

# On the impact of model simplification in input constrained optimal control: application to HEV energy-thermal management

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**Abstract**—In this paper, we propose a result allowing to simplify the statement of input constrained optimal control problems. In details, it is shown that perturbation terms of magnitude  $\varepsilon$  appearing in the dynamics and the cost function can be neglected, because they only yield an improvement of magnitude  $\varepsilon^2$  in the optimal cost. This result, which is handy for practical applications, is here proven by means of an interior penalty method to deal with input constraints. For illustration, an example of energy management system for a Hybrid Electric Vehicle (HEV) is treated. As is expected, the complexity of the problem can be reduced at very little expense of sub-optimality. Based on simulations, quantitative results in term of fuel consumption are provided.

## I. INTRODUCTION

Generally, optimal control problems (OCPs), which are straightforward and valuable transcriptions of engineering considerations, are deemed difficult to solve [1], [4], [14]. Among others, the number of state variables, and the occurrence of constraints greatly impact the resolution methods, by increasing the level of complexity and the induced computational burden. This observation holds for all methods, from dynamic programming [3], Pontryagin Minimum Principle (PMP) based methods [21], [22], or direct formulations (e.g. collocation methods) [10].

Prior to resolution, some simplifications of the problem may be used to reduce this complexity. In earlier works (see [2] and references therein), it has been shown how perturbation terms appearing in the dynamics and the cost of an *unconstrained* OCP could be neglected. More precisely, under some mild assumptions, if the error in the right hand-side of the dynamics and the cost function between the nominal model (which is generally used to calculate an optimal control) and the perturbed (real) model are of magnitude  $\varepsilon$ , then the error on the optimal trajectories of the state and the control is also in magnitude of  $\varepsilon$ . As a consequence, the sub-optimality in the cost is only of magnitude  $\varepsilon^2$ . Practically, these results can be used to support model simplification when determining a trade-off between accuracy, optimality and complexity.

However, in real situations, most OCPs have to include input (and also very often state constraints which we leave out of the discussion in this contribution). Numerous recent works have been proposing ways to deal with these

constraints by means of unconstrained representation of the variables, e.g. by saturation function [7], [8], [9]. Following this idea, the recent works in [16], [17], [18] have shown how to solve such constrained OCPs by generating a convergent sequence of OCPs without constraints, using a method based on interior penalties. By introducing penalties with a weight factor in the cost function, a new unconstrained problem can be defined for which the solution is determined from the usual stationarity condition. This solution is then shown to converge to the solution of the initial constrained problem when the weight on the penalty tends to zero. This result is built around the classic ideas of penalty in finite dimensional optimization [5].

In this contribution, we extend the results of [2] to input constrained case by using the results presented in [16], [17], [18]. Interestingly, we end up with a very similar result in its statement. The reason for this similarity is that we approach the solution in a manner which vastly differs from the usual asymptotic expansions in power of  $\varepsilon$  of the solutions of the PMP conditions under constraints (with switching condition) [6].

After establishing this general result, for illustration, we study a problem of energy management system for a parallel Hybrid Electric Vehicle (HEV). In this problem, it is desired to quantify the benefit of considering engine temperature dynamics in the minimization of the fuel consumption, as has been considered in [19], [20], [23], [25]. Based on simulation results presented in [15], the temperature state could be ignored from this problem. Similar observations have been given in [26] and [27]. We use the general result established in this paper to explain and support these experimental observations. The conclusion that can be drawn from this is an important simplification step towards the design of an effective and implementable energy management controller for an HEV.

The paper is organized as follows. Section II presents the problem statement. A nominal OCP and a perturbed OCP are described in a general manner. In Section III, the result of [2] in the unconstrained case is recalled. Then, the extension to input constrained OCPs is established. In Section IV, two numerical examples are given to illustrate the previous results. The first one is a linear quadratic (toy) problem under input constraints. The second one is the HEV optimal energy management system. Finally, some conclusions and perspectives are given in Section V.

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## II. PROBLEM STATEMENT AND NOTATIONS

The following perturbed input constrained OCP is considered:

$$\min_{u \in U^{ad}} \left[ J_\varepsilon(u) = \int_0^T L_\varepsilon(x, u) dt \right] \quad (1)$$

where  $L_\varepsilon$  is a Lipschitz function of its arguments,  $T$  is a fixed parameter (without loss of generality),  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  are the state and the control variables of the following nonlinear dynamics with prescribed initial conditions  $X_0$ :

$$\dot{x} = f_\varepsilon(x, u), \quad x(0) = X_0 \quad (2)$$

The control  $u$  is constrained to belong to the set  $U^{ad} \subset L^\infty[0, T]$  defined by:

$$u_{min}(t) \leq u(t) \leq u_{max}(t), \quad a.e. \quad t \in [0, T] \quad (3)$$

Throughout the paper, for convenience, we note  $\sigma \triangleq [x, u]$ . In OCP (1),  $\varepsilon$  is a parameter scaling the error terms (perturbation) in the cost function and the state dynamics, with respect to some ideal scenario  $\varepsilon = 0$ . This scenario is defined as follows. We assume that  $L_\varepsilon$  and  $f_\varepsilon$  are affine functions in  $\varepsilon$  of the form:

$$L_\varepsilon(\sigma) \triangleq L_0(\sigma) + \varepsilon L_1(\sigma), \quad f_\varepsilon(\sigma) \triangleq f_0(\sigma) + \varepsilon f_1(\sigma),$$

where  $f_0$  and  $L_0$  are of class  $C^1$  and  $L_1, f_1$  and their first and second derivatives are assumed to be bounded.

