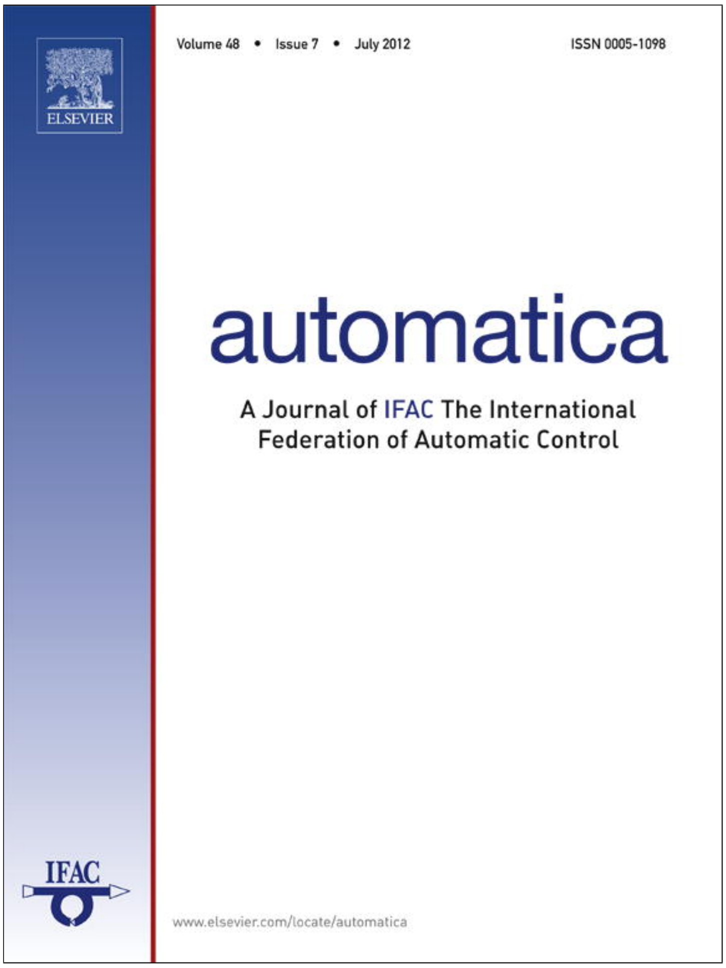
Volume 48 • Issue 7 • July 2012
ISSN 0005-1098
ELSEVIER
automatica
A Journal of IFAC The International Federation of Automatic Control
IFAC
www.elsevier.com/locate/automatica

Automatica 48 (2012) 1467–1472

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Periodic inputs reconstruction of partially measured linear periodic systems☆

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ARTICLE INFO

Article history:
Received 16 November 2010
Received in revised form
4 January 2012
Accepted 11 January 2012
Available online 5 June 2012

Keywords:
Time-periodic linear systems
Observers
Lyapunov function
Compact embeddings

ABSTRACT

In this paper, the problem of input reconstruction for the general case of periodic linear systems driven by periodic inputs $\dot{x} = A(t)x + A_0(t)w(t)$, $y = C(t)x$ is addressed where $x(t) \in \mathbb{R}^n$ and $A(t)$, $A_0(t)$, and $C(t)$ are T_0-periodic matrices and w is a periodic signal containing an infinite number of harmonics. The contribution of this paper is the design of a real-time observer of the periodic excitation $w(t)$ using only partial measurement. The employed technique estimates the (infinite) Fourier decomposition of the signal. Although the overall system is infinite dimensional, convergence of the observer is proven using a standard Lyapunov approach along with classic mathematical tools such as Cauchy series, Parseval equality, and compact embeddings of Hilbert spaces. This observer design relies on a simple asymptotic formula that is useful for tuning finite-dimensional filters. The presented result extends recent works where full-state measurement was assumed. Here, only partial measurement, through the matrix $C(t)$, is considered.

1. Introduction

As has been discussed by numerous authors (see Bittanti and Colaneri (2009) and references therein), linear time-varying systems driven by periodic input signals are ubiquitous in control systems : from natural sciences to engineering, economics, physics and the life science. For various reasons, including disturbance rejection or output regulation (see Ichikawa & Katayama, 2006; Shim, Kim, Kim, & Black, 2010; Zhang & Serrani, 2006) and diagnosis by analysis of the trajectories, estimation of their input signals is often desirable. In the present paper, a general method to address such problems is proposed.

