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Abstract

This paper investigates necessary and sufficient Lyapunov conditions for Input-to-State Stability (ISS) of Linear Integral Difference Equations in the presence of an additional exogenous signal. Building on recent research in the literature pertaining to necessary conditions for the exponential stability of difference equations, we introduce a quadratic Lyapunov functional that incorporates the derivative of the so-called Lyapunov matrix. We demonstrate that the ISS of the considered class of systems is contingent upon the existence of an ISS Lyapunov functional. The Lyapunov analysis hinges on the properties of the fundamental matrix and the delay Lyapunov matrix.

Key words: Linear Integral Difference Equations, Time-Delay Systems, Input-to-State Stability, Lyapunov theorem

1 Introduction

Delay equations and hyperbolic Partial Differential Equations (PDEs) are widely known to share a deep interconnection. Indeed, not only can a time-delay be represented by a simple transport equation [25], but, conversely, certain hyperbolic PDEs can be reformulated and understood as neutral delay systems as reported in [34]. A prominent and early example of this is d'Alembert's formula, which transforms a wave equation into a difference equation. Besides, recently, the exact relation between Linear First-Order Hyperbolic PDEs and Linear Integral Difference Equations (LIDEs) has been comprehensively investigated in [4], proving that the stability properties of Linear First-Order Hyperbolic PDEs are equivalent to those of a certain LIDE.

Yet, despite this assessment, difference equations have seldom been studied in the literature, in which they are often considered as a particular subcategory of neutral time-delay systems [19]. To our knowledge, some of the few studies specifically addressing difference systems are [10, 26], which study controllability properties of difference equations and propose spectral criteria, and [11, 12, 28] which aimed at obtaining necessary stability condition in terms of a Lyapunov functional.

This paper pursues such a Lyapunov approach to characterize the Input-to-State Stability of Linear Integral Difference Equations. This question arose in our recent work [2] investigating robust feedback for an underactuated network of PDEs, as the one appearing in mining ventilation systems [37]. Indeed, to study the cascade of these PDEs, one wishes to consider the equivalent LIDE and rely on Lyapunov ISS functionals to investigate the effect of the cascade, as commonly done in a small-gain context, for instance.

Nevertheless, while the Input-to-State Stability of retarded time-delay systems (see [9] for a comprehensive review of the field) and a large number of PDEs with bounded control operator or admissible boundary control is now well-grounded (see [30] for a complete survey of the domain) and their characterization with a coercive ISS Lyapunov function clearly investigated, it is not the case for Difference Equations. It is well-known [19] that the asymptotic stability of the homogeneous LDE is equivalent to the ISS of the non-homogeneous one (via Duhamel's principle). Nevertheless, its Lyapunov characterization has been scarcely studied: on the one hand, [20, 16] proposed Lyapunov ISS conditions of general nonlinear Difference Equations, but which are only

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sufficient; on the other hand, ISS Lyapunov characterizations for nonlinear continuous-time difference equations are provided in [32, 33] in terms of Lyapunov functional, but with a continuous-time difference operator, which may not be appropriate when considering cascaded or interconnected dynamics of different types, for instance.

In this paper, we extend our approach from [3] for Linear Difference equation to the case of a LIDE. Grounding on [36] which proposed necessary Lyapunov conditions for the exponential stability of Linear Difference Equations with pointwise delays, we propose to construct a Complete-Type Lyapunov functional for a LIDE, as common for time-delay systems [1, 6, 17, 22, 29, 31]. Following this methodology, we define a quadratic functional involving the derivative of the so-called delay Lyapunov matrix of the homogeneous system. A careful analysis of the regularity and discontinuities of this matrix enables us to prove that the ISS of a LIDE is equivalent to the existence of a quadratic ISS Lyapunov functional. This is the main contribution of the paper.

The paper is organized as follows. In Section 2, we introduce the problem under consideration and state the main result of the paper. Section 3 introduces some preliminary tools allowing us to construct an ISS Lyapunov functional. This design is performed in Section 4, which also concludes the proof of the main result. Finally, we conclude with directions of future works in Section 5.

Notations: Given $k \in \mathbb{N}$, we denote $C^{pw,k}([a,b],\mathbb{R}^n)$ the set of functions f mapping the interval $[a,b] \subset \mathbb{R}$ into $\mathbb{R}^n$ and which are k times piecewise continuously differentiable, that is, such that $[a,b]$ can be partitioned by a finite number of points $(t_i)_{0\leq i \leq N}$ so that f is k times continuously differentiable on each subinterval (t_{i-1},t_i) and $f, f', .., f^{(k)}$ admit finite right-hand and left-hand limits at each t_i, which we denote $f(t_i^+), \ldots, f^{(k)}(t_i^+)$ and $f(t_i^-), \ldots, f^{(k)}(t_i^-)$, respectively. We denote $C^{pw,k}([a,b),\mathbb{R}^n)$ the set of functions which are restrictions of elements of $C^{pw,k}([a,b],\mathbb{R}^n)$ to $[a,b)$. A function f is said to be piecewise continuous on $\mathbb{R}$ or $\mathbb{R}_+$ if its restriction to any line segment is piecewise continuous. For any fixed $\tau > 0$, we denote $C_\tau^{pw} = C^{pw,0}([-\tau,0),\mathbb{R}^n)$ the Banach space of piecewise continuous functions mapping the interval $[-\tau,0)$ into $\mathbb{R}^n$. For a function $\varphi : [-\tau,\infty) \mapsto \mathbb{R}^n$, we define its partial trajectory $\varphi_{[t]}$ by $\varphi_{[t]}(\theta) = \varphi(t+\theta), -\tau \leq \theta < 0$. The space C_τ^{pw} is endowed with the norm $\|\varphi\|^2_{C_\tau^{pw}} = \sup_{s\in[-\tau,0)} \varphi^T(s)\varphi(s)$ or with the L^2_τ norm $\|\varphi\|^2_{L^2_\tau} = \int_{-\tau}^0 \varphi^T(s)\varphi(s)ds$. The The space $C^{pw,k}([a,b],\mathbb{R}^n)$ is endowed with the norm $\|\varphi\|_\infty = \sum_{i=0}^k \sup_{s\in[a,b]} \sqrt{(\varphi^{(i)})^T(s)\varphi^{(i)}(s)}$. The identity matrix of size $n \in \mathbb{N}$ is denoted I_n or I, when no confusion arises. We denote $\|\cdot\|$ the usual Euclidean norm and, for real matrices, the Euclidean induced norm. The Dini upper right-hand derivative of a functional $v(\varphi_{[t]})$ is denoted by $D^+v(\varphi_{[t]})$.

