) + \mathbb{1}_{[t_0;r_{u_k}(t_0+T)]}(t)\cdot\partial_3 L([u_{k+1},\dot{u}_{k+1}]_D(r_{u_k}^{-1}(t)))(r_{u_k}^{-1})'(t) + \int_t^{\min(r_{u_k}^{-1}(t),t_0+T)} \partial_3 L([u_k,\dot{u}_k]_D(\tau))\dot{u_k}(\tau - D(\tau,u_k)) \frac{1}{\phi(u_k(\tau-D(\tau,u_k)))}\,\mathrm{d}\tau \frac{\partial\phi}{\partial u}(u_k(t)) = 0. \quad (17)$$

One easily sees that any fixed point of this recursion is a solution of (16). Denoting

$$F_k(t) = \int_t^{\min(r_{u_k}^{-1}(t),t_0+T)} \partial_3 L([u_k,\dot{u}_k]_D(\tau)) \dot{u_k}(\tau - D(\tau,u_k))\frac{1}{\phi(u_k(\tau-D(\tau,u_k)))}\,\mathrm{d}\tau\frac{\partial\phi}{\partial u}(u_k(t)),$$

we finally have $\forall t \in [t_0; t_0+T]$

$$\partial_2 L(t,u_{k+1}(t),u_{k+1}(t-D(t,u_k))) + \mathbb{1}_{[t_0;r_{u_k}(t_0+T)]}(t)\cdot \partial_3 L(t,u_{k+1}(r_{u_k}^{-1}(t)),u_{k+1}(t))(r_{u_k}^{-1})'(t) + F_k(t) = 0. \quad (18)$$

Let us now consider any given $t \in]\max(t_0, r_{u_k}(t_0+T)); t_0+T]$. There exists a unique $p\in\mathbb{N}$ such as

$$r_{u_k}^p(t) > t_0 \quad \text{and} \quad r_{u_k}^{p+1}(t) \leq t_0.$$

Then, evaluating (18) in $(t, r_{u_k}(t), ..., r_{u_k}^p(t))$, we have

$$\begin{cases} \partial_2 L(t,u_{k+1}(t),u_{k+1}(r_{u_k}(t))) + F_k(t) = 0 \\ \partial_2 L(t,u_{k+1}(r_{u_k}(t)),u_{k+1}(r_{u_k}^2(t)))+ \\ \quad \partial_3 L(t,u_{k+1}(t),u_{k+1}(r_{u_k}(t)))(r_{u_k}^{-1})'(r_{u_k}(t))+ \\ \qquad F_k(r_{u_k}(t)) = 0 \\ \qquad\qquad ... \\ \partial_2 L(t,u_{k+1}(r_{u_k}^p(t)),u_{k+1}(r_{u_k}^{p+1}(t)))+ \\ \quad \partial_3 L(t,u_{k+1}(r_{u_k}^{p-1}(t)),u_{k+1}(r_{u_k}^p(t)))(r_{u_k}^{-1})'(r_{u_k}^p(t)) \\ \qquad + F_k(r_{u_k}^p(t)) = 0 \end{cases} \quad (19)$$

By definition, $r_{u_k}^{p+1}(t) \leq t_0$ and $u_{k+1}(r_{u_k}^{p+1}(t)) = u(0)$ is a past input, hence fixed. As a consequence, (19) is a system of $3\times(p+1)$ scalar difference equations and variables that is fully determined. Hence, solving (17) to determine u_{k+1} is equivalent to solving a set of smaller independent sub-problems (19). Determining u_{k+1} from u_k is straightforward theoretically, and can be implemented with little computational expenses. We then iterate until we reach a (almost numerically) fixed point which we take as solution of problem (15).

The solution of this approach along with the associated output trajectory are presented in Figures 3c-3f. We witness a significant improvement as compared to the method using a fixed delay. Specifically, we know that once the initial content of the pipe has been flushed out, the system output can be set arbitrarily, see [18], and our optimization approach indeed achieves exact tracking of the output trajectory passed this point.

VII. Conclusions and Perspectives

We have derived first-order optimality conditions for two families of variably timed-delayed dynamic systems optimization problems, with a particular emphasis on so-called "hydraulic-delays" which is a class of practical importance.

Future works should be focused on the numerical treatment of these conditions. A fixed-point method has been employed here, to obtain the presented simulations results. Beyond natural theoretical considerations, establishing its convergence is a work that is needed prior to any real-life implementation. On the application side, robust, accurate and fast numerical schemes should be studied further.

