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Brief paper

Reconstruction of the Fourier expansion of inputs of linear time-varying systems[☆]

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ABSTRACT

In this paper we propose a general method to estimate periodic unknown input signals of finite-dimensional linear time-varying systems. We present an infinite-dimensional observer that reconstructs the coefficients of the Fourier decomposition of such systems. Although the overall system is infinite dimensional, convergence of the observer can be proven using a standard Lyapunov approach along with classic mathematical tools such as Cauchy series, Parseval equality, and compact embeddings of Hilbert spaces. Besides its low computational complexity and global convergence, this observer has the advantage of providing a simple asymptotic formula that is useful for tuning finite-dimensional filters. Two illustrative examples are presented.

1. Introduction

Linear time-varying systems driven by periodic input signals are ubiquitous in control systems. For various reasons, including disturbance rejection and diagnosis by analysis of the trajectories, estimation of their input signals is often desirable. In the present paper we propose a general method to address such problems.

Consider a T_0-periodic input signal denoted w. An easily understandable idea is to aim at reconstructing it by estimating its Fourier expansion coefficients. We assume that the period T_0 is perfectly known. Previously, the case of signals w that could be written as a sum of a *finite* number of harmonics was considered in Chauvin, Corde, Petit, and Rouchon (2007). In this context, a finite-dimensional linear time-varying observer was proposed. As a natural extension, we propose here an infinite-dimensional observer to reconstruct signals possessing an *infinite* Fourier expansion. Besides its improved generality and global convergence, this extension provides a simple asymptotic formula that, when truncated, serves as a tuning methodology for finite-dimensional filters.

This contribution is related to several research works found in the literature. Online estimation of the frequencies of a signal being the sum of a finite number of sinusoids with unknown magnitudes, frequencies, and phases has been addressed by numerous authors (one can refer to e.g. Hsu, Ortega, & Damn, 1999; Marino & Tomei, 2000; Xia, 2002). However, the problem we address is different. The signal we wish to estimate, and which is assumed to admit an infinite-dimensional Fourier decomposition, is not directly measured. It is filtered through a linear time-varying system. The filtered signal is the only available information. Secondly (and very importantly), its period is precisely known. This particularity suggests that a dedicated observation technique could be worth developing. Our approach can be considered close to the general class of methods aiming at identifying periodic disturbances in view of canceling them. Sinusoidal signals can be modeled as the output of linear exosystems (Ding, 2006). Recent progress has been made in rejecting such disturbances (Ding, 2001). Such approaches have been extended to general periodic signals (see e.g. Xi & Ding, 2007 and the references therein). Lately, in Ding (2006), estimation (and control for rejection) of general periodical disturbance has been addressed by exploiting their integral properties. Our method follows along different lines. It is focused on adapting a Fourier expansion of the signal. The main difficulty lies in determining a simple and mathematically consistent method to tune the gains of the infinite number of adaptation laws. As will appear, a simple solution is found.

The paper is organized as follows. In Section 2 we state the problem, present the observer structure and describe the explicit computations of the observation gains. The observer convergence proof is provided in Section 3. This proof relies on a Lyapunov analysis and uses Cauchy series, Parseval equality and compact embeddings of Hilbert spaces. The main result of this section is Proposition 3. For illustration, we propose two examples. First a simple automotive engine application is proposed in Section 4.1.

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We address the estimation of the (periodic) combustion torque of an automotive compression–ignition engine in real time from instantaneous crankshaft speed measurements (as considered in Rizzoni (1989)). Actual experimental estimation results for an engine are presented. The computational burden of our proposed technique is much lower than that for an extended Kalman filter. Then, in Section 4.2, we study a (celestial mechanics) perturbed two-body problem. By means of a classic perturbation analysis (Encke's method Bate, Mueller, & White, 1971), we show how to recover the perturbation acceleration generated by a remote attracting mass on an orbiting body. In this example, the input signal to recover has sharp transients, which implies that numerous harmonics must be considered to represent it well. We show that the calibration effort of our observer remains very low, thanks to the proposed tuning formula stemming from the convergence study performed in Section 3. Future work will consider the case of partial state measurement (as considered in a *finite-dimensional* case in Chauvin et al. (2007)). We expect that, as in Chauvin et al. (2007), averaging techniques will allow investigation of the convergence.

2. Statement of the problem and observer design

Notations

In the following, n and m are strictly positive integers, T_0 is a strictly positive real parameter, $\|\cdots\|_n$ refers to the Euclidean norm of $\mathbb{C}^n$, and $\|\cdots\|_{nm}$ refers to the Euclidean norm of $\mathcal{M}_{n,m}(\mathbb{R})$ the set of $n \times m$ matrices with real entries. The symbol † indicates the Hermitian transpose.

We define

$$\begin{cases} \ell_n^2 \triangleq \left\{ \{x_k\}_{k\in\mathbb{Z}} \in (\mathbb{C}^n)^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} \|x_k\|_n^2 < +\infty \right\} \\ \omega_n^{1,2} \triangleq \left\{ \{x_k\}_{k\in\mathbb{Z}} \in (\mathbb{C}^n)^{\mathbb{Z}} \Big/ \sum_{k\in\mathbb{Z}} (1+k^2) \|x_k\|_n^2 < +\infty \right\}. \end{cases}$$

Both ℓ_n^2 and $\omega_n^{1,2}$ are Hilbert spaces with the inner product $\langle x, y\rangle_{\ell_n^2} = \sum_{k\in\mathbb{Z}} \left\| x_k^\dagger y_k \right\|_n^2$, and $\langle x, y\rangle_{\omega_n^{1,2}} = \sum_{k\in\mathbb{Z}} (1+k^2) \left\| x_k^\dagger y_k \right\|_n^2$, respectively.[1]

We also consider the following functional spaces Adams (1975, pp. 23, 60):

$$\begin{cases} L_n^2[0, T_0] \triangleq \left\{ \{[0, T_0] \ni t \mapsto x(t) \in \mathbb{R}^n\} \right. \\ \qquad \left. \text{measurable over } [0, T_0] \text{ such that } \int_0^{T_0} \|x(t)\|_n^2 \,\mathrm{d}t < +\infty \right\} \\ W_n^{1,2}[0, T_0] \triangleq \left\{ \{[0, T_0] \ni t \mapsto x(t) \in \mathbb{R}^n\} \in L_n^2[0, T_0] \right. \\ \quad \left. \text{such that } Dx \in L_n^2[0, T_0] \right\}, \end{cases}$$

where Dx is the weak derivative of x.

As will become apparent, the functions considered in this paper have continuous partial derivatives (in the classical sense).

Again, both $L_n^2[0, T_0]$ and $W_n^{1,2}[0, T_0]$ are Hilbert spaces. Moreover, $W_n^{1,2}[0, T_0]$ is a Sobolev space.