In this paper, we wish to establish relationships between the solution of (1) for  $\varepsilon = 0$  and for  $\varepsilon > 0$ . For this and following the interior penalty approach presented in [16] and [17], we introduce a penalty function  $P(u)$  in the cost (1). This function is defined on  $]u_{min}, u_{max}[$ , is smooth, and grows unboundedly as its argument reaches either  $u_{min}$  or  $u_{max}$ . We use it to define a penalized constraints-free OCP:

$$\min_{u \in L^\infty[0, T]} \left[ J_\varepsilon^r(u) = \int_0^T [L_\varepsilon(\sigma) + rP(u)] dt \right], \quad r > 0 \quad (4)$$

*Assumption 1:* For simplicity, OCP (4) is supposed to possess a unique solution for any  $r > 0$  and  $\varepsilon \geq 0$ . We note  $u_\varepsilon^r$  the corresponding optimal control and  $x_\varepsilon^r$  the corresponding solution of the differential equation (2) for  $u = u_\varepsilon^r$ . We also note  $u_\varepsilon$  the optimal control of (1) for  $\varepsilon \geq 0$  and  $x_\varepsilon$  the corresponding solution of (2) for  $u = u_\varepsilon$ .

If the penalty function is chosen properly, i.e., satisfying some conditions given in [18], the optimal value of the modified cost in (4) converges to the optimal cost solution of OCP (1). Furthermore, the term  $rP(u)$  goes to zero when  $r$  tends to zero as has been shown in [5], [16] and [17]. The main advantage of the interior penalty methodology is that, for each value of  $r$ , the solution of OCP (4) is determined from simple stationarity conditions. This is of practical importance when implementing numerical methods.

As we have mentioned before, we wish to establish relationships between the solution of the OCP (1) for  $\varepsilon = 0$  and for  $\varepsilon > 0$ . To this end, we formulate the stationarity conditions of their penalized counterparts.

### A. Penalized nominal problem

The nominal problem is obtained for  $\varepsilon = 0$ . Using the PMP, the solution  $\sigma_0^r \triangleq [x_0^r, u_0^r]$  of this problem is given by the following two-point boundary value problem (TBVP):

$$\begin{aligned} \dot{x}_0^r &= f_0(\sigma_0^r), \quad x_0^r(0) = X_0 \\ -\dot{p}_0^{rT} &= \partial_x L_0(\sigma_0^r) + p_0^{rT} \partial_x f_0(\sigma_0^r), \quad p_0^{rT}(T) = 0 \end{aligned} \quad (5)$$

$$\partial_u L_0(\sigma_0^r) + r \partial_u P(u_0^r) + p_0^{rT} \partial_u f_0(\sigma_0^r) = 0 \quad (6)$$

where  $\partial_z K$  indicates the partial derivative of  $K$  with respect to  $z$ ,  $z = x, u$  and  $p_0^r$  is the adjoint state of  $x_0^r$ . The Hamiltonian associated with this problem is:

$$H_0^r(\sigma, p) = L_0(\sigma) + p^T f_0(\sigma) + rP(u)$$

### B. Penalized perturbed problem

In the case  $\varepsilon > 0$ , the TBVP is:

$$\begin{aligned} \dot{x}_\varepsilon^r &= f_\varepsilon(\sigma_\varepsilon^r), \quad x_\varepsilon^r(0) = X_0 \\ -\dot{p}_\varepsilon^{rT} &= \partial_x L_\varepsilon(\sigma_\varepsilon^r) + p_\varepsilon^{rT} \partial_x f_\varepsilon(\sigma_\varepsilon^r), \quad p_\varepsilon^{rT}(T) = 0 \end{aligned}$$

$$\partial_u L_\varepsilon(\sigma_\varepsilon^r) + r \partial_u P(u_\varepsilon^r) + p_\varepsilon^{rT} \partial_u f_\varepsilon(\sigma_\varepsilon^r) = 0$$

where  $p_\varepsilon^r$  is the adjoint state of  $x_\varepsilon^r$ . The Hamiltonian associated with this problem is:

$$H_\varepsilon^r(\sigma, p) = L_\varepsilon(\sigma) + p^T f_\varepsilon(\sigma) + rP(u) \quad (7)$$

It can be written as an affine function of  $\varepsilon$  as follows:

$$H_\varepsilon^r(\sigma, p) = H_0^r(\sigma, p) + \varepsilon H_1(\sigma, p) \quad (8)$$

where  $H_1(\sigma, p) = L_1(\sigma) + p^T f_1(\sigma)$  is independent of the penalty function.

### C. Problem under consideration

We wish to determine relationships between the solutions corresponding to  $u_0$  and  $u_\varepsilon$  which as a matter of fact may differ. However, their cost values will be close. The main contribution of this paper is an upper bound on  $J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon) > 0$  which will be constructed thanks to approaching sequences of interior penalty solutions  $u_\varepsilon^r$  and  $u_0^r$ .

## III. PERTURBATION METHODS

First, we recall the results of [2] in the unconstrained case giving an upper bound on the difference  $J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon)$ . Then, the main contribution of this paper for the input constrained case will be presented and detailed.