Consider a T_0-periodic input signal denoted w. This period T_0 is assumed to be perfectly known. As exposed in Chauvin, Corde, Petit, and Rouchon (2007) and Chauvin and Petit (2010), an easily understandable idea is to aim at reconstructing it by estimating its Fourier expansion coefficients. Previously, the case of signals w that could be written as a sum of a *finite* number of harmonics was considered in Chauvin et al. (2007). In this context, a finite-dimensional linear time-varying observer was proposed. As a natural extension, Chauvin and Petit (2010) has proposed an infinite-dimensional observer to reconstruct signals possessing an *infinite* Fourier expansion. Besides its improved generality and global convergence, this extension provides a simple asymptotic formula that, when truncated, serves as a tuning methodology for finite-dimensional filters. A first step was reached in Chauvin and Petit (2010) where an infinite-dimensional observer was proposed in the full-state measurement case. Here, the general case of partially measured systems, which is of importance for applications, is addressed.

Generally, these contributions are related to several research works found in the literature (e.g. Ding, 2001, 2006; Ichikawa & Katayama, 2006; Shim et al., 2010; Xi & Ding, 2007; Zhang & Serrani, 2006). In particular, online estimation of the frequencies of a signal being the sum of a finite number of sinusoids with unknown magnitudes, frequencies, and phases has been widely addressed earlier by numerous authors (one can refer to e.g. Hsu, Ortega, and Damn (1999), Marino and Tomei (2000) and Xia (2002)). However, the problem under consideration here is different. The signal we wish to estimate, and which is assumed to admit an infinite-dimensional Fourier decomposition, is not directly measured. It is filtered through a linear time-varying system. The filtered signal is the only available information. Secondly (and very importantly), its period is precisely known. This assumption is motivated by the examples we have put our attention on: automotive engines (see Heywood, 1988; Rizzoni, 1989; Stotsky & Kolmanovsky, 2002), and oscillating water columns (see Brook, 2003; Laitone & Wehausen, 1960; Pitt & Tucker, 2001), among other possible applications. Our method consists of reconstructing the Fourier expansion of the

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Andrea Serrani under the direction of Editor Miroslav Krstic.

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doi:10.1016/j.automatica.2012.05.020

input signal. The main difficulty lies in determining a simple and mathematically consistent method to select the gains of the infinite number of adaptation laws. The proof of this result blends arguments of Chauvin et al. (2007) (change of variables) and (Chauvin & Petit, 2010) (Rellich–Kondrachov theorem on compact imbedding for the study of LaSalle's invariant set in an infinite-dimensional case), and a new formulation of the observer.

The paper is organized as follows. In Section 2, the problem statement is exposed and the observer structure presented. Explicit computations of the observation gains are detailed. The observer convergence proof is given in Section 3. The main result is Theorem 1.

Notations. In the following, n and m are strictly positive integers, T_0 is a strictly positive real parameter, $\|\cdots\|_n$ refers to the Euclidean norm of $\mathbb{C}^n$, and $\|\cdots\|_{nm}$ refers to the Euclidean norm of $\mathcal{M}_{n,m}$ the set of $n\times m$ matrices. The symbol † indicates the Hermitian transpose. One defines

$$\begin{cases} \ell_n^2 \triangleq \left\{ \{x_k\}_{k\in\mathbb{Z}} \in (\mathbb{C}^n)^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} \|x_k\|_n^2 < +\infty \right\} \\ \omega_n^{1,2} \triangleq \left\{ \{x_k\}_{k\in\mathbb{Z}} \in (\mathbb{C}^n)^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} (1+k^2)\|x_k\|_n^2 < +\infty \right\}. \end{cases}$$

Both ℓ_n^2 and $\omega_n^{1,2}$ are Hilbert spaces with the inner product $\langle x,y\rangle_{\ell_n^2} = \sum_{k\in\mathbb{Z}} \|x_k^\dagger y_k\|_n$, and $\langle x,y\rangle_{\omega_n^{1,2}} = \sum_{k\in\mathbb{Z}}(1+k^2)\|x_k^\dagger y_k\|_n$, respectively.[2] By the same way, one defines for matrices

$$\begin{cases} \ell_{nm}^2 \triangleq \left\{ \{X_k\}_{k\in\mathbb{Z}} \in (\mathcal{M}_{nm})^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} \|X_k\|_{nm}^2 < +\infty \right\} \\ \omega_{nm}^{1,2} \triangleq \left\{ \{X_k\}_{k\in\mathbb{Z}} \in (\mathcal{M}_{nm})^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} (1+k^2)\ \|X_k\|_{nm}^2 < +\infty \right\}. \end{cases}$$