We consider the space $\mathbb{E} \triangleq \mathbb{R}^n \times \omega_m^{1,2}$ and note its elements $\mathcal{X} = (x, c)$. The norm on $\mathbb{E}$ we consider is $\|\mathcal{X}\|_{\mathbb{E}}^2 = \|x\|_n^2 + \|c\|_{\omega_m^{1,2}}^2$.

2.1. Estimation problem and definitions

Consider the following linear time-varying system driven by an unknown periodic input signal $w(t)$:

$$\dot{x} = A(t)x + A_0(t)w(t), \qquad y = C(t)x,$$

where the state $x(t)$ and the output $y(t)$ belong to $\mathbb{R}^n$ and $A(t)$, $A_0(t)$, $C(t)$ are continuous matrices in $\mathcal{M}_{n,n}(\mathbb{R})$, $\mathcal{M}_{n,m}(\mathbb{R})$ and $\mathcal{M}_{n,n}(\mathbb{R})$, respectively, with entries that are uniformly bounded (not necessary periodic), locally integrable functions of t. The matrix $A_0(t)$ has T_0-periodic coefficients. We assume that T_0 is perfectly known, and we want to estimate the T_0-periodic KC^1 (continuous and with piecewise continuous derivative) input signal $t \mapsto w(t) \in \mathbb{R}^m$, with $m = \dim(w) \leq n = \dim(y) = \dim(x)$, through its Fourier decomposition[2]:

$$w(t) \triangleq \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

In the last expression, each vector c_k admits m complex entries. The state of this model is $\mathcal{X} = (x, c) \in \mathbb{E}$. Because w is real valued, for any $k \in \mathbb{Z}$, $c_{-k} = c_k^\dagger$. Because w is KC^1, $c \triangleq \{c_k\}_{k\in\mathbb{Z}}$ belongs to $\omega_m^{1,2}$ (as implied by Parseval equality, $\|c\|_{\ell_m^2}^2 = \frac{1}{T_0} \|w\|_{L_m^2[0,T_0]}^2$, and $\|c\|_{\omega_m^{1,2}}^2 = \frac{1}{T_0} \left((1 - \frac{1}{w_0^2}) \|w\|_{L_m^2[0,T_0]}^2 + \frac{1}{w_0^2} \|w\|_{W_m^{1,2}[0,T_0]}^2 \right)$. Simple rewriting yields:

$$\begin{cases} \dot{x} = A(t)x + A_0(t) \left(\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t} \right), \qquad y = C(t)x \\ \dot{c}_k = 0, \quad \forall k \in \mathbb{Z} \end{cases} \tag{1}$$

Furthermore, following (Chauvin et al., 2007), we make some general assumptions.

H 1. *We assume that there exist two strictly positive numbers (ρ_m, ρ_M) such that, for all $t \geq 0$:*

$$\begin{cases} A^T(t)A(t) \leq \rho_M^2 I_n \\ \rho_m^2 I_m \leq A_0^T(t)A_0(t) \leq \rho_M^2 I_m \\ \rho_m^2 I_n \leq C(t)C^T(t) \leq \rho_M^2 I_n. \end{cases} \tag{2}$$

In particular, we can deduce from H1 that $A_0(t)$ has a non-singular pseudo-inverse.[3]

2.2. Observer definition

Corresponding to state-space model (1), we define a time-varying Luenberger type observer:

$$\begin{cases} \dot{\hat{x}} = A(t)\hat{x} + A_0(t) \left(\sum_{k\in\mathbb{Z}} \hat{c}_k \mathrm{e}^{\imath k\omega_0 t} \right) - L(t)(C(t)\hat{x} - y) \\ \dot{\hat{c}}_k = -\mathrm{e}^{-\imath k\omega_0 t} L_k(t)(C(t)\hat{x} - y(t)), \quad \forall k \in \mathbb{Z} \\ \left(\hat{x}(0), \hat{c}(0) \triangleq \{\hat{c}_k(0)\}_{k\in\mathbb{Z}}\right) \in \mathbb{E}. \end{cases} \tag{3}$$

The state is $\hat{\mathcal{X}} \triangleq (\hat{x}, \hat{c}) \in \mathbb{E}$. The gain matrices $L(t)$ (with real entries) and $\{L_k\}_{k\in\mathbb{Z}}$ (with complex entries) are defined in the following subsection [see (4) and (5)].

[1] These inner products implicitly define the norms $\|x\|_{\ell_n^2} = \langle x, x\rangle_{\ell_n^2}$, and $\|x\|_{\omega_n^{1,2}} = \langle x, x\rangle_{\omega_n^{1,2}}$.

[2] This is how we account for the periodic nature of the signal w. By making this decomposition early in the study, we are immediately left with an infinite number of variables. This method is quite different from the control and observation approach developed in Vazquez and Krstic (2008) where discretizations and such decompositions would be postponed as much as possible, in order to handle functions instead of infinite sets of variables.

[3] The interested reader might notice that this assumption would enable a least-square approach to estimate $w(t)$ from past measurements of x (from the inversibility of C) through the differential equation (1). Such methods, which usually do not explicitly take advantage of the periodic nature of the signal $w(t)$, are out of the scope of the paper but would be totally relevant here, especially if they are appropriately tuned to account for noises.

2.3. Design of L and $\{L_k\}_{k\in\mathbb{Z}}$

By assumption (2), for all $t \geq 0$, $C(t)$ is a square invertible matrix. Let H be a Hurwitz matrix in $\mathcal{M}_{n,n}(\mathbb{R})$. We set

$$L(t) \triangleq (A(t) - H)C^{-1}(t). \tag{4}$$

$\mathbb{R}^+ \ni t \mapsto L(t) \in \mathcal{M}_{n,n}(\mathbb{R})$ is a bounded function because $t \mapsto A(t)$; H and $t \mapsto C^{-1}(t)$ are also bounded. Consider P the unique symmetric definite solution in $\mathcal{M}_{n,n}(\mathbb{R})$ of the Lyapunov equation $PH + H^TP = -I_n$. We use P to compute the observer gains as follows: for all $k \in \mathbb{Z}$:

$$L_k(t) \triangleq \frac{\alpha}{k^2+1}A_0^T(t)PC(t)^{-1}, \tag{5}$$

where α is a strictly positive constant. Let $\tilde{\mathcal{X}} \triangleq \mathcal{X} - \hat{\mathcal{X}} \triangleq (\tilde{x}, \tilde{c}) \in \mathbb{E}$. The error dynamics is:

$$\begin{cases} \dot{\tilde{x}} = H\tilde{x} + A_0(t)\left(\sum_{k\in\mathbb{Z}} \tilde{c}_k e^{ik\omega_0 t}\right) \\ \dot{\tilde{c}}_k = -\dfrac{\alpha}{k^2+1}e^{-ik\omega_0 t}A_0^T(t)P\tilde{x}, \quad \forall k \in \mathbb{N}, \\ \text{where } (\tilde{x}(0), \tilde{c}(0)) \in \mathbb{E}. \end{cases} \tag{6}$$

Under this form, the roles of the tuning parameters (the Hurwitz matrix H, and the strictly positive constant α) are distinct. H controls the convergence rate of the error state $\tilde{x}$, and α impacts on the convergence of the Fourier coefficient estimates.