2 Problem under consideration

2.1 Linear Integral Difference system

Let us consider the following inhomogeneous Linear Integral Difference system

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi + f(t)\,, \quad t \geq 0 \tag{1}$$

where $X(t) \in \mathbb{R}^n$ ($n \in \mathbb{N}$), $A_k \in \mathbb{R}^{n\times n}$, $\tau_k > 0$ ($1 \leq k \leq M$ with $M \in \mathbb{N}$) are positive time-delays ordered as $0 < \tau_1 < \tau_2 < \ldots < \tau_M$ and $F \in C^{pw,1}([-\tau_M,0])$ with corresponding partition $(h_j)_{0\leq j\leq N}$ of $[-\tau_M,0]$. The function f is an exogenous signal which belongs to $C^{pw}([0,\infty),\mathbb{R}^n)$. The corresponding initial condition is given as

$$X_{[0]} = X^0 \in C_{\tau_M}^{pw}\,. \tag{2}$$

A function $X : [-\tau_M,\infty) \to \mathbb{R}^n$ is called a solution of the initial value problem (1)–(2) if $X_{[0]} = X^0$ and if equation (1) is satisfied for $t \geq 0$. The solution at time t is denoted by $X(t,X^0)$, and the dependence with respect to X^0 may be dropped when no confusion arises. We are interested in characterizing the stability properties of such solutions, and in particular their Input-to-State Stability (ISS) defined as follows [1].

Definition 1 *The system* (1) *is said to be (L_2-) exponentially* Input-to-State Stable *(ISS) if there exist $R, \lambda, \gamma > 0$ such that, for any $X^0 \in C_{\tau_M}^{pw}$ and any piecewise continuous function f, the solution to* (1)–(2) *satisfies*

$$\|X_{[t]}\|_{L^2_{\tau_M}} \leq Re^{-\lambda t}\|X^0\|_{L^2_{\tau_M}} + \gamma \sup_{s\in[0,t]} \|f(s)\|\,,\ t \geq 0\,. \tag{3}$$

It is emphasized in [18] that the exponential stability of a Linear Difference Equation is independent of the functional space under consideration (provided the original system is well-posed for such a functional space). Hence, this property extends to the Input-to-State Stability of LIDEs, and property (3) is, for instance, equivalent to the L_∞-exponentially ISS, i.e, the existence of $R, \lambda, \gamma > 0$ such that, for any $X^0 \in C_{\tau_M}^{pw}$ and any piecewise continuous function f,

$$\|X(t)\| \leq Re^{-\lambda t}\|X^0\|_{C_{\tau_M}^{pw}} + \gamma \sup_{s\in[0,t]} \|f(s)\|\,,\ t \geq 0\,.$$

[1] Notice that, as the system under consideration is linear, exponential ISS is equivalent to the standard Input-to-State Stability in terms of comparison functions defined for instance in [9] for time-delay systems.

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Besides, it is proven in [19][Theorem 3.5] that the exponential stability of the homogeneous system associated with equation (1):

$$X(t)=\sum_{k=1}^{M}A_kX(t-\tau_k)+\int_{-\tau_M}^{0}F(\xi)X(t+\xi)d\xi, \quad (4)$$

with the initial data $X^0\in C^{pw}_{\tau_M}$, implies[2] the exponential-ISS of (1)–(2). Hence, in the sequel, when the context is clear, we will refer to system (1) indifferently as exponentially stable or exponentially-ISS without specifying the norm under consideration.

2.2 Objective of the paper

For a Linear Difference Equation, that is, in the absence of the distributed-delay term ($F\equiv 0$), [36] designed a functional with prescribed derivative using the so-called delay Lyapunov matrix. This design was then adjusted in [3] to characterize the ISS property of such Linear Difference Equations in terms of the existence of a Lyapunov functional, that is equivalent to the L^2-norm of the state.

Our main result consists in extending this ISS characterization for the LIDE (1)–(2), with the following theorem.