We also consider the following functional spaces (Adams, 1975, pp. 23, 60):

$$\begin{cases} L_n^2[0,T_0] \triangleq \Big\{ \{[0,T_0] \ni t \mapsto x(t) \in \mathbb{R}^n\} \\ \qquad \text{measurable over } [0,T_0] \text{ such that} \\ \qquad \int_0^{T_0} \|x(t)\|_n^2 dt < +\infty \Big\} \\ W_n^{1,2}[0,T_0] \triangleq \{\{[0,T_0] \ni t \mapsto x(t) \in \mathbb{R}^n\} \in L_n^2[0,T_0] \\ \qquad \text{such that } Dx \in L_n^2[0,T_0]\}, \end{cases}$$

where Dx is the weak derivative of x.[3] We consider the space $\mathbb{E} \triangleq \mathbb{R}^n \times \omega_m^{1,2}$ and note its elements $\mathcal{X} = (x,c)$. The considered norm on $\mathbb{E}$ is $\|\mathcal{X}\|_{\mathbb{E}}^2 = \|x\|_n^2 + \|c\|_{\omega_m^{1,2}}^2$.

2. Problem statement and observer design

Consider the following linear time-varying system driven by an unknown periodic input signal $w(t)$

$$\dot{x} = A(t)x + A_0(t)w(t), \qquad y = C(t)x$$

where the state $x(t)$ belongs to $\mathbb{R}^n$ and A, A_0, C are T_0-periodic matrices in $\mathcal{M}_{n,n}$, $\mathcal{M}_{n,m}$ and $\mathcal{M}_{p,n}$ with real entries, respectively. T_0 is assumed to be perfectly known, and the goal is to estimate the T_0-periodic input signal $t \mapsto w(t) \in \mathbb{R}^m$, with $m \triangleq \dim(w)$, $p \triangleq \dim(y)$, and $n \triangleq \dim(x)$, through its Fourier decomposition

$$w(t) \triangleq \sum_{k\in\mathbb{Z}} c_k e^{ik\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

It is assumed that $t \mapsto w(t)$ is continuous and that its derivative is piecewise continuous: w is thus KC^1. In the last expression, each vector c_k admits m complex entries. The state of this model is $\mathcal{X} = (x,c) \in \mathbb{E}$. Because w is real valued, for any $k \in \mathbb{Z}$, $c_{-k} = c_k^\dagger$. Because w is KC^1, its Fourier coefficients decay rapidly. More precisely, $c \triangleq \{c_k\}_{k\in\mathbb{Z}}$ belongs to $\omega_m^{1,2}$ as implied by Parseval equality. A simple rewriting yields

$$\begin{cases} \dot{x} = A(t)x + A_0(t)\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\omega_0 t}\right), \qquad y = C(t)x. \\ \dot{c}_k = 0, \quad \forall k \in \mathbb{Z} \end{cases} \tag{1}$$

All through the paper, the two following hypothesis (referred to H 1 and H 2, respectively, in the rest of the paper) are assumed to hold.

Hypothesis 1 (*Matrices Properties*)**.** The matrices A, A_0, C are continuous and their derivatives are piecewise continuous.

This hypothesis leads to the existence of a strictly positive real ρ_M such that for any time t,

$$\begin{cases} A^T(t)A(t) \le \rho_M^2 I_n \\ A_0^T(t)A_0(t) \le \rho_M^2 I_m \\ C(t)C^T(t) \le \rho_M^2 I_p. \end{cases}$$

Hypothesis 2 (*Zero Observability*)**.** The only solution $t \mapsto (x(t), c_k(t))$ of Eq. (1) for which the output $y(t) = C(t)x(t)$ is identically zero on $[0,T_0]$, is the zero solution ($x \equiv 0$, and $c_k \equiv 0$ for all $k \in \mathbb{Z}$).

Notice that H 2 is a relaxed version of the injectivity property assumed in Chauvin and Petit (2010), which also appears in Chauvin et al. (2007). It simply formulates an observability property of the system.

2.1. Observer structure

Corresponding to state-space model (1), one defines a time-varying Luenberger type observer

$$\begin{cases} \dot{\hat{x}} = A(t)\hat{x} + A_0(t)\left(\sum_{k\in\mathbb{Z}} \hat{c}_k e^{ik\omega_0 t}\right) \\ \qquad - (L_s(t) + L_{per}(t))(C(t)\hat{x} - y) \\ \dot{\hat{c}}_k = -L_k(t)(C(t)\hat{x} - y(t)), \quad \forall k \in \mathbb{N}, \hat{c}_{-k} = \hat{c}_k^\dagger \end{cases}$$
$$(\hat{x}(0), \hat{c}(0) \triangleq \{\hat{c}_k(0)\}_{k\in\mathbb{Z}}) \in \mathbb{E}. \tag{2}$$

The gain matrices $L_s(t)$, $L_{per}(t)$ and $\{L_k(t)\}_{k\in\mathbb{Z}}$ are T_0-periodic functions defined in the following sub-section. In (2), the roles of these tuning parameters appear distinctly. On the one hand, L_s controls the convergence rate of the error state $(x - \hat{x})$, i.e. L_s solves the estimation problem of x from y when $w \equiv 0$. The $\{L_k\}_{k\in\mathbb{Z}}$ impact on the convergence rate of the Fourier coefficient estimates. Finally, L_{per} serves to coordinate the two dynamics. Its role is underlined in the convergence analysis of the observer (2).