### A. Perturbation methods in the unconstrained case

For now we consider  $U^{ad} = L^\infty[0, T]$ . Note  $x_0, u_0$  and  $\sigma_0 \triangleq [x_0, u_0]$  the nominal solution to OCP (1). For any  $x$  and  $u$ , we note:

$$\delta x = x - x_0, \quad \delta u = u - u_0, \quad \delta \sigma = \sigma - \sigma_0$$

Classically, the TBVP associated to the nominal problem is

$$\begin{aligned} \dot{x}_0 &= f_0(\sigma_0), \quad x_0(0) = X_0 \\ -\dot{p}_0^T &= \partial_x L_0(\sigma_0) + p_0^T \partial_x f_0(\sigma_0), \quad p_0^T(T) = 0 \end{aligned}$$

$$\partial_u L_0(\sigma_0) + p_0^T \partial_u f_0(\sigma_0) = 0$$

while for the perturbed problem, we have:

$$\begin{aligned}\dot{x}_\varepsilon &= f_\varepsilon(\sigma_\varepsilon), x_\varepsilon(0) = X_0 \\ -\dot{p}_\varepsilon^T &= \partial_x L_\varepsilon(\sigma_\varepsilon) + p_\varepsilon^T \partial_x f_\varepsilon(\sigma_\varepsilon), p_\varepsilon^T(T) = 0 \\ \partial_u L_\varepsilon(\sigma_\varepsilon) + p_\varepsilon^T \partial_u f_\varepsilon(\sigma_\varepsilon) &= 0\end{aligned}$$

where  $p_0$  and  $p_\varepsilon$  are the adjoint states of  $x_0$  and  $x_\varepsilon$  respectively. With these notations, we have:

*Theorem 1:* [Bensoussan][2] Assume that:

- $\exists k > 0$  such that  $\|L_1\| \leq k$  and  $\|f_1\| \leq k$ ,
- $\exists \beta > 0$  such that:

$$\begin{cases} (\partial_{xx}H_0^0 - \partial_{xu}H_0^0[\partial_{uu}H_0^0]^{-1}\partial_{ux}H_0^0)(\sigma, p_0) \geq 0 \\ \partial_{uu}H_0^0(\sigma, p_0) \geq \beta I \text{ uniformly in } \sigma \end{cases}$$

then we have, for  $\varepsilon$  small enough and for some positive constants  $c_x$ ,  $c_u$  and  $K$ :

$$\begin{aligned}|x_\varepsilon - x_0| &\leq c_x \varepsilon \\ |u_\varepsilon - u_0| &\leq c_u \varepsilon \\ |J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)| &\leq K \varepsilon^2\end{aligned}$$

### B. Perturbation methods in the input constrained case

Now, we go back to the problem of interest where  $U^{ad}$  is defined in (3). The result that we shall establish is the extension of Theorem 1 for the input constrained case. Since we have reformulated the input constraints under the form of a penalty problem, the OCP (4) is constraint-free and we can directly apply Theorem 1. However, a central question would arise: do the parameters  $c_x$ ,  $c_u$  and  $K$  in the input constrained case depend on  $rP(\cdot)$ ? If they do, this would prevent us from studying the limit  $r \rightarrow 0$ . To answer this question, we first need to rewrite the cost function in a particular form.

*Proposition 1:* For any control  $u$ :

$$\begin{aligned}J_\varepsilon^r(u) &= \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt + \int_0^T [N^0 \delta u^r + P^0 \delta x^r] dt \\ &+ \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda \mu \delta \sigma^r, p_0^r) (\delta \sigma^r)^2 d\lambda d\mu dt \quad (9)\end{aligned}$$

where

$$\begin{aligned}N^0 &\triangleq \varepsilon \partial_u H_1(\sigma_0^r, p_0^r), P^0 \triangleq \varepsilon \partial_x H_1(\sigma_0^r, p_0^r) \\ \delta u^r &= u - u_0^r, \delta x^r = x - x_0^r, \delta \sigma^r = \sigma - \sigma_0^r\end{aligned}$$

In (9), the term  $\int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt$  is constant, it depends only on the solution of the nominal problem. The second term  $\int_0^T [N^0 \delta u^r + P^0 \delta x^r] dt$  represents the first order variation of the cost due to the variation of the state and the control trajectories.

*Proof:* Classically, for a smooth function  $F$ , the Taylor expansion for any  $y$  and  $y_0$  is noted:

$$\begin{aligned}F(y) &= F(y_0) + \partial_y F(y_0)(y - y_0) \\ &+ \int_0^1 \int_0^1 \lambda (y - y_0)^T \partial_{yy} F(y_0 + \lambda \mu (y - y_0))(y - y_0) d\lambda d\mu \quad (10)\end{aligned}$$

From this, we can write:

$$J_\varepsilon^r(u) = \int_0^T [L_\varepsilon(\sigma_0^r) + \partial_x L_\varepsilon(\sigma_0^r) \delta x^r + \partial_u L_\varepsilon(\sigma_0^r) \delta u^r] dt$$

$$\begin{aligned}&+ \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} L_\varepsilon(\sigma_0^r + \lambda \mu \delta \sigma^r) (\delta \sigma^r)^2 d\lambda d\mu dt \\ &+ r \int_0^T [P(u_0^r) + \partial_u P(u_0^r) \delta u^r] dt \\ &+ r \int_0^T \int_0^1 \int_0^1 \lambda \partial_{uu} P(u_0^r + \lambda \mu \delta u^r) (\delta u^r)^2 d\lambda d\mu dt \quad (11)\end{aligned}$$