Theorem 2 *Consider system* (1) *with initial condition* $X^0\in C^{pw}_{\tau_M}$. *Assume that* f *belongs to* $C^{pw}([0,\infty),\mathbb{R}^n)$. *The two following statements are equivalent:*

(1) the solution to (1) *is* L_2*-ISS;*
(2) there exists a quadratic function $v_1: C^{pw}_{\tau_M}\to\mathbb{R}_+$ *such that*
  (a) $\exists\,\rho,\sigma>0\quad D^+v_1(X_{[t]})\le-\rho v_1(X_{[t]})+\sigma\|f(t)\|^2$;
  (b) $\exists\,\alpha_l,\alpha_u>0\quad\forall\varphi\in C^{pw}_{\tau_M}\quad \alpha_\ell\|\varphi\|^2_{L^2_{\tau_M}}\le v_1(\varphi)\le\alpha_u\|\varphi\|^2_{L^2_{\tau_M}}$.

This result is the direct counterpart for LIDEs of the Lyapunov characterization of the ISS of (differential) time-delay systems, such as [9][Theorem 23]. Observe that such a ISS Lyapunov functional can be chosen quadratic, owing to the linear nature of the system.

Such a necessary and sufficient condition can reveal helpful to study robustness for cascaded or coupled dynamics. This is the case for instance in our work [2] investigating stabilization of a network of hyperbolic PDEs. This problem was reformulated as a LIDE with a delayed input, that is, the cascade of a transport PDE into a LIDE. Such a Lyapunov characterization is then instrumental to study delay uncertainties.

[2] It also implies that $\det(I_n-\int_{-\tau_M}^{0}F(\xi)d\xi-\sum_{k=1}^{M}A_k)\neq 0$, which is used in Lemma 3.

The rest of the paper is dedicated to the proof of Theorem 2, and more specifically, Section 4, in which an explicit expression of the quadratic functional v_1 is provided. To achieve this goal, we first need to introduce some preliminary properties and intermediate analysis tools.

3 Preliminaries – Fundamental and delay Lyapunov matrices

In this section, we first present the two intermediate matrices that play a crucial role in our definition of an ISS Lyapunov functional. These matrices are standardly used to construct Complete-Type Lyapunov functional for time-delay systems [23, 27].

3.1 The fundamental matrix

The first variable of interest is the fundamental matrix associated with system (4). In the sequel, we gather some of its properties shown in [35] for a Linear Difference Equation, that is, in the absence of distributed delay ($F\equiv 0$) and state without proof the lemmas which are only marginal variations of the ones given in [35].

Lemma 3 (see Theorem 1 in [35]) *The* $n\times n$ *matrix* $K(t)$ *defined for all* $t\ge 0$ *by*

$$K(t)=\sum_{k=1}^{M}K(t-\tau_k)A_k+\int_{-\tau_M}^{0}K(t+\xi)F(\xi)d\xi, \quad (5)$$

with initial condition

$$K(\theta)=K_0=\left(\sum_{k=1}^{M}A_k+\int_{-\tau_M}^{0}F(\xi)d\xi-Id\right)^{-1}, \quad (6)$$

for $\theta\in[-\tau_M,0)$, *is called the fundamental matrix of system* (4). *For any initial condition* $X^0\in C^{pw}_{\tau_M}$, *the solution of the homogeneous equation* (4) *verifies*

$$\begin{aligned}X(t)=&\sum_{k=1}^{M}\frac{d}{dt}\int_{-\tau_k}^{0}K(t-\theta-\tau_k)A_kX^0(\theta)d\theta\\&+\frac{d}{dt}\int_{-\tau_M}^{0}\int_{\xi}^{0}K(t-\theta+\xi)F(\xi)X^0(\theta)d\theta d\xi. \quad (7)\end{aligned}$$

Formula (7) is known as **Cauchy's formula**.

Lemma 4 (see Corollary 1 in [35]) *Let* K *be defined in* (5)-(6). *It holds, for all* $t\ge 0$,

$$K(t)=\sum_{k=1}^{M}A_kK(t-\tau_k)+\int_{-\tau_M}^{0}F(\xi)K(t+\xi)d\xi\,. \quad (8)$$

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The regularity of K is specified in the following lemma.

Lemma 5 *The fundamental matrix K defined in (5)-(6) is piecewise continuously differentiable on $[-\tau_M, +\infty)$. The set of its discontinuity points is $\mathcal{I}_{\mathcal{K}} = \{t_k\}_{k\in\mathbb{N}}$, where*

$$t_k = \min_{p_k^1,\dots p_k^m} \left\{ \sum_{i=1}^{M} p_k^j \tau_j \mid \sum_{i=1}^{M} p_k^j \tau_j > t_{k-1},\ p_k^j \in \mathbb{N} \right\}. \tag{9}$$

and its derivative discontinuity points are given by the sequence (f_k), defined by $f_0 = 0$ and, for $k \in \mathbb{N}$,

$$f_{k+1} = \min_{\substack{k_1,\dots,k_M\in\mathbb{N}\\ l_1,\dots,l_N\in\mathbb{N}}} \left\{ a = \sum_{i=1}^{M} k_i\tau_i - \sum_{j=1}^{N} l_j h_j \mid a > f_k \right\}.$$

PROOF. For $k \in \mathbb{N}$, and $t \in [t_k, t_{k+1})$, equation (8) can be rewritten as

$$K(t) = H(t) + \int_{t_k}^{t} F(\xi - t)K(\xi)d\xi,$$

where $H(t) = \sum_{k=1}^{M} A_k K(t-\tau_k) + \int_{t-\tau_M}^{t_k} F(\xi - t)K(\xi)d\xi$. This form corresponds to a Volterra integral equation of the second kind which ensures that K is continuous on $[t_k, t_{k+1})$ (see [39, Chapter 4]). The first part of Lemma 5 can then be proven recursively.