3. Convergence proof

In this section, we prove the convergence of (6). This dynamics is infinite dimensional. In contrast to finite-dimensional cases, the existence, uniqueness and precompactness of trajectories need to be proven before using LaSalle's theorem (e.g., Coron, 2007; Coron & d'Andréa Novel, 1998; Dafermos & Slemrod, 1973; Luo, Guo, & Morgül, 1999). We follow a different path requiring only elementary arguments. First, we prove that (6) defines a unique solution $\tilde{\mathcal{X}}$ valued in $\mathbb{E}$ over the time range $[0, +\infty[$. Then we prove that $\tilde{\mathcal{X}}$ asymptotically converges towards a limit $\bar{\mathcal{X}}$. We conclude by proving that this limit is $\bar{\mathcal{X}} = 0_{\mathbb{R}^n\times\ell_m^2}$.

3.1. Existence and uniqueness of the error system trajectories

Because c is constant, the existence and uniqueness of the trajectories of the reference system (1) can be proven by showing that the function $\mathbb{R}\times\mathbb{R}^n \ni (t, x) \mapsto f(t, x) = A(t)x + A_0(t)w(t) \in \mathbb{R}^n$ is uniformly Lipschitz over $\mathbb{R}^n$ using the Cauchy Lipschitz theorem on $\mathbb{R}^n$ as explained in Coddington and Levinson (1955).

Similarly, the existence and uniqueness of trajectories of the error dynamics (6) can be proven by showing that its right-hand side is uniformly Lipschitz over $\mathbb{E}$. For this purpose, we note

$$\begin{aligned} &(\mathbb{R}, \mathbb{E}) \ni (t, \mathcal{X} = (x, c)) \mapsto f_x(t, \mathcal{X}) \\ &f_x(t, \mathcal{X}) \triangleq Hx + A_0(t)\left(\sum_{k\in\mathbb{Z}} c_k e^{ik\omega_0 t}\right) \in \mathbb{R}^n \end{aligned} \tag{7}$$

$$\begin{aligned} &(\mathbb{R}, \mathbb{E}) \ni (t, \mathcal{X} = (x, c)) \mapsto f_c(t, \mathcal{X}) \\ &f_c(t, \mathcal{X}) \triangleq \{-\frac{\alpha}{k^2+1}e^{-ik\omega_0 t}A_0^T(t)Px\}_{k\in\mathbb{Z}} \in \omega_m^{1,2} \end{aligned} \tag{8}$$

and

$$(\mathbb{R}, \mathbb{E}) \ni (t, \mathcal{X} = (x, c)) \mapsto f_{\mathcal{X}}(t, \mathcal{X}) \triangleq \begin{pmatrix} f_x(t, \mathcal{X}) \\ f_c(t, \mathcal{X}) \end{pmatrix} \in \mathbb{E}. \tag{9}$$

We first focus on f_x. Note

$$\kappa \triangleq \sqrt{\sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}}$$

For all $(\mathcal{X}_1, \mathcal{X}_2) \in \mathbb{E}^2$, and all $t \in \mathbb{R}$:

$$\begin{aligned} &\|f_x(t, \mathcal{X}_1) - f_x(t, \mathcal{X}_2)\|_n^2 \\ &\quad \leq 2\,\|H\|_{nn}^2\,\|x_1 - x_2\|_n^2 + 2\,\|A_0(t)\|_{nm}^2 \left\| \sum_{k\in\mathbb{Z}} (c_{1,k} - c_{2,k}) e^{ik\omega_0 t} \right\|_m^2 \\ &\quad \leq 2\,\|H\|_{nn}^2\,\|x_1 - x_2\|_n^2 + 2\rho_M^2\kappa^2\,\|c_1 - c_2\|_{\omega_m^{1,2}}. \end{aligned}$$

Consider $\rho_x \triangleq 2\max\{\|H\|_{nn}, \kappa\rho_M\}$. It follows that $\|f_x(t, \mathcal{X}_1) - f_x(t, \mathcal{X}_2)\|_n \leq \rho_x\,\|\mathcal{X}_1 - \mathcal{X}_2\|_{\mathbb{E}}$ and the following result holds.

Lemma 1. *For all t, the function $\mathbb{E} \ni \mathcal{X} \mapsto f_x(t, \mathcal{X}) \in \mathbb{R}^n$ is (globally) ρ_x-Lipschitz over $\mathbb{E}$, where ρ_x is a constant (independent of t).*

Now we focus on f_c. For all $(\mathcal{X}_1, \mathcal{X}_2) \in \mathbb{E}^2$ and all $t \in \mathbb{R}$:

$$\|f_c(t, \mathcal{X}_1) - f_c(t, \mathcal{X}_2)\|_{\omega_m^{1,2}}^2 \leq \alpha^2\kappa^2\,\left\|A_0^T(t)P\right\|_{mn}^2\,\|x_1 - x_2\|_n^2.$$

We define $\rho_c \triangleq \alpha\rho_M\,\|P\|_{nn}\,\kappa$. We thus have

$$\|f_c(t, \mathcal{X}_1) - f_c(t, \mathcal{X}_2)\|_{\omega_m^{1,2}} \leq \rho_c\,\|\mathcal{X}_1 - \mathcal{X}_2\|_{\mathbb{E}}$$

and the following result holds.

Lemma 2. *For all t, the function $\mathbb{E} \ni \mathcal{X} \mapsto f_c(t, \mathcal{X}) \in \omega_m^{1,2}$ is (globally) ρ_c-Lipschitz over $\mathbb{E}$, where ρ_c is a constant (independent of t).*

The error dynamics (6) can be written as $\dot{\tilde{\mathcal{X}}} = f_{\mathcal{X}}(t, \tilde{\mathcal{X}})$. $\mathbb{E}$ is a Banach space, and $f_{\mathcal{X}}$ is $\rho_{\mathcal{X}}$-Lipschitz over $\mathbb{E}$ (with $\rho_{\mathcal{X}} \triangleq \max\{\rho_x, \rho_c\}$ by Lemmas 1 and 2), where $\rho_{\mathcal{X}}$ is a constant (independent of t). This allows us to conclude that the solution of (6) exists and is unique by Brézis (1983, Theorem VII.3) (see also Aubin, 2000). The regularity mentioned in the above proposition follows from the same theorem. System (3) shares the same properties.