Note

$$S \triangleq \partial_x L_\varepsilon(\sigma_0^r) \delta x^r + \partial_u L_\varepsilon(\sigma_0^r) \delta u^r + r \partial_u P(u_0^r) \delta u^r$$

Using the stationarity conditions (5, 6), we can write  $S$ :

$$\begin{aligned}S &= [-\dot{p}_0^{rT} - p_0^{rT} \partial_x f_\varepsilon(\sigma_0^r) + \varepsilon \partial_x L_1(\sigma_0^r) + \varepsilon p_0^{rT} \partial_x f_1(\sigma_0^r)] \delta x^r \\ &+ [-p_0^{rT} \partial_u f_\varepsilon(\sigma_0^r) + \varepsilon \partial_u L_1(\sigma_0^r) + \varepsilon p_0^{rT} \partial_u f_1(\sigma_0^r)] \delta u^r\end{aligned}$$

By integration, we get:

$$\begin{aligned}\int_0^T S(t) dt &= - \int_0^T \dot{p}_0^{rT} \delta x^r dt - \int_0^T p_0^{rT} \partial_{\sigma} f_\varepsilon(\sigma_0^r) \delta \sigma^r dt \\ &+ \varepsilon \int_0^T \partial_{\sigma} H_1(\sigma_0^r, p_0^r) \delta \sigma^r dt\end{aligned}$$

which can be rewritten thanks to integration by parts as

$$\begin{aligned}\int_0^T S(t) dt &= - \overbrace{p_0^{rT}(T)}^{=0} \delta x^r(T) + p_0^{rT} \overbrace{\delta x^r(0)}^{=0} + \int_0^T p_0^{rT} (\dot{x} - \dot{x}^r) dt \\ &+ \int_0^T [-p_0^{rT} \partial_{\sigma} f_\varepsilon(\sigma_0^r) \delta \sigma^r + \varepsilon \partial_{\sigma} H_1(\sigma_0^r, p_0^r) \delta \sigma^r] dt\end{aligned}$$

yielding

$$\begin{aligned}\int_0^T S(t) dt &= \varepsilon \int_0^T \partial_{\sigma} H_1(\sigma_0^r, p_0^r) \delta \sigma^r dt \\ &+ \int_0^T p_0^{rT} (\dot{x} - \dot{x}^r - \partial_{\sigma} f_\varepsilon(\sigma_0^r) \delta \sigma^r) dt \quad (12)\end{aligned}$$

By using the Taylor expansion (10) again, we get:

$$\begin{aligned}\dot{x} - \dot{x}^r - \partial_{\sigma} f_\varepsilon(\sigma_0^r) \delta \sigma^r &= \varepsilon f_1(\sigma_0^r) \\ &+ \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} f_\varepsilon(\sigma_0^r + \lambda \mu \delta \sigma^r) (\delta \sigma^r)^2 d\lambda d\mu \quad (13)\end{aligned}$$

Remembering that from the definition of  $H_\varepsilon^r$  in (7), we have

$$L_\varepsilon(\sigma_0^r) + rP(u_0^r) = H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \frac{dx_0^r}{dt} - \varepsilon p_0^{rT} f_1(\sigma_0^r) \quad (14)$$

We replace (12, 13, 14) in the expansion (11) and we obtain the expression (9). This concludes the proof. ■

We are now ready to state the main contribution of this paper.

*Theorem 2:* Assume that:

- $\exists k > 0$  such that  $\|L_1\| \leq k$  and  $\|f_1\| \leq k$ ,
- $\exists \beta > 0$  such that:

$$\begin{cases} (\partial_{xx}H_0^r - \partial_{xu}H_0^r[\partial_{uu}H_0^r]^{-1}\partial_{ux}H_0^r)(\sigma, p_0^r) \geq 0 \\ \partial_{uu}H_0^r(\sigma, p_0^r) \geq \beta I \text{ uniformly in } \sigma \end{cases} \quad (15)$$

then we have, for  $\varepsilon$  small enough and for some positive constants  $c_x$ ,  $c_u$  and  $K$  independent from  $rP(\cdot)$ :

$$\begin{aligned}|x_\varepsilon^r - x_0^r| &\leq c_x \varepsilon \\ |u_\varepsilon^r - u_0^r| &\leq c_u \varepsilon \\ |J_\varepsilon^r(u_\varepsilon^r) - J_\varepsilon^r(u_0^r)| &\leq K \varepsilon^2\end{aligned}$$

As the penalized cost  $J_\varepsilon^r$  converges to the optimal value of  $J_\varepsilon$  under input constraints when  $r$  tends to zero and the parameter  $K$  is independent from  $rP(\cdot)$ , we have:

$$|J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0)| \leq K\varepsilon^2$$