To characterize the derivative discontinuities, we follow an induction approach. With this aim in view, define the property (H_k) as, for all $k \in \mathbb{N}$, "the function K is piecewise continuously differentiable on $[-\tau_M, t_k]$", in which the sequence (t_k) is defined in Lemma 5.

First, observe that, for $t \in [-\tau_M, 0)$, it holds $K(t) = K_0$, which immediately implies the desired property for $k = 0$. Let us now assume that (H_k) holds for a certain $k \in \mathbb{N}$. Consider $t \in (t_k, t_{k+1})$, and notice that, for all $j \in \{1,\dots,M\}$, $t-\tau_j < t_k$, as $|t_{k+1}-t_k| \le \tau_1$. Moreover, given the definition of (t_k), $t-\tau_j \neq t_q\ \forall q \in \mathbb{N}$. Therefore, K is differentiable on (t_k, t_{k+1}) and (5) yields

$$\begin{aligned} K'(t) &= \sum_{j=1}^{M} K'(t-\tau_j)A_j + K(t)F(0) \qquad (10)\\ &- \sum_{j=1}^{N} \left(K((t+h_j)^+)F(h_j^+) - K((t+h_j)^-)F(h_j^-) \right)\\ &- K(t-\tau_M)F(-\tau_M) - \int_{t-\tau_M}^{t} K(\nu)F'(\nu - t)d\nu\,. \end{aligned}$$

From this expression, one deduces that K' jumps if $t + h_j \in \mathcal{I}_{\mathcal{K}}$ for a certain j or if $t \in \mathcal{I}_{\mathcal{K}}$. It thus follows that the discontinuity points of K' on (t_k, t_{k+1}) are given by $(f_i)_{i\in\mathbb{N}} \cap (t_k, t_{k+1})$. Besides, from (10), K' admits limits at t_k^+, t_{k+1}^-, that is, (H_{k+1}) holds. This concludes the proof. □

In the sequel, when necessary, we will identify K' with its right-continuous extension. For all $t \ge 0$, we define ΔK, the discontinuity jumps of the fundamental matrix K, as

$$\Delta K(t) = K(t^+) - K(t^-)\,. \tag{11}$$

It can be easily verified that $\Delta K(0) = I_n$. Besides, the following result holds.

Lemma 6 *Assume (4) is exponentially stable. Then, $K, \Delta K$ and K' are exponentially stable.*

PROOF. Eq. (8) implies that each column of $K(t)$ is a solution of (4). Therefore, if the difference equation (4) is exponentially stable, in turns, the matrix K is exponentially stable and thus ΔK as well.

Now, consider $t > 0$ and define the sequence $(\tilde{t}_k)$ of the discontinuity points of K lying in the interval $(t-\tau_M, t)$. We first prove that this sequence is finite. Let us consider $p \in \mathbb{N}$ such that $t \in [p\tau_1, (p+1)\tau_1]$. Observe that the number of elements of $\mathcal{I}_{\mathcal{K}}$ that are smaller than $(p+1)\tau_1$ is bounded by $(p+2)^M$ (the maximal number of linear combinations of the τ_i with integer coefficients and such that each coefficient is smaller than $p+1$), and thus so is the number of elements of $\mathcal{I}_{\mathcal{K}}$ in $(t-\tau_M, t)$. Let us denote $L(t) \in \mathbb{N}$ the cardinal of $(\tilde{t}_k)$.

For a.a. $t > 0$, we obtain, from (10) and performing an integration by parts,

$$K'(t) = \sum_{i=1}^{M} K'(t-\tau_i)A_i + \int_{-\tau_M}^{0} K'(t+\xi)F(\xi)d\xi + H(t)\,, \tag{12}$$

with $H(t) = \sum_{i=1}^{L(t)} \Delta K(\tilde{t}_i)\ F(\tilde{t}_i - t) - \sum_{i=1}^{N} K(t+h_i)\ \Delta F(h_j)$. The exponential stability of K and ΔK implies that the function $H(t)$ exponentially converges to zero as, for certain $R, \beta > 0$,

$$\begin{aligned} \left\| \sum_{i=1}^{L(t)} \Delta K(\tilde{t}_i) \right\| &\le L(t)Re^{-\beta(t-\tau_M)}\|\Delta K(0)\|\\ &\le (p+2)^M e^{\tau_M - \beta p\tau_1} R\|\Delta K(0)\| \underset{p\to\infty}{\to} 0\,. \end{aligned} \tag{13}$$

Consequently, since equation (1) is exponentially-ISS, we obtain the exponential convergence of $K'(t)$ to zero. □

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3.2 The delay Lyapunov matrix

The second variable of interest is the delay Lyapunov matrix associated with system (4). Again, in the sequel, we state without proof the lemmas which are close variations of the ones proven in [36] for the case $F \equiv 0$.

Definition 7 (see Lemma 3 in [36]) *Let W be a $n \times n$ symmetric positive definite matrix and consider the fundamental matrix K associated to system (4), defined in (3). Assume (4) is exponentially stable. Then, for any $\tau \in [-\tau_M, \tau_M]$, the **delay Lyapunov matrix***

$$U(\tau) = \int_0^\infty (K(t) - K_0)^T W K(t+\tau) dt, \tag{14}$$

is well-defined.