[2] These inner products implicitly define the norms $\|x\|_{\ell_n^2} = \sqrt{\langle x,x\rangle_{\ell_n^2}}$, and $\|x\|_{\omega_n^{1,2}} = \sqrt{\langle x,x\rangle_{\omega_n^{1,2}}}$.

[3] Again, by the same way, we define the same functional spaces for matrices in $\mathcal{M}_{nm}$ noted $L_{nm}^2[0,T_0]$ and $W_{nm}^{1,2}[0,T_0]$ respectively.

2.2. Gains design

2.2.1. Design of L_s

The zero observability hypothesis (H 2) implies that the observability Gramian of (1) on $[0, T_0]$ is definite positive. Therefore, there exists a T_0-periodic gain matrix $L_s(t)$ such that $P(t) \triangleq A(t) - L_s(t)C(t)$ is exponentially stable (see Anderson and Moore (1971) Section 14.2 and Ikeda, Maeda, and Kodama (1975), for example). A constructive choice is given, for example, by the Kalman filter. Then, we have at our disposal a continuous T_0-periodic gain $L_s(t)$ (with real entries) such as the time-periodic system $\dot{\xi} = P(t)\xi$ is exponentially stable. This means that using the gain L_s solves the real-time estimation of x from y when $w \equiv 0$.

2.2.2. Design of $\{L_k\}_{k\in\mathbb{Z}}$ and L_{per}

For $k \in \mathbb{Z}$, note $W_k(t)$ a $n \times m$ matrix with complex entries, a solution of the continuous T_0-periodic differential equation

$$\begin{cases} \dot{W}_k(t) = P(t)W_k(t) + e^{\imath k\omega_0 t}A_0(t) \\ W_k(0) = W_k(T_0). \end{cases} \tag{3}$$

In fact, the $\{W_k\}_{k\in\mathbb{Z}}$ are unique. They can be defined explicitly[4] and belong to ℓ^2_{nm} (i.e. $\exists \rho_W > 0$, $\|W_k\|^2_{\ell^2_{nm}} \le \rho^2_W$). Now, set

$$\forall k \in \mathbb{Z}, \quad L_k(t) \triangleq \frac{\alpha}{k^2+1}(C(t)W_k(t))^\dagger \tag{4}$$

where α is a strictly positive number. Then, for all t,

$$\sum_{k\in\mathbb{Z}} (1+k^2)\|L_k(t)\|^2_{mp} \le \rho_L^2 \triangleq \alpha^2\rho_M^2\rho_W^2. \tag{5}$$

This shows that $\{L_k\}_{k\in\mathbb{Z}}$ belongs to $\omega^{1,2}_{mp}$. Finally, let

$$\begin{aligned} L_{per}(t) &\triangleq \sum_{k\in\mathbb{Z}} W_k(t)L_k(t) \\ &= \sum_{k\in\mathbb{Z}} \frac{\alpha}{k^2+1} W_k(t)(C(t)W_k(t))^\dagger. \end{aligned} \tag{6}$$

From the previous inequalities, the matrix L_{per} (with real entries) is well defined.

2.3. Main result

One can now state the main contribution of this paper.

Theorem 1. *Consider (1) under hypothesis* H 1 *and* H 2. *Consider the observer (2) with gains L_s, L_{per} and L_k chosen as follows: $L_s(t)$ is a T_0-periodic gain stabilizing the pair (A, C), L_{per} is given by (3)–(6), and L_k is given by (3)–(4). Then, the error dynamics between (1) and (2) asymptotically converges to zero.*

3. Convergence proof

As a preliminary result, following the arguments in Chauvin and Petit (2010), one can easily prove existence and uniqueness of the trajectories of (1) and (2), along with the uniform continuity of the solutions. In details, one gets the following conclusion.

Proposition 1. *Consider system (2) with initial condition $\hat{\mathcal{X}}(0) = \hat{\mathcal{X}}_0 \in \mathbb{E}$. This Cauchy problem admits a unique solution over $[0, +\infty[$ in $\mathbb{E}$. Moreover, the solution is uniformly continuous.*

The following convergence proof consists of three parts. First of all, a change of variables is made. Its purpose is to make a triangular structure appear. The resulting triangular dynamics consists in a linear periodic dynamics in $\mathbb{R}^n$ and a T_0-periodic operator in $\omega^{1,2}_m$, as is established using the decay rate of Fourier coefficients of the considered input signal. Then, in a second part, focus is put on the dynamics of the periodic operator. Finally, convergence of the whole dynamics is proven using Input-to-State Stability (ISS Sontag, 1989)-like arguments. The convergence proof scheme is detailed in Fig. 1.