*Proof:* From Proposition (1), we can rewrite the penalized cost function  $J_\varepsilon^r$  for  $u_0^r$ :

$$\begin{aligned} J_\varepsilon^r(u_0^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt &= \int_0^T [P^0(\xi_0^r - x_0^r)] dt \\ + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{xx} H_\varepsilon^r(x_0^r + \lambda \mu (\xi_0^r - x_0^r), u_0^r, p_0^r) (\xi_0^r - x_0^r)^2 d\lambda d\mu dt & \end{aligned} \quad (16)$$

where  $\xi_0^r$  is the solution of the differential equation:

$$\dot{\xi}_0^r = f_\varepsilon(\xi_0^r, u_0^r), \quad \xi_0^r(0) = X_0$$

Since the first derivatives of  $L_1$  and  $f_1$  are bounded by assumption, we can write, for some positive constants  $k_1$  and  $k_2$ :

$$|P^0| \leq k_1 \varepsilon, \quad |N^0| \leq k_2 \varepsilon \quad (17)$$

Using the fact that  $\partial_{xx} H_\varepsilon^r$  is bounded and  $|\xi_0^r - x_0^r| < c_0 \varepsilon$  (using the comparison lemma [11]), we derive from (16, 17), for  $\varepsilon$  small enough that:

$$|J_\varepsilon^r(u_0^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt| \leq c\varepsilon^2 \quad (18)$$

where  $c$  depends on  $k_1$ ,  $c_0$  and the upper bound of  $\partial_{xx} H_\varepsilon^r$  but not on  $rP(\cdot)$ . This property is the result of  $\partial_{xx} H_\varepsilon^r$  being independent from  $rP(\cdot)$ .

Let  $u_\varepsilon^r$  be the optimal control for the perturbed problem (4). By definition, we have:

$$J_\varepsilon^r(u_\varepsilon^r) \leq J_\varepsilon^r(u_0^r)$$

which can be rewritten using (18) as:

$$\begin{aligned} J_\varepsilon^r(u_\varepsilon^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt \\ \leq J_\varepsilon^r(u_0^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt \leq c\varepsilon^2 \end{aligned}$$

By using Proposition 1 again, we derive that:

$$\begin{aligned} c\varepsilon^2 \geq \int_0^T [N^0 \delta u_\varepsilon^r + P^0 \delta x_\varepsilon^r] dt \\ + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 d\lambda d\mu dt \end{aligned} \quad (19)$$

We now look for a bound on  $\partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2$ . To proceed, we replace every factor of  $\delta u_\varepsilon^r$  in the second order variation of the cost  $J_\varepsilon^r(u_\varepsilon^r)$  by terms in  $z$  and  $\delta x_\varepsilon^r$ , where  $z$  is given by :

$$z(\lambda, \mu, t) = \delta u_\varepsilon^r + [\partial_{uu} H_0^r(\cdot)]^{-1} \partial_{ux} H_0^r(\cdot) \delta x_\varepsilon^r \quad (20)$$

In this latter,  $z$  is well defined because  $\partial_{uu} H_0^r(\cdot)$  is assumed to be positive definite from (15). This allows us to handle diagonal quadratic forms in terms of  $z$  and  $\delta x_\varepsilon^r$ . Thus, we can write  $\partial_{\sigma\sigma} H_0^r(\cdot) (\delta \sigma_\varepsilon^r)^2$  as follows:

$$\begin{aligned} \partial_{\sigma\sigma} H_0^r(\cdot) (\delta \sigma_\varepsilon^r)^2 &= z^T \partial_{uu} H_0^r(\cdot) z \\ + \delta x_\varepsilon^{rT} [\partial_{xx} H_0^r - \partial_{xu} H_0^r [\partial_{uu} H_0^r]^{-1} \partial_{ux} H_0^r] (\cdot) \delta x_\varepsilon^r \end{aligned} \quad (21)$$

From the second order optimality conditions (15), we derive that:

$$\partial_{\sigma\sigma} H_0^r(\cdot) (\delta \sigma_\varepsilon^r)^2 \geq \beta \|z\|^2 \quad (22)$$

Now, equation (19) can be rewritten using (8, 22) to give

$$\begin{aligned} c\varepsilon^2 \geq \int_0^T [N^0 \delta u_\varepsilon^r + P^0 \delta x_\varepsilon^r] dt \\ + \beta \int_0^T \int_0^1 \int_0^1 \lambda \|z\|^2 d\lambda d\mu dt \\ + \int_0^T \int_0^1 \int_0^1 \varepsilon \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 d\lambda d\mu dt \end{aligned} \quad (23)$$

The next step is to find an upper bound on  $|\delta x_\varepsilon^r|$  depending on  $z$  and  $\varepsilon$ . From (20), we have:

$$\delta u_\varepsilon^r = z - [\partial_{uu} H_0^r(\cdot)]^{-1} \partial_{ux} H_0^r(\cdot) \delta x_\varepsilon^r \quad (24)$$

Using the Taylor expansion (10), the dynamic of  $\delta x_\varepsilon^r$  can be written as:

$$\frac{d(\delta x_\varepsilon^r)}{dt} = A \delta x_\varepsilon^r + B \delta u_\varepsilon^r + \varepsilon f_1(x_\varepsilon^r, u_\varepsilon^r)$$

where  $A$  and  $B$  are given by:

$$A = \int_0^1 \partial_x f_0(\sigma_0^r + \lambda \delta \sigma_\varepsilon^r) d\lambda, \quad B = \int_0^1 \partial_u f_0(\sigma_0^r + \lambda \delta \sigma_\varepsilon^r) d\lambda$$