Proposition 1. *Consider system (6) with initial condition $\tilde{\mathcal{X}}(0) = \tilde{\mathcal{X}}_0 \in \mathbb{E}$. This Cauchy problem admits a unique solution over $[0, +\infty[$. This solution continuously depends on the initial condition $\tilde{\mathcal{X}}_0$.*

3.2. Definition of a Lyapunov function

We define

$$V(\tilde{\mathcal{X}}) \triangleq \tilde{x}^TP\tilde{x} + \frac{1}{\alpha}\,\|\tilde{c}\|_{\omega_m^{1,2}}^2. \tag{10}$$

With respect to the norm on $\mathbb{E}$, the function V is radially unbounded. By derivation with respect to t, we obtain

$$\dot{V} = -\,\|\tilde{x}\|_n^2.$$

The function V defined by (10) is a Lyapunov function because it is continuously differentiable and satisfies

$$\begin{cases} V(0) = 0 \\ V(\tilde{\mathcal{X}}) > 0 \quad \text{for } \tilde{\mathcal{X}} \in \mathbb{E} \setminus \{0\} \\ \dot{V}(\tilde{\mathcal{X}}) \leq 0 \quad \text{in } \mathbb{E}. \end{cases}$$

Furthermore, V is decreasing and bounded below by 0. Then, V has a limit as t tends to ∞. Moreover, because V is decreasing, for all $t \in \mathbb{R}^+$, $V(\tilde{\mathcal{X}}(t)) \leq V(\tilde{\mathcal{X}}(0))$. The state $\tilde{\mathcal{X}}$ is then bounded. In addition, $f_{\mathcal{X}}$ is $\rho_{\mathcal{X}}$-Lipschitz over $\mathbb{E}$, where $\rho_{\mathcal{X}}$ is a constant (independent of t), then $\dot{\tilde{\mathcal{X}}}$ is bounded, and $\tilde{\mathcal{X}}$ is thus uniformly continuous.

By Barbalat's lemma (e.g., Popov, 1973)

$$\lim_{t\to\infty} \dot{V}(t) = -\lim_{t\to\infty} \left\|\tilde{x}(t)\right\|_n^2 = 0$$

because V and $\dot{V}$ are uniformly continuous on $[0, +\infty[$ from the uniform continuity of $\tilde{\mathcal{X}}$. Since $\int_0^t \left\|\tilde{x}(\tau)\right\|_n^2 \mathrm{d}\tau = V(\tilde{\mathcal{X}}(0)) - V(\tilde{\mathcal{X}}(t)) \le V(\tilde{\mathcal{X}}(0))$ for $t \ge 0$, the mapping $t \mapsto \tilde{x}(t)$ belongs to $L_n^2[0, +\infty[$. We can state the following result.

Proposition 2. *Consider $\tilde{\mathcal{X}} = (\tilde{x}, \tilde{c})$ the state of the error dynamics (6). Both $t \mapsto \tilde{x}(t)$ and $t \mapsto \tilde{c}(t)$ are bounded in $\mathbb{R}^n$ and $\omega_m^{1,2}$, respectively. Furthermore, the mapping $t \mapsto \tilde{x}(t)$ is uniformly continuous, belongs to $L_n^2[0, +\infty[$, and $\lim_{t\to+\infty} \tilde{x}(t) = 0$.*

3.3. State convergence

We denote $\Omega^+ \triangleq \{\bar{c} \in \ell_m^2 \,/\, \exists (t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$ s.t. $\left\|\tilde{c}(t_l) - \bar{c}\right\|_{\ell_m^2} \to 0$ as $l \to +\infty\}$ as the positive limit set of $\tilde{c}$. In this subsection, we first prove that $\tilde{c}(t)$ converges to Ω^+ when $t \to +\infty$. Then we prove that this set is reduced to $0_{\ell_m^2}$.

3.3.1. $\tilde{c}(t)$ converges to Ω^+ when $t \to +\infty$

A fundamental property is that $W_n^{1,2}[0, T_0]$ is compactly embedded in $L_n^2[0, T_0]$. This is a consequence of the Rellich–Kondrachov theorem Adams (1975, p. 168).[4] This property can be transposed to the sequence spaces $\omega_n^{1,2}$ and ℓ_n^2. Indeed, $\omega_n^{1,2}$ is compactly embedded in ℓ_n^2. This point is detailed in the Appendix.

As is proved in the Appendix, $\omega_m^{1,2}$ is compactly embedded in ℓ_m^2, i.e., $\omega_m^{1,2} \hookrightarrow \ell_m^2$. Since the mapping $\{t \to \tilde{c}(t)\}$ is bounded in $\omega_m^{1,2}$, it is possible to construct a Cauchy sequence in ℓ_m^2 that is convergent (because ℓ_m^2 is a Hilbert space) w.r.t. the ℓ_m^2-norm. Hence, the positive limit set Ω^+ is non-empty. To show that $\tilde{c}(t)$ converges to Ω^+ as $t \to +\infty$, we use a contradiction argument. Suppose that this is not the case; then there is an $\epsilon > 0$ and a sequence $(t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$ such that $\mathrm{dist}(\tilde{c}(t_l), \Omega^+)_{\ell_m^2} > \epsilon$, where dist is the distance defined by the ℓ_m^2-norm. Since the sequence $\tilde{c}(t_l)$ is also bounded in $\omega_m^{1,2}$, we can deduce from the compact injection property ($\omega_m^{1,2} \hookrightarrow \ell_m^2$) that there exists $\bar{c}^\star \in \ell_m^2$ and a subsequence $(t'_l)_{l\in\mathbb{N}}$ of $(t_l)_{l\in\mathbb{N}}$ such that $\left\|\tilde{c}(t'_l) - \bar{c}^\star\right\|_{\ell_m^2} \to 0$ as $l \to +\infty$. By definition, the point $\bar{c}^\star$ belongs to Ω^+ and at the same time must be at a distance ϵ from Ω^+, i.e., $\mathrm{dist}(\bar{c}^\star, \Omega^+)_{\ell_m^2} \ge \epsilon$, which is a contradiction. Thus, the following lemma holds.