3.1. Changes of variables for the error dynamics

The error dynamics between (1) and (2) is, with $z \triangleq x - \hat{x}$, $z_k \triangleq c_k - \hat{c}_k$,

$$\begin{cases} \dot{z} = P(t)z - L_{per}(t)C(t)z + \sum_{k\in\mathbb{Z}} e^{\imath k\omega_0 t}A_0(t)z_k \\ \dot{z}_k = -L_k(t)C(t)z, \quad \forall k \in \mathbb{N}. \end{cases} \tag{7}$$

Following Javid (1980, 1982), a series expansion w.r.t. z_k on (7) is performed. Let $\bar{z} \triangleq z - \sum_{k\in\mathbb{Z}} W_k(t)z_k$. In detail, $\sum_{k\in\mathbb{Z}} W_k(t)z_k$ corresponds to the solution of z in (7) for $\alpha = 0$ (i.e. z_k are constants). One gets

$$\begin{cases} \dot{\bar{z}} = \dot{z} - \sum_{k\in\mathbb{Z}} \frac{d(W_k(t))}{dt} z_k - \sum_{k\in\mathbb{Z}} W_k(t)\dot{z}_k \\ \quad = P(t)z - L_{per}(t)C(t)z + \sum_{k\in\mathbb{Z}} e^{\imath k\omega_0 t}A_0(t)z_k \\ \qquad - \sum_{k\in\mathbb{Z}} \frac{d(W_k(t))}{dt} z_k + \sum_{k\in\mathbb{Z}} W_k(t)L_k(t)C(t)z \\ \quad = P(t)\bar{z}. \end{cases}$$

One can notice here the main role of the L_{per} matrix gain. It leads to a triangularization of the system, and yields a simple $\bar{z}$-dynamics, independent of the z_k-dynamics.

Then, in the $(\bar{z}, \{z_k\}_{k\in\mathbb{N}})$ coordinates, system (7) is rewritten, using (4),

$$\begin{cases} \dot{\bar{z}} = P(t)\bar{z} \\ \dot{z}_k = -\frac{\alpha}{k^2+1}\sum_{l\in\mathbb{Z}} P_k^\dagger(t)P_l(t)z_l - \frac{\alpha}{k^2+1}P_k^\dagger(t)C(t)\bar{z}, \quad \forall k \end{cases} \tag{8}$$

where $P_k(t) \triangleq C(t)W_k(t)$. Notice that $\mathscr{P}(t) \triangleq \{P_k(t)\}_{k\in\mathbb{Z}}$ belongs to ℓ^2_{nm}. This change of coordinates stresses the first part of the dynamics (8) as an exponentially stable system by the design of the matrix L_s. Establishing the stability of the second part of (8) requires further investigations. Gathering z_k for k in $\mathbb{Z}$ into $\phi(t) = \{z_k(t)\}_{k\in\mathbb{Z}} \in \omega^{1,2}_m$, $z_{-k} = z_k^\dagger$ one finally rewrites system (8) under the form

$$\begin{cases} \dot{\bar{z}} = P(t)\bar{z} \\ \dot{\phi} = -E(t)\phi - E_{\bar{z}}(t)\bar{z} \end{cases} \tag{9}$$

where $t \mapsto E(t)$ and $t \mapsto E_{\bar{z}}(t)$ are linear periodic bounded operators. For all $(k_1, k_2) \in \mathbb{Z}^2$, their components are written

$$\begin{cases} E_{k_1,k_2}(t) \triangleq \frac{\alpha}{k_1^2+1}P_{k_1}(t)^\dagger P_{k_2}(t) \\ E_{\bar{z},k_1}(t) \triangleq \frac{\alpha}{k_1^2+1}P_{k_1}^\dagger(t)C(t). \end{cases} \tag{10}$$

[4] Direct computations show that their expressions are

$$\begin{cases} W_k(t) = \int_0^t \Phi_P(t,\tau)e^{\imath k\omega_0\tau}A_0(\tau)d\tau \\ \quad +\Phi_P(t,0)(I - \Phi_P(T_0,0))^{-1}\int_0^{T_0} \Phi_P(T_0,\tau)e^{\imath k\omega_0\tau}A_0(\tau)d\tau \end{cases}$$

where $\Phi_P(t,\tau)$ denotes the state-transition matrix associated with $P(t)$. This formula is obtained by a lifting technique (Khargonekar, Poolla, & Tannenbaum, 1985; Yamamoto, 1994; Yamamoto & Araki, 1994) in continuous time, thanks to the fact that the input of (3) has the same period as the filter dynamics $P(t)$.