We replace  $\delta u_\varepsilon^r$  given by (24) in the dynamic of  $\delta x_\varepsilon^r$ :

$$\frac{d(\delta x_\varepsilon^r)}{dt} = \underbrace{(A - B[\partial_{uu} H_0^r(\cdot)]^{-1} \partial_{ux} H_0^r(\cdot))}_{\triangleq A_{mod}} \delta x_\varepsilon^r + Bz + \varepsilon f_1(\sigma_\varepsilon^r)$$

with  $\delta x_\varepsilon^r(0) = 0$ . Since the term  $\partial_{ux} H_0^r(\cdot)$  is independent of  $rP(\cdot)$  and  $\partial_{uu} H_0^r(\cdot) > \beta I$ , the matrix  $A_{mod}$  is bounded independently from  $rP(\cdot)$ . As a result and using the comparison lemma [11], we can write:

$$|\delta x_\varepsilon^r| \leq c_{2x} \left[ \int_0^T \|z\|^2 dt \right]^{\frac{1}{2}} + c_{3x} \varepsilon \quad (25)$$

for some positive constants  $c_{2x}$  and  $c_{3x}$ . This implies:

$$|\delta x_\varepsilon^r|^2 \leq 4c_{2x}^2 \int_0^T \int_0^1 \int_0^1 \lambda \|z\|^2 d\lambda d\mu dt + 2c_{3x}^2 \varepsilon^2 \quad (26)$$

In (25),  $c_{2x}$  depends on the upper bounds of  $A_{mod}$  and  $B$  and  $c_{3x}$  depends on the upper bounds of  $A_{mod}$  and  $f_1$ . These parameters are independent from  $rP(\cdot)$ .

We can now determine an upper bound on  $|\delta u_\varepsilon^r|$ . From equation (24), we deduce:

$$|\delta u_\varepsilon^r|^2 \leq c_{2u} \int_0^T \int_0^1 \int_0^1 \lambda \|z\|^2 d\lambda d\mu dt + c_{3u} \varepsilon^2 \quad (27)$$

where  $c_{2u}$  and  $c_{3u}$  depend on  $(c_{2x}, c_{3x})$ , but not on  $P(\cdot)$ .

Since  $P^0$  and  $N^0$  are bounded and using (26, 27), we can write (23) as:

$$\begin{aligned} c\varepsilon^2 \geq \beta_1 \int_0^T \int_0^1 \int_0^1 \lambda \|z\|^2 d\lambda d\mu dt \\ + \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 d\lambda d\mu dt \end{aligned}$$

By using the assumption that  $\partial_{\sigma\sigma}H_1(\cdot)$  is bounded (it is independent from  $rP(\cdot)$ ) and the term  $\varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma}H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt$  leads to a factor in  $\varepsilon^3$ , we get:

$$\int_0^T \int_0^1 \int_0^1 \lambda \|z\|^2 d\lambda d\mu dt \leq \frac{c}{\beta_1} \varepsilon^2$$

The two equations (26) and (27) can be written in the form:

$$|\delta x_\varepsilon^r|^2 \leq c_x^2 \varepsilon^2, \quad |\delta u_\varepsilon^r|^2 \leq c_u^2 \varepsilon^2 \quad (28)$$

Finally, we can easily find an upper bound of  $J_\varepsilon^r(u_\varepsilon^r) - J_\varepsilon^r(u_0^r)$ . In details:

$$\begin{aligned} |J_\varepsilon^r(u_\varepsilon^r) - J_\varepsilon^r(u_0^r)| &\leq \left| J_\varepsilon^r(u_\varepsilon^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt \right| \\ &\quad + \underbrace{\left| J_\varepsilon^r(u_0^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt \right|}_{\leq c\varepsilon^2} \end{aligned}$$

Using Proposition 1 and equations (28), we derive for  $\varepsilon$  small enough that:

$$|J_\varepsilon^r(u_\varepsilon^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt| \leq c_1 \varepsilon^2 \quad (29)$$

where  $c_1$  depends on  $(k_1, k_2, c_x, c_u)$ ,  $c \geq c_1$  and is independent from  $rP(\cdot)$ . From equation (18) which has the same form as (29), we conclude that:

$$J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \leq (c + c_1) \varepsilon^2 = K \varepsilon^2 \quad (30)$$

As we can see it from the previous calculations, the constants  $c_x, c_u$  and  $K$  are independent from  $r$  and  $P(\cdot)$ . This concludes the proof. ■

The main conclusion is the following: if the error in the dynamics and the cost function between the nominal model (which is generally used to calculate a controller) and the perturbed (real) model are of magnitude  $\varepsilon$ , then the suboptimality in the presence of the input constraints is only of magnitude  $\varepsilon^2$ . This result is very similar in its statement to the result given in [2] for the unconstrained case. The interior penalty approach is used here as a proof tool to deal with input constraints.

#### IV. ILLUSTRATIVE EXAMPLES

To illustrate the proposed result, we consider two problems.