Lemma 3. *Consider $\tilde{\mathcal{X}} = (\tilde{x}, \tilde{c})$ the state of the error dynamics (6). Its positive limit set $0_{\mathbb{R}^n} \times \Omega^+$ is non-empty. Furthermore, $\tilde{\mathcal{X}}(t)$ approaches $0_{\mathbb{R}^n} \times \Omega^+$ as $t \to +\infty$.*

3.3.2. $\Omega^+ = 0_{\ell_m^2}$

To prove that $\Omega^+ = 0_{\ell_m^2}$, we take an element $\bar{c} \in \Omega^+$, denote $\bar{\phi} : \mathbb{R}^+ \ni t \mapsto \bar{\phi}(t) \triangleq \sum_{k\in\mathbb{Z}} \bar{c}_k \mathrm{e}^{\imath k\omega_0 t} \in \mathbb{R}$ as the associated function in $L_m^2[0, T_0]$ and prove that $\bar{\phi} = 0_{L_m^2[0,T_0]}$ almost everywhere. This yields $\bar{c} = 0_{\ell_m^2}$.

For all $\bar{c} \in \Omega^+$, there exits a sequence $(t_l)_{l\in\mathbb{N}}$ with $t_l \to +\infty$ as $l \to +\infty$ such that $\left\|\tilde{c}(t_l) - \bar{c}\right\|_{\ell_m^2} \to 0$ as $l \to +\infty$. For all $l \in \mathbb{N}$, we can define

$$\begin{cases} \phi_l : \mathbb{R}^+ \ni t \mapsto \phi_l(t) \triangleq \sum\limits_{k\in\mathbb{Z}} \tilde{c}_k(t_l)\mathrm{e}^{\imath k\omega_0 t} \in \mathbb{R}^m \\ \bar{\phi} : \mathbb{R}^+ \ni t \mapsto \bar{\phi}(t) \triangleq \sum\limits_{k\in\mathbb{Z}} \bar{c}_k\mathrm{e}^{\imath k\omega_0 t} \in \mathbb{R}^m \end{cases}$$

These two functions belong to $L_m^2[0, T_0]$ and we have

$$\lim_{l\to+\infty} \left\|\phi_l - \bar{\phi}\right\|_{L_m^2[0,T_0]} = 0. \tag{11}$$

Then,

$$\begin{aligned} \dot{\tilde{x}}(t) &= H\tilde{x}(t) + A_0(t)\sum_{k\in\mathbb{Z}} \tilde{c}_k(t_l)\mathrm{e}^{\imath k\omega_0 t} \\ &\quad + A_0(t)\sum_{k\in\mathbb{Z}} (\tilde{c}_k(t) - \tilde{c}_k(t_l))\mathrm{e}^{\imath k\omega_0 t} \\ &= H\tilde{x}(t) + A_0(t)\phi_l(t) + A_0(t)\psi_l(t), \end{aligned} \tag{12}$$

where

$$\psi_l(t) \triangleq \sum_{k\in\mathbb{Z}} \left(\int_{t_l}^t -\frac{\alpha}{k^2+1} (\mathrm{e}^{\mathrm{i}k\omega_0 s} A_0(s))^\dagger P\tilde{x}(s)\mathrm{d}s \right) \mathrm{e}^{\imath k\omega_0 t}.$$

By successive maximization, we have

$$\begin{aligned} &\left\|\psi_l(t)\right\|_n^2 \\ &\le |t - t_l| \int_{t_l}^t \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{k^2+1}\right) (\mathrm{e}^{\mathrm{i}k\omega_0 s} A_0(s))^\dagger P\tilde{x}(s) \right\|_n^2 \mathrm{d}s \\ &\le |t - t_l| \int_{t_l}^t \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\alpha}{k^2+1}\right) \mathrm{e}^{-k\omega_0 s} \right\|^2 \left\|P\right\|_{nn}^2 \rho_M^2 \left\|\tilde{x}(s)\right\|_n^2 \mathrm{d}s \\ &\le |t - t_l| \,\alpha^2\kappa^4 \left\|P\right\|_{nn}^2 \rho_M^2 \left(\int_{t_l}^t \left\|\tilde{x}(s)\right\|_n^2 \mathrm{d}s \right). \end{aligned}$$

We define $\rho_\psi \triangleq \sqrt{T_0}\alpha\kappa^2 \left\|P\right\|_{nn} \rho_M$, from which it follows that

$$\sup_{t\in[t_l-T_0,t_l+T_0[} \left\|\psi_l(t)\right\|_n^2 \le \rho_\psi^2 \int_{t_l-T_0}^{t_l+T_0} \left\|\tilde{x}(s)\right\|_n^2 \mathrm{d}s. \tag{13}$$

To exploit the periodicity of several terms appearing in (12), we consider this differential equation over the time range $\mathcal{I}_l \triangleq [k(t_l)T_0, (k(t_l) + 1)T_0]$, where $k(t_l)$ is the integer part of the ratio t_l/T_0. We now proceed in three steps.

Step 1: $\lim_{l\to+\infty} \sup_{t\in[0,T_0]} \left(\left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \right) = 0.$

From Proposition 2 and inequality (13), we have

$$\lim_{l\to+\infty} \sup_{t\in\mathcal{I}_l} \left\|\psi_l(t)\right\|_n = 0$$

and

$$\lim_{l\to+\infty} \sup_{t\in\mathcal{I}_l} \left\|\tilde{x}(t)\right\|_n = 0.$$

Gathering these two limits and (12) gives

$$\lim_{l\to+\infty} \sup_{t\in\mathcal{I}_l} \left\|\dot{\tilde{x}}(t) - A_0(t)\phi_l(t)\right\|_n = 0. \tag{14}$$

Thus, by integration over the time range $[k(t_l)T_0, t]$, with $k(t_l)T_0 \le t \le (k(t_l) + 1)T_0$,

$$\begin{aligned} \mathbb{I}_l &\triangleq \int_{k(t_l)T_0}^t \left\|\dot{\tilde{x}}(s) - A_0(s)\phi_l(s)\right\|_n \mathrm{d}s \\ &\ge \left\| x(t) - x(k(t_l)T_0) - \int_{k(t_l)T_0}^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n . \end{aligned}$$

[4] The cited theorem gives the conclusion using (with the notations found in the reference) Part II with $n = 1, j = 0, m = 1, p = 2$, whereas $[0, T_0]$ satisfies the cone condition Adams (1975, p. 82).