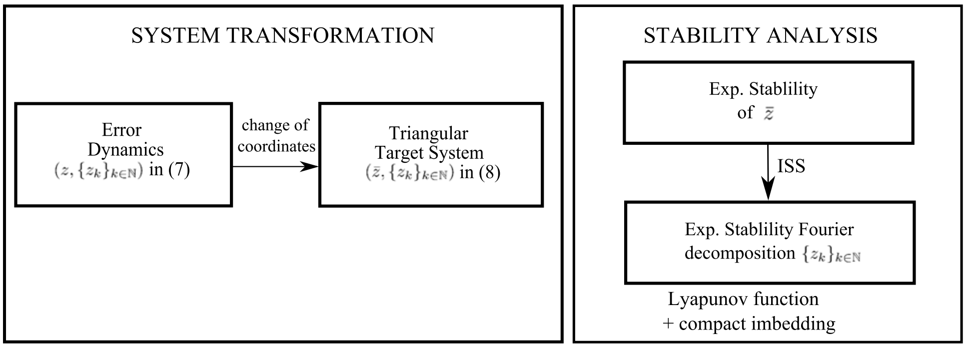

Fig. 1. Scheme of the convergence proof.

It is easy to show that E is a bounded operator, because, for all $\phi \in \omega_m^{1,2}$

$$\begin{aligned}\|E(t)\|\phi^2_{\omega_m^{1,2}} &= \sum_{k\in\mathbb{Z}}(1+k^2)\left\|\sum_{l\in\mathbb{Z}} E_{k,l}(t)\phi_l\right\|_m^2 \\ &= \sum_{k\in\mathbb{Z}}\frac{\alpha^2}{k^2+1}\left\|\sum_{l\in\mathbb{Z}} P_k(t)^\dagger P_l(t)^\dagger \phi_l\right\|_m^2 \\ &\le (\alpha\rho_M^2\rho_W^2)^2\|\phi\|^2_{\omega_m^{1,2}}.\end{aligned}$$

The second equation of (9) can then be seen as a periodic linear system driven by an exponentially decaying input. To prove the convergence of ϕ, its behavior when there is no extraneous input, i.e. when $\bar{z} \equiv 0$, is firstly studied. Then, we prove that the convergence is not impacted when the exponentially decaying extraneous input is present.

3.2. Stability of $\dot{\phi} = -E(t)\phi$

3.2.1. Lyapunov function candidate.

Let us pose

$$V(\phi) \triangleq \frac{1}{2\alpha}\|\phi\|^2_{\omega_m^{1,2}}. \tag{11}$$

Then, by derivation w.r.t. time,

$$\dot{V} = -\sum_{(k,l)\in\mathbb{Z}^2}(P_k(t)\phi_k)^\dagger(P_l(t)\phi_l).$$

Let $\psi(t,\phi(t)) \triangleq \sum_{k\in\mathbb{Z}} P_k(t)\phi_k(t)$. Thus, the previous expression can be regrouped under the following form

$$\dot{V} = -\|\psi(t,\phi(t))\|_m^2. \tag{12}$$

Then, the following properties hold: (i) $V(\phi) > 0$ for $\phi \in \omega_m^{1,2}\setminus\{0\}$ and $V(0) = 0$, (ii) V is radially unbounded, and (iii) $\dot{V}(\phi) \le 0$ for $\phi \in \omega_m^{1,2}$.

Further, V is decreasing and lower-bounded by 0. Therefore, it has a limit as t goes to $+\infty$. As V is decreasing, for all $t \in \mathbb{R}^+$, the state ϕ is then bounded. Moreover, as shown in Proposition 1, ϕ is uniformly continuous (because $\dot{\phi}$ is bounded). Furthermore, the operator $t \mapsto \mathcal{P}(t) = \{P_k(t)\}_{k\in\mathbb{Z}}$ is a continuous T_0-periodic operator in ℓ^2_{nm}. This yields the uniform continuity of ψ and thus the uniform continuity of $\dot{V}$ on $[0,+\infty[$. Finally, and classically, by Barbalat's lemma (see e.g. Popov, 1973)

$$-\lim_{t\to\infty}\dot{V}(t) = \lim_{t\to\infty}\|\psi(t,\phi(t))\|_m^2 = 0.$$

Further, since

$$\int_0^t \|\psi(\tau,\phi(\tau))\|_m^2 d\tau = V(\phi(0)) - V(\phi(t)) \le V(\phi(0))$$

for $t \ge 0$, the mapping $t \mapsto \psi(t,\phi(t))$ is square integrable. This result will be important in the following discussion. Let us note $\Omega^+ \triangleq \{\bar{\phi} \in \ell_m^2 / \exists (t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$ s.t. $\|\phi(t_l) - \bar{\phi}\|_{\ell_m^2} \to 0$ as $l \to +\infty\}$ the positive limit set of ϕ. We now establish that $\phi(t)$ converges to Ω^+ when $t \to +\infty$. Then, we prove that this set is reduced to $0_{\ell_m^2}$.