##### A. Input constrained Linear Quadratic (LQ) problem

Consider the following LQ problem:

$$J_\varepsilon(u) = \frac{1}{2} \int_0^{10} (u^2 + x_1^2) dt \quad (31)$$

where  $x_1, x_2$  and  $u$  are the state and the control variables of the following linear system:

$$\dot{x}_1 = x_2 - \frac{\varepsilon}{5} x_1, \quad x_1(0) = 4 \quad (32)$$

$$\dot{x}_2 = -(1 - \frac{\varepsilon}{4}) x_2 + u, \quad x_2(0) = 4 \quad (33)$$

The control  $u$  is constrained to belong to the set  $U^{ad}$  defined by:

$$-2 \leq u(t) \leq 2 \quad (34)$$

By using the PMP, the optimal control is given by:

$$u_\varepsilon^* = \min(2, \max(-2, -\lambda_2)) \quad (35)$$

where  $\lambda_1$  and  $\lambda_2$  are the adjoint states associated to  $x_1$  and  $x_2$  respectively. They are given by:

$$\dot{\lambda}_1 = -x_1 + \frac{\varepsilon}{5} \lambda_1, \quad \lambda_1(10) = 0 \quad (36)$$

$$\dot{\lambda}_2 = (1 - \frac{\varepsilon}{4}) \lambda_2 - \lambda_1, \quad \lambda_2(10) = 0 \quad (37)$$

The equations (35, 36, 37) define a TBVP which has been solved using `bvp4c` [13].

Let  $u_0$  the optimal control minimizing  $J_\varepsilon$  under the dynamics (32, 33) when  $\varepsilon = 0$  and  $u_\varepsilon$  the solution of the OCP defined by (31, 32, 33) when  $\varepsilon > 0$ . Figure 1 reports the difference  $\Delta J = J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon)$  between the two costs and an upper bound which kindly fits a parabola, as expected from Theorem 2. Figures 2 and 3 give the states and the control trajectories.

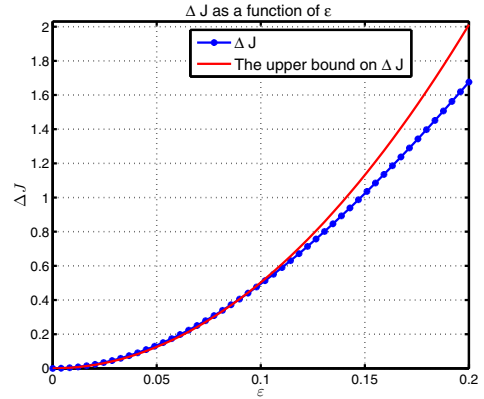


Fig. 1. Upper bound on  $\Delta J$  as a function of  $\varepsilon$  in the LQ case

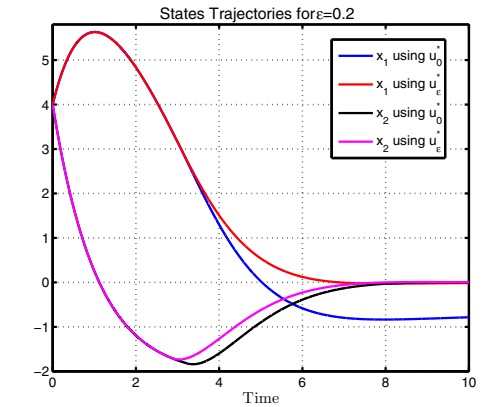


Fig. 2. States trajectories for  $\varepsilon = 0.2$

From Figure 1, we can see that the difference between  $J_\varepsilon(u_\varepsilon)$  and  $J_\varepsilon(u_0)$  is bounded by  $K\varepsilon^2$ . The maximum relative

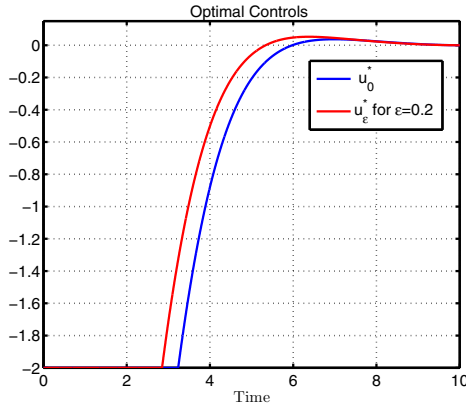


Fig. 3. Optimal controls for  $\varepsilon = 0.2$

error between the two costs is less than 3% for  $\varepsilon \in [0, 0.2]$ , which is considered acceptable.

### B. Thermal management Problem for a parallel hybrid electric vehicle

The numerical results presented in [15] are the main motivation for this study. Based on extensive numerical tests and experiments, these results show that, in the optimization of an energy management system for a parallel Hybrid Electric Vehicle (HEV), neglecting the engine temperature leads to an acceptable sub-optimal fuel consumption. Interestingly, this observation can be justified by Theorem 2. To illustrate our point, we formulate the corresponding nominal and perturbed OCPs.

The cost function under consideration is the fuel consumption over a fixed time window corresponding to a given driving cycle of duration  $T$ :

$$J(u) = \int_0^T c(u, t) e(\theta) dt$$

where  $u$  is the control variable (the engine torque),  $\theta$  is the engine temperature and  $c(u, t)$  is the fuel consumption rate when the engine is warm. The time variable accounts for the dependence of the consumption on the engine speed, which is a varying set points assumed to be perfectly tracked.