However, from (14) we can deduce that $\lim_{l\to+\infty} \mathbb{I}_l = 0$, and we already know that $\lim_{l\to+\infty} x(k(t_l)T_0) = 0 = \lim_{l\to+\infty} x(t \geq k(t_l)T_0)$. This yields:

$$\lim_{l\to+\infty} \sup_{t\in\mathbb{I}_l} \left(\left\| \int_{k(t_l)T_0}^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \right) = 0.$$

Noting that both A_0 and ϕ_l are T_0-periodic functions, this last equality reduces to

$$\lim_{l\to+\infty} \sup_{t\in[0,T_0]} \left(\left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \right) = 0. \tag{15}$$

Step 2: $\lim_{l\to+\infty} \sup_{t\in[0,T_0]} \left(\left\| \int_0^t A_0(s)\bar{\phi}(s)\mathrm{d}s \right\|_n \right) = 0.$

Using a triangular inequality, we can deduce the convergence of the following quantity:

$$\begin{aligned}
&\left\| \int_0^t A_0(s)\bar{\phi}(s)\mathrm{d}s \right\|_n \\
&\quad \leq \left\| \int_0^t A_0(s)(\phi_l(s) - \bar{\phi}(s))\mathrm{d}s \right\|_n + \left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \\
&\quad \leq \rho_M \int_0^t \left\| \phi_l(s) - \bar{\phi}(s) \right\|_n \mathrm{d}s + \left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \\
&\quad \leq \rho_M \sqrt{t} \left(\int_0^t \left\| \phi_l(s) - \bar{\phi}(s) \right\|_n^2 \mathrm{d}s \right)^{1/2} + \left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n \\
&\quad \leq \rho_M \sqrt{t} \left\| \phi_l - \bar{\phi} \right\|_{\ell_m^2[0,T_0]} + \left\| \int_0^t A_0(s)\phi_l(s)\mathrm{d}s \right\|_n
\end{aligned}$$

from Cauchy–Schwarz inequality. Therefore, from (11) and (15) we have

$$\lim_{l\to+\infty} \sup_{t\in[0,T_0]} \left(\left\| \int_0^t A_0(s)\bar{\phi}(s)\mathrm{d}s \right\|_n \right) = 0. \tag{16}$$

Step 3: $\bar{\phi} = 0_{L_m^2[0,T_0]}$.

By continuity of the integrand, we deduce from the preceding limit that $\left\| A_0(t)\bar{\phi}(t) \right\| = \bar{\phi}(t)^T A_0^T(t) A_0(t)\bar{\phi}(t) = 0$ for all $t \in [0, T_0]$. From the injectivity assumption of A_0 (see (2)), we simply deduce that $\bar{\phi}(t) = 0$ for all $t \in [0, T_0]$ and that therefore $\bar{c} = 0$. The following lemma holds.

Lemma 4. $\Omega^+ = 0_{\ell_m^2}$.

3.4. Conclusion and main result

We can now formulate our main result.

Proposition 3. *Consider system (1). Assume that (2) holds. Consider the observer (3) with L and $\{L_k\}_{k\in\mathbb{Z}}$ defined in (4) and in (5). Then the error dynamics (6) asymptotically converges to zero in $\mathbb{R}^n \times \ell_m^2$.*

The frequent case for which A, A_0 and C are time-invariant is addressed in the following corollary.

Corollary 1. *Consider system (1). Assume that C is invertible and A_0 is injective. Consider the observer (3) with L and $\{L_k\}_{k\in\mathbb{Z}}$ defined in (4) and in (5). Then the error dynamics (6) asymptotically converges toward $0_{\mathbb{R}^n\times\ell_m^2}$.*

4. Examples

4.1. Crankshaft dynamics reference model

For illustration, we present an automotive system observation problem already considered in Rizzoni (1989), Chauvin et al. (2004) and van Nieuwstadt and Kolmanovsky (1997). Consider an n_{cyl}-cylinder engine. Following Kiencke and Nielsen (2000), the torque balance on the crankshaft can be written as $\frac{\mathrm{d}\left(\frac{1}{2}J(\theta)\omega^2\right)}{\mathrm{d}\theta} = T$, where θ is the crank angle, ω is the instantaneous engine speed, J is the $\frac{4\pi}{n_{cyl}}$-periodic inertia, and T is the combustion torque. In the variable θ time scale, T is $\frac{4\pi}{n_{cyl}}$-periodic and has zero mean at steady state. This system defines a first-order periodic dynamics with a periodic input signal T. The state $x(\theta) = \frac{1}{2}J(\theta)\omega^2$ is fully measured through the equation $y = \omega^2$. The periodic input signal T admits a Fourier series expansion $T \triangleq \sum_{k\in\mathbb{Z},k\neq 0} c_k e^{\imath k\frac{\theta}{2}}$. The reference dynamics is

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sum_{k\in\mathbb{Z},k\neq 0} c_k e^{\imath k\frac{\theta}{2}}.$$

Then, the observer we propose is, following (3),

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}\theta}\hat{x} = \displaystyle\sum_{k\in\mathbb{Z},k\neq 0} \hat{c}_k e^{\imath k\frac{\theta}{2}} + H\left(\hat{x} - \frac{1}{2}J(\theta)y\right) \\
\dfrac{\mathrm{d}}{\mathrm{d}\theta}\hat{c}_k = \dfrac{\alpha}{2(k^2+1)H} e^{-\imath k\frac{\theta}{2}} \left(\hat{x} - \frac{1}{2}J(\theta)y\right), \quad \forall k \neq 0
\end{cases} \tag{17}$$

$\hat{x}(0) = 0, \quad \hat{c}_k(0) = 0, \quad \forall k \neq 0.$

Assumption (2) is easily verified with

$$\begin{cases}
\rho_m = \min\left\{1, \displaystyle\min_{\theta\in[0,\frac{4\pi}{n_{cyl}}]} \frac{2}{J(\theta)}\right\} \\
\rho_M = \max\left\{1, \displaystyle\max_{\theta\in[0,\frac{4\pi}{n_{cyl}}]} \frac{2}{J(\theta)}\right\}.
\end{cases}$$

To estimate T, we can use the observer (17) with, e.g., $H = -100$ and $\alpha = 50$ (these are the values used to obtain the experimental results presented below). This gives the estimate

$$\hat{T} = \sum_{k\in\mathbb{Z},k\neq 0} \hat{c}_k e^{\imath k\frac{\theta}{2}}.$$

Experimental results

In practice, only a finite number of Fourier expansion coefficients can be included. However, numerous harmonics need to be considered to reconstruct the signal (at least 5). Very conveniently, using the results of this paper, the observer design can easily be updated when the number of harmonics considered is changed. Indeed, without modifying the tuning parameters H and α, new gains are computed from (4) and (5). These formulae remain valid when the number of harmonics asymptotically approaches infinity (as proven by Proposition 3). Fig. 1 shows experimental observer results for a four-cylinder diesel engine. It is possible to compare our results to a high-accuracy estimate obtained from in-cylinder pressure sensors.