3.2.2. $\phi(t)$ converges to Ω^+ when $t \to +\infty$

A fundamental property of the considered functional spaces is that $W_n^{1,2}[0,T_0]$ is compactly imbedded into $L_n^2[0,T_0]$. This is a consequence of the Rellich–Kondrachov theorem (see Adams (1975, page 168)). This property can be transposed to the sequence spaces $\omega_n^{1,2}$ and ℓ_n^2. Indeed, $\omega_n^{1,2}$ is compactly imbedded into ℓ_n^2. Non-emptiness and convergence toward Ω^+ is detailed in Chauvin and Petit (2010) using a contradiction argument.

3.2.3. The set Ω^+ equals $0_{\ell_m^2}$

To prove that $\Omega^+ = 0_{\ell_m^2}$, we take an element $\bar{\phi} \in \Omega^+$ and prove that it can only be $0_{\ell_m^2}$.

For all $\bar{\phi} \in \Omega^+$, there exists, by definition, a sequence $(t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$ such that $\|\phi(t_l) - \bar{\phi}\|_{\ell_m^2} \to 0$ as $l \to +\infty$. We thus define the functions[5]

$$\begin{cases}\psi(t) \triangleq \sum_{k\in\mathbb{Z}} P_k(t)\phi_k(t), \\ \psi_l(t) \triangleq \sum_{k\in\mathbb{Z}} P_k(t)\phi_k(t_l), \quad \text{for all } l \in \mathbb{N} \\ \bar{\psi}(t) \triangleq \sum_{k\in\mathbb{Z}} P_k(t)\bar{\phi}_k.\end{cases} \tag{13}$$

Step 1. $\lim_{l\to+\infty}\|\psi(t_l) - \bar{\psi}\|_{L^2_m[0,T_0]} = 0$. The difference $\bar{\psi} - \psi_l$ is written

$$\bar{\psi}(t) - \psi_l(t) = \sum_{k\in\mathbb{Z}} P_k(t)(\bar{\phi}_k - \phi_k(t_l)).$$

Thus,

$$\begin{aligned}\|\bar{\psi}(t) - \psi_l(t)\|_m^2 &= \left\|\sum_{k\in\mathbb{Z}} P_k(t)(\bar{\phi}_k - \phi_k(t_l))\right\|_m^2 \\ &\le \left(\sum_{k\in\mathbb{Z}}\|P_k(t)\|_{pm}^2 dt\right)(\|\bar{\phi} - \phi(t_l)\|^2_{\ell_m^2}).\end{aligned}$$

Using (5), one has $\sum_{k\in\mathbb{Z}}\|P_k(t)\|_{pm}^2 < \rho_W^2\rho_M^2$. Thus, the following lemma holds.

[5] These functions are similar to inverse Fourier transforms of the sequences $\{\phi_k(t_l)\}$ and $\bar{\phi}_k$ respectively.

Lemma 1. *Consider the functions ψ_l and $\bar{\psi}$ as defined in (13); then, for any sequence $(t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$, one has $\|\bar{\psi} - \psi(t_l)\|_{L^2_m[0,T_0]} \to 0$ as $l \to +\infty$.*

Step 2. $\lim_{l\to+\infty} \int_{t_l}^{t_l+T_0} \|\psi_l(t)\|_m^2 dt = 0$. First, for all $l \in \mathbb{N}$, one has

$$\int_{t_l}^{t_l+T_0} \|\psi_l(t)\|_m^2 dt \le 2\int_{t_l}^{t_l+T_0} \|\psi(t)\|_m^2 dt + 2\int_{t_l}^{t_l+T_0} \|\psi(t) - \psi_l(t)\|_m^2 dt.$$

Now, let us focus on the second term

$$\psi(t) - \psi_l(t) = \sum_{k\in\mathbb{Z}} P_k(t)(\phi_k(t) - \phi_k(t_l)) = -\sum_{k\in\mathbb{Z}} \frac{\alpha}{k^2+1} P_k(t) \int_{t_l}^{t} P_k^{\dagger}(u)\psi(u)du.$$