In this model,  $e(\cdot)$  is a correction factor of fuel consumption with respect to the engine temperature  $\theta$ . It can be approximated by:

$$e(\theta, \varepsilon) = \begin{cases} (1 - \frac{\theta}{\theta_w})\varepsilon + 1, & \theta_c \leq \theta \leq \theta_w \\ 1, & \theta > \theta_w \end{cases}$$

The considered dynamics are:

- The dynamics of the State Of Charge of the battery (SOC)

$$\frac{d\xi}{dt} = f(u, t), \quad \xi(0) = \xi_0$$

One operational constraint requires that the final value of  $\xi$  should be equal to its initial value:

$$\xi(T) = \xi(0).$$

- Engine temperature dynamics

$$\frac{d\theta}{dt} = g(u, t, \theta), \quad \theta(0) = \theta_0$$

The constraints on the control are given by:

$$u_{min}(t) \leq u(t) \leq u_{max}(t)$$

where  $u_{min}(t)$  and  $u_{max}(t)$  are determined by the driving conditions, and physical limitations of the engine and the electric motor. For more details, one can refer to [15], [26], [27].

The simplification  $\varepsilon = 0$  represents a case where the engine thermal efficiency is independent from its temperature, e.g. because the engine has reached its stabilized temperature (it is *warm*). In this case, the engine temperature can be left out of the equations describing the OCP.

Generally, the cost function to be minimized is:

$$J_\varepsilon(u) = \int_0^T c(u, t) e(\theta, \varepsilon) dt$$

We now define two following OCPs:

$$(P_\varepsilon) \begin{cases} \min_u \left[ J_\varepsilon(u) = \int_0^T c(u, t) e(\theta, \varepsilon) dt \right] \\ \frac{d\xi}{dt} = f(u, t), \quad \xi(0) = \xi_0 \\ \frac{d\theta}{dt} = g(u, t, \theta), \quad \theta(0) = \theta_0 \\ u_{min}(t) \leq u(t) \leq u_{max}(t) \\ \xi(T) = \xi(0) \end{cases} \quad (38)$$

$$(P_0) \begin{cases} \min_u \left[ J_0(u) = \int_0^T c(u, t) dt \right] \\ \frac{d\xi}{dt} = f(u, t), \quad \xi(0) = \xi_0 \\ u_{min}(t) \leq u(t) \leq u_{max}(t) \\ \xi(T) = \xi(0) \end{cases} \quad (39)$$

$(P_0)$  is a simplification of  $(P_\varepsilon)$  for  $\varepsilon = 0$ , where the effect of the engine temperature on the fuel consumption is neglected.

On the application side, the problem (38) is the right problem to solve. We shall note that in (39), the state  $\theta$  has been omitted. Interestingly, this simplification is not due to an argument of singular perturbations as in [12], but rather of regular perturbations. Thus, the number of states is reduced to 1 which is appealing as it reduces the complexity of numerical methods.

These two problems  $(P_\varepsilon)$  and  $(P_0)$  are solved for the NEDC cycle. We proceed as follows :

- For a given  $\varepsilon$ , compute the optimal control  $u_\varepsilon$  solution of  $(P_\varepsilon)$ .
- For  $\varepsilon = 0$ , compute the optimal control  $u_0$  solution of  $(P_0)$ .
- Compare the two costs  $J_\varepsilon(u_\varepsilon)$  and  $J_\varepsilon(u_0)$ .

Again, we define  $\Delta J$  as the error on the cost function:

$$\Delta J = J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon)$$

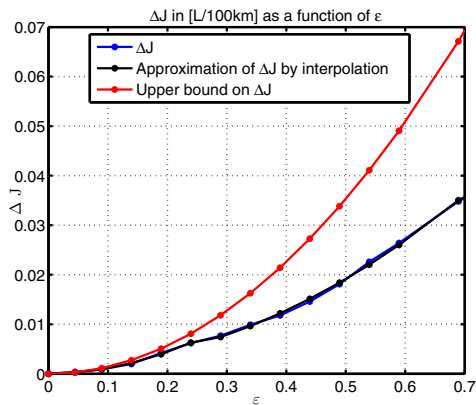


Fig. 4. Upper bound on  $\Delta J$  as a function of  $\varepsilon$

Theorem 2 informs us that the sub-optimality effect of neglecting the engine temperature is bounded by a quadratic form in  $\varepsilon$ . Indeed, we observe this phenomena. The blue graph in Figure (4) represents a theoretical upper bound

$$\Delta J \leq K\varepsilon^2 \quad (40)$$

where  $K$  is determined experimentally to fit values for  $\varepsilon$  small enough, its value is 0.15. In the vicinity of  $\varepsilon = 0$ , the results fit (40). For higher value of  $\varepsilon$ ,  $\Delta J$  remains below the quadratic conservative estimation given by (40).

On the practical side, this result indicates that the error in the optimal cost function will be less than 1%, and suggests that it is sufficient to consider the sub-optimal solution obtained by neglecting the engine temperature in the OCP. This result is an important step towards the design of an effective and simple energy management controller that is suitable for implementation in real-time because the number of the state variable in the optimization has a great impact on the time needed for resolution. As a concluding remark we can note that other similar conclusion can be formulated on battery related problems in [24].

## V. CONCLUSION

The result of [2] has been extended to the input constrained case by using the interior penalty methods. The obtained result provides a conservative upper bound on the error in the optimal cost, which is quadratic in the magnitude of uncertainties. This upper bound can be used to select the right level of complexity of the OCP to solve to optimize the accuracy/complexity trade-off by establishing an estimation as a function of the system parameters. The extension of the obtained results to include state constraints, which is more involved because modeling errors in the dynamics can lead to a violation of the state constraints, is the subject of current investigation.

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