This well-posedness follows from the exponential stability of K given by Lemma 6. Observe that, unlike the matrix K, this matrix function is only defined on the interval $[-\tau_M, \tau_M]$, which turns out to be the only interval of interest in the sequel.

While U is continuous on $[-\tau_M, \tau_M]$, it is only almost everywhere differentiable on this interval, as stated in the following result. In the sequel, for any $-\tau_M < t_0 < t_1 < \tau_M$, let us denote $\mathcal{I}((t_0, t_1))$ the set of discontinuity points of the function U' that belong to (t_0, t_1) and define the derivative's jump of U' as

$$\Delta U'(\tau) = U'(\tau^+) - U'(\tau^-), \quad \tau \in [-\tau_M, \tau_M]. \tag{15}$$

Lemma 8 *Assume (4) is exponentially stable. For all $\tau \in [0, \tau_M]$, the delay Lyapunov matrix U satisfies the following difference equation*

$$U(\tau) = \sum_{k=1}^{M} U(\tau - \tau_k) A_k + \int_{-\tau_M}^{0} U(\tau + \xi) F(\xi) d\xi. \tag{16}$$

Besides, U is twice continuously differentiable almost everywhere on $[-\tau_M, \tau_M]$, the set $\mathcal{I}((-\tau_M, \tau_M))$ of the discontinuities of its derivative is countable and, for all $\tau \in [-\tau_M, \tau_M] \setminus \mathcal{I}((-\tau_M, \tau_M))$, it holds

$$\begin{aligned} U'(\tau) = &\sum_{k \geq 0} (K^T(t_k - \tau) - K_0^T) W \Delta K(t_k) \\ &+ \int_0^\infty (K(t) - K_0)^T W K'(t+\tau) dt, \end{aligned} \tag{17}$$

and

$$\begin{aligned} U''(\tau) = &- \sum_{k \geq 0} (K'^T(t_k - \tau) + K'^T(t_k + \tau)) W \Delta K(t_k) \\ &- \int_0^\infty K'(t)^T W K'(t+\tau) dt. \end{aligned} \tag{18}$$

Finally, for all $\tau \in [-\tau_M, \tau_M]$, it holds

$$\Delta U'(\tau) = - \sum_{k \geq 0} \Delta K^T(t_k - \tau) W \Delta K(t_k). \tag{19}$$

and the quantity $\sum_{\tau_c \in \mathcal{I}((-\tau_M, \tau_M))} \|\Delta U'(\tau_c)\|$ is finite.

PROOF. The difference equation (16) is a straightforward consequence of (5) and the use of Fubini's theorem.

The expression of the derivative (17) can be directly obtained from (14) in the sense of distribution, using the Jump rule [15, 38]. Observe that the well-posedness of the sum and integral in (17) follows from the exponential stability of $K, \Delta K$ and K'.

From (17), one can observe that U' is continuous at any τ such that when $t_k - \tau \notin \mathcal{I}_\mathcal{K}$ for all $k \in \mathbb{N}$. The proof of [3, Lemma 5] can then be directly adjusted to conclude that $\mathcal{I}((-\tau_M, \tau_M))$ is countable and that $\sum_{\tau_c \in \mathcal{I}((-\tau_M, \tau_M))} \|\Delta U'(\tau_c)\|$ is finite.

Besides, using the change of variable $s = t + \tau$, it follows that the integral in (17) can be rewritten as

$$\int_\tau^\infty (K(s-\tau) - K_0)^T W K'(s) ds.$$

The second-order derivative (18) follows, taking a time-derivative and using the Jump rule. Besides, observe that Lemma 6 ensures that $\int_0^\infty K'(t)^T W K'(t+\tau) dt$ in (18) is well-defined. Finally, the expression of $\Delta U'$ in Eq. (19) follows from its definition (15) and (17), as $\Delta K'$ is null almost everywhere. □

In the sequel, when necessary, we will identify U' and U'' with their right-continuous extensions. We now recall some valuable properties of the delay Lyapunov matrix derivative's jump discontinuities.

Lemma 9 (see Lemma 5 in [35]) *Assume (4) is exponentially stable. The matrix function $\Delta U'(\tau)$ satisfies*

- *the Symmetry property*

$$\Delta U'(-\tau) = [\Delta U'(\tau)]^T, \tag{20}$$

- *the Dynamic property*

$$\Delta U'(\tau) = \begin{cases} \sum_{k=1}^{M} \Delta U'(\tau - \tau_k) A_k, & \tau > 0, \\ \sum_{k=1}^{M} A_k^T \Delta U'(\tau + \tau_k), & \tau < 0, \end{cases} \tag{21}$$

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- *the Generalized algebraic property for all* $\tau \geq 0$

$$W\Delta K(\tau) = \sum_{i,j=1}^{M} A_i^T \Delta U'(\tau + \tau_i - \tau_j) A_j - \Delta U'(\tau)\,. \tag{22}$$

Finally, in addition, the second-order derivative of the Lyapunov matrix satisfies the following properties.