In practice, it is observed that the solutions we obtain tend to violate the preliminary assumptions of twice differentiability, on a subset of zero measure. Certainly, this fact is important as it could result into spurious oscillations in the numerical solutions. Further investigations should also focus on this point.

Appendix

Proof of Proposition 1

With the notations of Proposition 1, following (1), $\forall t \in r_u^{-1}(\Omega)\cap\Omega$ for δ small enough, one has

$$\int_{t-D(t,u)}^t \phi(u(\tau))\,\mathrm{d}\tau = \int_{t-D(t,u+\delta h)}^t \phi(u(\tau)+\delta h(\tau))\,\mathrm{d}\tau.$$

As a consequence

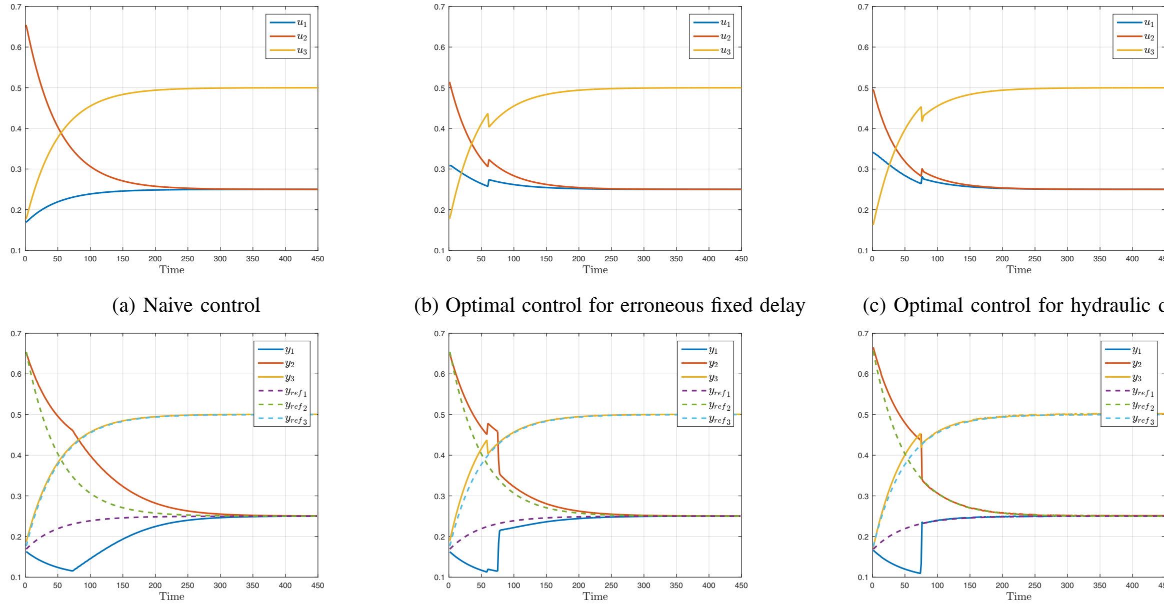
$$\int_{t-D(t,u)}^{t-D(t,u+\delta h)} \phi(u(\tau))\,\mathrm{d}\tau = \int_{t-D(t,u+\delta h)}^t \frac{\partial\phi}{\partial u}(u(\tau))\delta h(\tau) + \epsilon(\delta h(\tau))\delta h(\tau)\,\mathrm{d}\tau,$$

where ϵ is a twice differentiable function such as

$$\forall\tau\in[t-D(t,u+\delta h);t], \quad \lim_{\delta\to 0}\epsilon(\delta h(\tau)) = 0.$$

Then, since ϕ is assumed to be a strictly positive function, necessarily

$$\lim_{\delta\to 0} D(t,u+\delta h) = D(t,u).$$



(a) Naive control (b) Optimal control for erroneous fixed delay (c) Optimal control for hydraulic delay

(d) Output for strategy (a) (e) Output for strategy (b) (f) Output for strategy (c)

Fig. 3: Quadratic tracking of a reference trajectory for the "paint" problem

Finally

$$\frac{1}{\delta}\int_{t-D(t,u)}^{t-D(t,u+\delta h)} \phi(u(\tau))\,\mathrm{d}\tau = \int_{t-D(t,u)}^{t} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau) + \epsilon(\delta h(\tau))h(\tau)\,\mathrm{d}\tau + \int_{t-D(t,u+\delta h)}^{t-D(t,u)} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau) + \epsilon(\delta h(\tau))h(\tau)\,\mathrm{d}\tau,$$

and in the limit, we get the desired conclusion.

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