The results for the observer presented are good. For similar accuracy, the computational burden of our observer compares very favorably to an extended Kalman filter (as previously used Chauvin et al., 2004). The results in Fig. 2 demonstrate that this burden increases quadratically as a function of the number of harmonics for the extended Kalman filter, whereas it simply increases linearly for our observer. The interested reader will also note that another advantage of the proposed method is its proof of convergence. Interestingly, convergence of the EKF for this system can also be established (see Chauvin et al., 2004), but it requires a careful investigation of observability and controllability Grammians to guarantee uniform (with respect to the time variable) properties of this periodic system. Through the classic results of Bittanti (1986) and

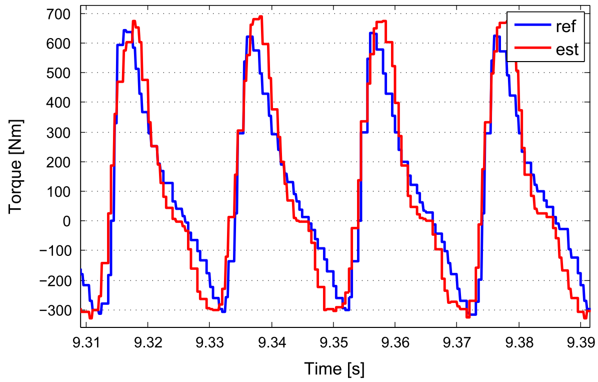

Fig. 1. Combustion torque. Continuous line, reference combustion torque obtained from in-cylinder pressure sensors; dashed line, combustion torque estimated by the proposed observer.

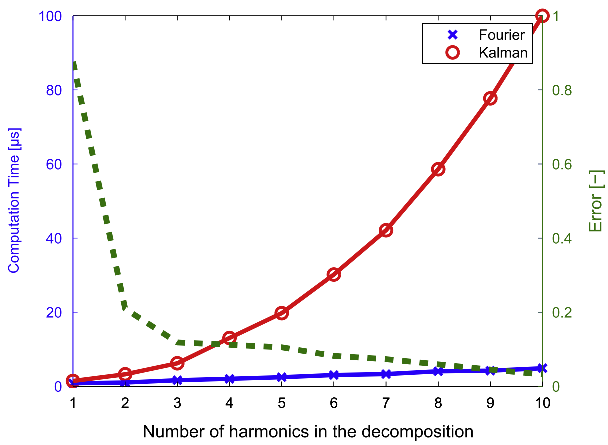

Fig. 2. Normalized CPU time as a function of the number of terms considered in the Fourier expansion. Comparison between an extended Kalman filter and the proposed observer (3). Right axis, reconstruction error.

Bittanti, Colaneri, and De Nicolao (1988), these properties guarantee existence and uniqueness of a symmetric periodic positive solution to the discrete periodic Riccati equation that serves in the proof of convergence.

Besides its convergence, the tuning simplicity and the relatively low computational cost are the two points of interest of the proposed technique.

4.2. Perturbed planar two-body problem

As a second illustrative example, we consider a perturbed planar two-body problem such as pictured in Fig. 3 (see e.g. Bate et al., 1971). Due to gravitational effects, one celestial body is orbiting around another one, defining a (relative motion) reference periodic orbit ρ. The reference orbit can be analytically computed: it is an ellipse. A remote attracting mass generates a perturbing acceleration a_p on this celestial body. Following Encke's special perturbation method of analysis (see again Bate et al., 1971), at first order, the deviation $\delta r(t) \in \mathbb{R}^2$ from the reference orbit $\rho(t) \in \mathbb{R}^2$ satisfies the following differential equation

$$\ddot{\delta r} = a_p + \mu \left(\frac{1}{\|\rho(t)\|^3} - \frac{1}{\|r(t)\|^3} \right) r - \frac{\delta r}{\|\rho(t)\|^3}$$

where $r(t) \in \mathbb{R}^2$ is the perturbed orbit, and μ is the gravitational parameter. To handle the natural unbounded growth of the solution of the preceding equation (secular drift, see e.g. Prussing & Conway, 1993), a normalization is performed. Introduce $z = \frac{\delta r}{t}$,

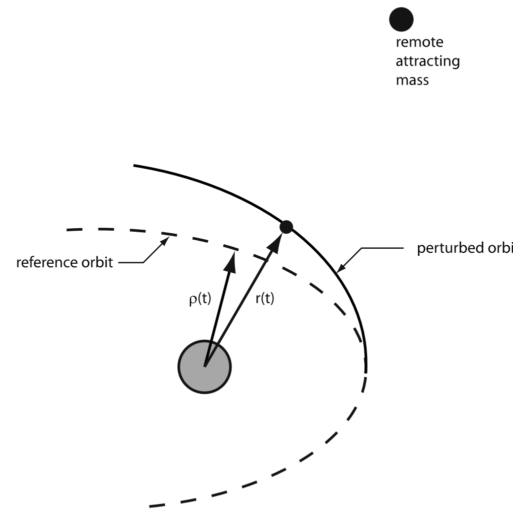

Fig. 3. Perturbed two-body problem.

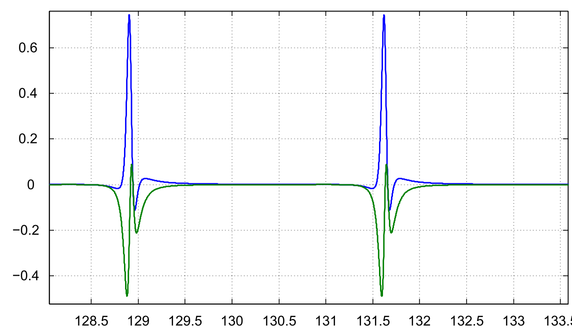

Fig. 4. Normalized perturbing gravitational effects.

then one obtains the equivalent dynamics

$$\ddot{z} = -\frac{z}{\|\rho(t)\|^3} - \frac{2\dot{z}}{t} + w(t) \tag{18}$$

where $w(t)$ is a periodic signal containing a_p, the reference orbit ρ and the true orbit r. In details, $z(t) \in \mathbb{R}^2$, $w(t) \in \mathbb{R}^2$. The problem we address here, is to determine the gravitational effects of the third body (e.g. its position or mass) by means of observation of the perturbed orbit (both position and velocities being available, possibly through distinct sensor systems, as is classically considered, see e.g. Bate et al., 1971). Equivalently, we wish to determine $w(t)$ from measurements of $z(t)$ and $\dot{z}(t)$. The two-dimensional (normalized) signal w is reported in Fig. 4. It has been obtained from numerical simulation on a normalized celestial problem. Considering the differential equation (18) away from $t = 0$, we simply apply the presented method to reconstruct the periodic input signal $w(t)$ from full measurements of the two parallel oscillators (18) with (periodic) time-varying stiffness and time-varying damping. The results are presented in Fig. 5 (for only one of the two oscillators for sake of brevity). To represent the sought-after signal w accurately, a large number of Fourier coefficients are needed. Here, $n_h = 100$ is considered. Normally, tuning an observer with so many states would require an enormous calibration effort. Yet, for each oscillator, we simply follow the tuning rule (5), and pick the following tuning parameters: $H = -10I$, where I is the two-dimensional identity matrix, and $\alpha = -20\,000$. After a limited number of periods (approximately 15 to 20), the input signal w is reconstructed well, then, a_p can be determined. For sake of simplification, no noise was present in the measurements. In practical cases, signal/noise ratio would certainly require that a slower convergence rate be considered, which can be achieved by lowering the values of the coefficients of H and α.