Lemma 10 *Assume* (4) *is exponentially stable and consider the associated Lyapunov matrix defined in* (14)*. The second-order derivative* U'' *of the Lyapunov matrix verifies the symmetry property, i.e., for all* $-\tau_M \leq \tau \leq \tau_M$

$$U''(-\tau) = U''^T(\tau) \tag{23}$$

and, for all $\tau \in [0, \tau_M]$*, satisfies*

$$U''(\tau) = \sum_{i=1}^{M} U''(\tau - \tau_i) A_i + \int_{-\tau_M}^{0} U''(\tau + \xi) F(\xi) d\xi + \sum_{\tau_c \in \mathcal{I}((\tau - \tau_M, \tau))} \Delta U'(\tau_c) F(\tau_c - \tau)\,. \tag{24}$$

PROOF. Let us consider (18) to compute $U''(-\tau)$ and observe that, with the change of variable $s = t - \tau$,

$$\int_0^\infty K'(t)^T W K'(t - \tau) dt = \int_{-\tau}^\infty K'(s + \tau)^T W K'(s) ds\,, \tag{25}$$

Besides, if $\tau \geq 0$, $K'(s) = 0$ for $s \in [-\tau, 0]$ and, if $\tau \leq 0$, $K'(s + \tau) = 0 = 0$ for $s \in [-\tau, 0]$. This leads to (23). Finally, Eq. (24) follows from taking a second-order derivative of (16), and from the Jump rule. □

4 Lyapunov-Krasovskii functionals and Proof of Theorem 2

We can now provide the proof of Theorem 2. Inspired by [36], where the Lyapunov functional candidate is obtained following a converse Lyapunov approach, we directly define the functional of interest based on Cauchy formula (7) for the homogeneous case. We then prove it satisfies suitable properties for ISS, by explicitly computing its time-derivative along the system trajectories.

4.1 Lyapunov-Krasovkii functionals

Following [35], we introduce the functional $v_0(\varphi)$ defined for all $\varphi \in C^{pw}_{\tau_M}$ by

$$v_0(\varphi) = a_1(\varphi) + a_2(\varphi) + 2a_3(\varphi)\,, \tag{26}$$

where

$$a_1(\varphi) = \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \varphi(\xi)^T A_i^T D_\theta^+ D_\xi^+ U(-\theta - \tau_j + \xi + \tau_i) A_j \varphi(\theta) d\theta d\xi\,, \tag{27}$$

$$a_2(\varphi) = \int_{-\tau_M}^{0} \int_{\xi_1}^{0} \int_{-\tau_M}^{0} \int_{\xi_2}^{0} \varphi(\theta_1)^T F^T(\xi_1) D_{\theta_1}^+ D_{\theta_2}^+ (U(-\xi_1 + \xi_2 + \theta_1 - \theta_2)) F(\xi_2) \varphi(\theta_2) d\theta_2 d\xi_2 d\theta_1 d\xi_1\,, \tag{28}$$

$$a_3(\varphi) = \sum_{k=1}^{M} \int_{-\tau_k}^{0} \int_{-\tau_M}^{0} \int_{\xi}^{0} \varphi(\theta)^T A_k^T D_\theta^+ D_\mu^+ U(\theta + \tau_k + \xi - \mu) F(\xi) \varphi(\mu) d\mu d\xi d\theta\,, \tag{29}$$

where we have denoted D_θ^+ the Dini derivative with respect to the variable θ. We emphasize that the definition of this functional requires system (4) to be exponentially stable, for the Lyapunov matrix U introduced in Lemma 7 to be well-defined. We now prove that v_0 has interesting properties for constructing a Lyapunov functional.

Lemma 11 *If system* (4) *is exponentially stable, then there exists* $\alpha_1 > 0$*, such that the functional* v_0 *defined in* (26) *satisfies, for all* $\varphi \in C^{pw}_{\tau_M}$*,*

$$0 \leq v_0(\varphi) \leq \alpha_1 \|\varphi\|^2_{L^2_{\tau_M}}\,. \tag{30}$$

PROOF. Define $Y(t, \varphi)$ as the solution of (4) with the initial data φ (to avoid any confusion with the solution to the inhomogeneous equation, denoted as X). Define $\Psi(\varphi) = \int_0^\infty Y^T(t, \varphi) W Y(t, \varphi) dt$, which is well-defined since Y exponentially converges to zero. Replacing Y with Cauchy's formula (7), using Fubini's theorem and the definition of the Lyapunov matrix (14), one obtains $\Psi(\varphi) = v_0(\varphi)$. Thus $v_0(\varphi) \geq 0$, as W is definite positive. Consider $N \in \mathbb{N}$ and define $g(N) = \int_0^{N\tau_M} Y^T(t, \varphi) W Y(t, \varphi) dt$. We have

$$\begin{aligned} g(N) &\leq \|W\| \sum_{k=0}^{N-1} \int_{-\tau_M}^{0} \|Y(\nu + (k+1)\tau_M, \varphi)\|^2 d\nu \\ &\leq \|W\| \sum_{k=0}^{N-1} R^2 \mathrm{e}^{-2\lambda(k+1)\tau_M} \|\varphi\|^2_{L^2_{\tau_M}}\,, \end{aligned}$$

where the constants R and λ are defined in (3). This implies the expected result by taking $N \to \infty$. □