By majoration, for all t in $[t_l, t_l + T_0]$,

$$\|\psi(t) - \psi_l(t)\|_m^2 \le \left(\sum_{k\in\mathbb{Z}} \alpha^2 \|P_k(t)\|_{pm}^2\right) \times \left(\sum_{k\in\mathbb{Z}} \left\| \int_{t_l}^{t} P_{k(u)}^{\dagger} \psi(u)du \right\|_m^2\right).$$

Then, by integration,

$$\int_{t_l}^{t_l+T_0} \|\psi(t) - \psi_l(t)\|_m^2 dt \le \rho_\psi^2 \int_{t_l}^{t_l+T_0} \|\psi(t)\|_m^2 dt$$

with $\rho_\psi \triangleq \sqrt{T_0}\alpha\rho_W^2\rho_M^2$. Moreover, the square integrability of ψ implies that $\int_{t_l}^{t_l+T_0} \|\psi(t)\|_m^2 dt$ converges to 0 when l goes to ∞. Then, it follows that

$$\lim_{l\to+\infty} \int_{t_l}^{t_l+T_0} \|\psi_l(t)\|_m^2 dt = 0. \tag{14}$$

Step 3. $\|\bar{\psi}(t)\|_{L^2_m[0,T_0]} = 0$. Secondly,

$$\int_{t_l}^{t_l+T_0} \|\bar{\psi}(t)\|_m^2 dt \le 2\int_{t_l}^{t_l+T_0} \|\bar{\psi}(t) - \psi_l(t)\|_m^2 dt + 2\int_{t_l}^{t_l+T_0} \|\psi_l(t)\|_m^2 dt.$$

Therefore, from the previous majoration and Lemma 1, one has

$$\lim_{l\to+\infty} \int_{t_l}^{t_l+T_0} \|\bar{\psi}(t)\|_m^2 dt = 0.$$

As $\bar{\psi}$ is independent of l, one has $\int_0^{T_0} \|\bar{\psi}(t)\|_m^2 dt = 0$.

Step 4. $\bar{\phi} = 0_{\ell^2_m}$. By continuity of the integrand, one deduces from the preceding limit that, for all $t \in [0, T_0]$,

$$\left\| \sum_{k\in\mathbb{Z}} \frac{\alpha}{k^2+1} P_k(t)\bar{\phi}_k \right\| = 0.$$

It follows that $\bar{\phi} = 0_{\ell^2_m}$. Yet, necessarily, otherwise H 2 would be violated, for any $\bar{\Gamma}$ solution of $\|\sum_{k\in\mathbb{Z}} \frac{\alpha}{k^2+1} P_k(t)\bar{\Gamma}_k\| = 0$ for all $t \in [0, T_0]$, $x = \sum_{k\in\mathbb{Z}} W_k(t)\frac{\alpha}{k^2+1}\bar{\Gamma}_k$ and $c_k = \frac{\alpha}{k^2+1}\bar{\Gamma}_k$,[6] is a solution of (1) with $y = Cx \equiv 0$ and the following lemma holds.

Lemma 2. *The dynamics $\dot{\phi} = -E(t)\phi$ uniformly asymptotically converges toward $0_{\ell^2_m}$.*

[6] As $\bar{\Gamma}$ is a solution in ℓ^2_m, $\{\frac{\alpha}{k^2+1}\bar{\Gamma}_k\}_{k\in\mathbb{Z}}$ is in $\omega_m^{1,2}$.

3.3. Exponential stability of $\dot{\phi} = -E(t)\phi - E_{\bar{z}}(t)\bar{z}(t)$

The solution of (9) is written

$$\phi(t) = \Gamma(t, 0)\phi(0) - \int_0^t \Gamma(t, \tau)E_{\bar{z}}(\tau)\bar{z}(\tau)d\tau$$

where $\Gamma(t, \tau)$ is the transition operator of the system $\dot{\phi} = -E(t)\phi$ between τ and t. Because the system is linear and periodic, its uniform asymptotic convergence, which is established in Lemma 2 is equivalent to its exponential convergence. Therefore $\exists(k_1, k_2) > 0, \forall(t, \tau) \ge 0, \forall\phi_0 \in \ell^2_m \|\Gamma(t, \tau)\phi_0\|_{\ell^2_m} \le k_1 e^{-k_2(t-\tau)}\|\phi_0\|_{\ell^2_m}$. This allows us to use the same proof technique as usually employed in Input-to-State Stability (ISS) (Sontag, 1989). It leads to the convergence of $\|\phi(t)\|_{\ell^2_m}$ towards 0 when t goes to $+\infty$. All the changes of coordinates are linear, time-periodic and smooth, and thus uniformly continuous; therefore Theorem 1 is proven.

Acknowledgments

The authors are indebted to the reviewers for their insightful and constructive remarks and comments.

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