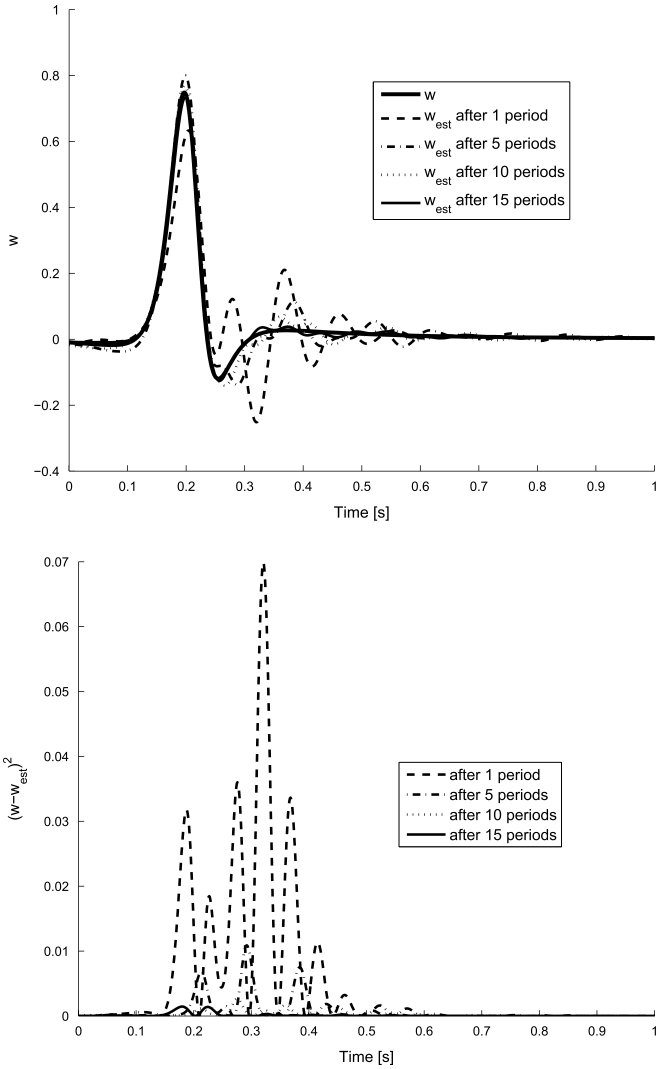

Fig. 5. Reconstruction of the perturbing gravitational effects (first component). Top: reconstructed signal. Bottom: squared error between the reconstruction and the true signal.

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Appendix. Compact embedding $\omega_n^{1,2} \hookrightarrow \ell_n^2$

According to the definition of embedded normed spaces Adams (1975, page 9), we need to prove that:

(1) $\omega_n^{1,2}$ is a vector subspace of ℓ_n^2;
(2) There exists a constant $K > 0$ such that, for all $c \in \omega_n^{1,2}$, $\|c\|_{\omega_n^{1,2}} \geq K \|c\|_{\ell_n^2}$; and
(3) Any bounded set in $\omega_n^{1,2}$ is precompact in ℓ_n^2, i.e., every bounded sequence in $\omega_n^{1,2}$ has a subsequence that is Cauchy in ℓ_n^2.

Point (1) is straightforward. Let us now focus on point (2). Clearly, any $(c_k)_{k\in\mathbb{Z}} = c \in \omega_n^{1,2}$ can be associated with a unique $\varphi \in {W_n}^{1,2}[0, T_0]$ by the Fourier reconstruction formula

$$\varphi(t) \triangleq \sum_{k\in\mathbb{Z}} c_k e^{\imath k\omega_0 t} \in \mathbb{R}^n.$$

By definition of the norm in $\omega_n^{1,2}$, it follows that

$$\|c\|^2_{\omega_n^{1,2}} = \frac{1}{T_0}\left(\left(1-\frac{1}{w_0^2}\right)\|\varphi\|^2_{L_n^2[0,T_0]} + \frac{1}{w_0^2}\|\varphi\|^2_{{W_n}^{1,2}[0,T_0]}\right). \quad (A.1)$$

From Parseval equality, we have

$$\|\varphi\|^2_{L_n^2[0,T_0]} = T_0 \|c\|^2_{\ell_n^2}.$$

From the compact embedding ${W_n}^{1,2}[0, T_0] \hookrightarrow L_n^2[0, T_0]$, we deduce that there exists $K_1 > 0$ such that

$$\|\varphi\|^2_{{W_n}^{1,2}[0,T_0]} \geq K_1 \|\varphi\|^2_{L_n^2[0,T_0]}.$$

Moreover, $K_1 \geq 1$. Gathering this last inequality with the previous equality, we easily obtain

$$\|c\|_{\omega_n^{1,2}} \geq K \|c\|_{\ell_n^2}, \quad (A.2)$$

with $K = \sqrt{1 + \frac{K_1 - 1}{\omega_0^2}}$. Point (2) is proven.

Let us now prove point (3).
Consider a bounded sequence $c \in \omega_n^{1,2}$ and compute its Fourier reconstruction φ as previously described. From (A.1), it follows that φ is bounded in ${W_n}^{1,2}[0, T_0]$. Therefore, from the compact embedding ${W_n}^{1,2}[0, T_0] \hookrightarrow L_n^2[0, T_0]$, we can construct, from any sequence of sequences bounded in $\omega_n^{1,2}$, a sequence (φ) that is Cauchy in $L_n^2[0, T_0]$.

Every element of this last sequence belongs to $L_n^2[0, T_0]$ and can be associated with an element in ℓ_n^2 (its Fourier transform from Parseval inequality). The Fourier transforms of the elements of (φ) define a Cauchy sequence. This property holds because, for any φ_n and φ_m elements of the Cauchy sequence (φ), their Fourier transforms, c_n and c_m, respectively, satisfy

$$\|c_n - c_m\|_{\ell_n^2} = \|\phi_n - \phi_m\|_{L_n^2[0,T_0]}.$$

In summary, we have constructively shown how, from any sequence bounded in $\omega_n^{1,2}$, a subsequence that is Cauchy in ℓ_n^2 can be defined. This proves point (3) and concludes the proof. □

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