Lemma 12 *Assume that system* (4) *is exponentially stable. Consider the functional* v_0 *defined in* (26) *and* $X_{[t]}$ *the solution of the initial value problem* (1)–(2)*. Then, for* $t \geq 0$*, we have*

$$D^+ v_0(X_{[t]}) = -X^T(t) W X(t)$$

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$$
\begin{aligned}
&- 2X^T(t)\Delta U'(0)f(t) + f^T(t)\Delta U'(0)f(t) \\
&- 2\sum_{i=1}^{M} \sum_{\tau_c \in \mathcal{I}((0,\tau_i))} X^T(t+\tau_c-\tau_i)A_i^T \Delta U'(\tau_c) f(t) \\
&- 2f^T(t)\sum_{\tau_c \in \mathcal{I}((\xi,0))} \int_{-\tau_M}^{0} \Delta U'(\tau_c)F(\xi)X(t+\xi-\tau_c)d\xi \\
&- 2f^T(t)\int_{-\tau_M}^{0}\int_{\xi}^{0} U''(\tau)F(\xi)X(t+\xi-\tau)d\tau d\xi \\
&- 2\sum_{k=1}^{M}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k)A_k^T U''(\tau)d\tau f(t)\,. \quad (31)
\end{aligned}
$$

PROOF. The proof is given in Appendix A. □

Notice that when f is equal to zero, we obtain $D^+v_0(X_{[t]}) = -X^T(t)WX(t) \leq 0$, which is the homogeneous case obtained in [35] for a Linear Difference Equation ($F \equiv 0$).

However, v_0 is not a strict Lyapunov and is only upper-bounded by the L_2-norm of the system state, according to Lemma 11, but not lower-bounded a priori. Hence, to obtain these two properties, we follow the approach of [3] and first introduce the intermediate functional $\bar{v}_0(\varphi) = v_0(e^{\cdot\frac{\rho}{2}}\varphi)$, where $\rho > 0$, and the difference one $\tilde{v}_0(\varphi) = \bar{v}_0(\varphi) - v_0(\varphi)$.

Lemma 13 *Assume that equation* (4) *is exponentially stable, and consider $X_{[t]}$ the solution of* (1). *Then there exists $K_1, K_2, K_3, K_4 > 0, a, \bar{a}, \tilde{a} > 0$, a sequence of positive coefficient $\bar{d}_q$ such that $\sum_{q\geq 0}\bar{d}_q$ converges, an increasing sequence of delays $\bar{\tau}_q$ with $\bar{\tau}_0 = 0$, independent from ρ, such that for all $\varepsilon, \varepsilon' > 0,\ t \geq 0$*

$$
\begin{aligned}
&D^+(\bar{v}_0(X_{[t]}) \leq -X^T(t)WX(t) + \left(\frac{a}{\varepsilon} + \frac{\bar{a}}{\varepsilon'} + \tilde{a}\right)\|f(t)\|^2 \\
&+ (K_1(1-e^{-\rho\tau_M}) + K_2\varepsilon)\sum_{q\geq 0}\bar{d}_q\|X(t-\bar{\tau}_q)\|^2 \qquad (32) \\
&+ (K_3(1-e^{-\rho\tau_M}) + K_4\varepsilon')\|X_{[t]}\|^2_{L^2_{\tau_M}} - \rho\bar{v}_0(X_{[t]})\,.
\end{aligned}
$$

PROOF. The proof is given in Appendix B. □

Let us now modify $\bar{v}_0$ to obtain an exponential decay, by removing the pointwise terms in $\|X_{[t]}\|^2$ and $\|X(t-\tau_q)\|^2$ in the derivative. Consider a sequence of positive coefficient b_q such that the series $\sum_{q\geq 1} b_q$ converges, and two positive coefficients $b, b' > 0$. For all $\varphi \in C^{pw}_{\tau_M}$, define the functional v_1 as

$$
\begin{aligned}
v_1(\varphi) = \bar{v}_0(\varphi) + \sum_{q\geq 1} b_q \int_{-\bar{\tau}_q}^{0} \|\varphi(u)\|^2 e^{\rho u}du + b\int_{-\tau_M}^{0} \\
\|\varphi(u)\|^2 e^{\rho u}du + b'\int_{-\tau_M}^{0}\int_{s}^{0}\|\varphi(u)\|^2 e^{\rho u}du ds\,. \qquad (33)
\end{aligned}
$$

Lemma 14 *For all $\varphi \in C^{pw}_{\tau_M}$, $v_1(\varphi) \geq 0$. Assume the system* (4) *is exponentially stable and let $X_{[t]}$ be the solution of* (1). *There exist $\rho, b, b', (b_q)_{q\in\mathbb{N}^\star}, \varepsilon, \varepsilon' > 0$ such that*

$$
\begin{aligned}
D^+v_1(X_{[t]}) \leq &-\rho v_1(X_{[t]}) + \left(\frac{a}{\varepsilon} + \frac{\bar{a}}{\varepsilon'} + \tilde{a}\right)\|f(t)\|^2 \\
&- be^{-\rho\tau_M}\|X(t-\tau_M)\|^2. \qquad (34)
\end{aligned}
$$

PROOF. Since $\bar{v}_0(\varphi) \geq 0$ it follows that $v_1(\varphi) \geq 0$. We can then take the time derivative of (33), which leads to

$$
\begin{aligned}
&D^+v_1(X_{[t]}) \leq D^+\bar{v}_0(X_{[t]}) + \sum_{q\geq 1} b_q\|X(t)\|^2 \\
&- \sum_{q\geq 1} b_q\left(e^{-\rho\bar{\tau}_q}\|X(t-\bar{\tau}_q)\|^2 + \rho\int_{-\bar{\tau}_q}^{0}\|X(t+u)\|^2 e^{\rho u}du\right) \\
&+ b'\tau_M\|X(t)\|^2 - \rho b'\int_{-\tau_M}^{0}\int_{s}^{0}\|X(t+u)\|^2 e^{\rho u}du \\
&- b'e^{-\rho\tau_M}\|X_{[t]}\|^2_{L^2_{\tau_M}} + b\|X(t)\|^2 - be^{-\rho\tau_M}\|X(t-\tau_M)\|^2 \\
&- \rho b\int_{-\tau_M}^{0}\|X(t+u)\|^2 e^{\rho u}du\,. \qquad (35)
\end{aligned}
$$

Let us choose $b, b', \varepsilon, \varepsilon', \rho > 0$ and $(b_q)_{q\geq 1} \in {\mathbb{R}_+^\star}^{\mathbb{N}}$ such that

$$
\begin{aligned}
&\bar{d}_q(K_1(1-e^{-\rho\tau_M}) + K_2\varepsilon) - b_q e^{-\rho\tau_M} < 0,\ \forall q \geq 1\,, \\
&\bar{d}_0(K_1(1-e^{-\rho\tau_M}) + K_2\varepsilon) - w + \sum_{q\geq 1} b_q + b'\tau_M + b < 0\,, \\
&K_3(1-e^{-\rho\tau_M}) + K_4\varepsilon' - b'e^{-\rho\tau_M} < 0\,,
\end{aligned}
$$

where $w > 0$ is the smallest eigenvalue of W. As these inequalities hold for $\rho = \varepsilon = \varepsilon' = 0$, and any $b, b' > 0$ and any $(b_q)_{q\geq 1} \in {\mathbb{R}_+^\star}^{\mathbb{N}}$ such that $\sum_{q\geq 1} b_q < \infty$ which are small enough to satisfy $\sum_{q\geq 1} b_q + b'\tau_M + b < w$, by continuity, these conditions are always feasible. Injecting the result of Lemma 13 into equation (35) leads to the expected result. □

In addition, the presence of a modified L_2 norm in (33) allows to obtain the desired lower bound.

Lemma 15 *Assume* (4) *is exponentially stable. There exist $\alpha_l, \alpha_u > 0$ such that, for all $\varphi \in C^{pw}_{\tau_M}$,*

$$
\alpha_l\|\varphi\|^2_{L^2_{\tau_M}} \leq v_1(\varphi) \leq \alpha_u\|\varphi\|^2_{L^2_{\tau_M}}\,. \qquad (36)
$$

PROOF. The upper bound can be achieved pretty easily using Lemma 11 and noticing that $\bar{v}_0(\varphi) = v_0(e^{\cdot\frac{\rho}{2}}\varphi)$.

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The proof of the left-hand side inequality is adjusted from[36]. We define the functional $\tilde{v}(\varphi)$ such that $\tilde{v}(\varphi) = v_1(\varphi) - \alpha_\ell \|\varphi\|^2_{L^2_{\tau_M}}$. We define Y_t as the solution of (4) with the initial data φ. We obtain from equation (34)

$$D^+ v_1(Y_t) \leq -\rho v_1(Y_t) - b e^{-\rho\tau_M} \|Y(t-\tau_M)\|^2, \quad (37)$$

since $f \equiv 0$ for solutions of the homogeneous equation. Thus, we have

$$\begin{aligned} D^+ \tilde{v}(Y_t) &\leq -b e^{-\rho\tau_M} \|Y(t-\tau_M)\|^2 \\ &- \alpha_\ell [Y^T(t)Y(t) - Y^T(t-\tau_M)Y(t-\tau_M)] \, . \end{aligned} \quad (38)$$

Choosing $0 < \alpha_\ell \leq b e^{-\rho\tau_M}$, we obtain $D^+\tilde{v}(Y_t) \leq 0$. Integrating between 0 and T, we have $\tilde{v}(Y_T) \leq \tilde{v}(\varphi)$. The exponential stability of Y_T allows taking $T \to \infty$. We can conclude that $v_1(\varphi) \geq \alpha_\ell \|\varphi\|^2_{L^2_{\tau_M}}$. □.

4.2 *Proof of Theorem 2*

We can now finalize the proof of Theorem 2.

The proof that (1) implies (2) follows from the fact that the exponential ISS of (1) implies that the homogeneous equation (4) is exponentially stable. Hence, one can consider the functional v_1 as defined in (33), and apply Lemmas 14 and 15. Let us now prove the converse statement and assume that (2) holds. From (2)(a), using the comparison principle, one obtains

$$v_1(X_{[t]}) \leq \mathrm{e}^{-\rho t} v_1(X^0) + \int_0^t \sigma \mathrm{e}^{\rho(\nu - t)} ||f(\nu)||^2 d\nu \, .$$

Using (2)(b), we thus have

$$\begin{aligned} ||X_{[t]}||^2_{L^2_{\tau_M}} &\leq \frac{\alpha_u}{\alpha_\ell} \mathrm{e}^{-\rho t} ||X^0||^2_{L^2_{\tau_M}} + \frac{\sigma}{\alpha_\ell} \int_0^t \mathrm{e}^{\rho(\nu-t)} ||f(\nu)||^2 d\nu \\ &\leq \frac{\alpha_u}{\alpha_\ell} \mathrm{e}^{-\rho t} ||X^0||^2_{L^2_{\tau_M}} + \frac{\sigma}{\rho\alpha_\ell} \sup_{s\in[0,t]} ||f(s)||^2. \end{aligned}$$

Taking the square root we obtain the expected result with $R = \sqrt{\frac{\alpha_u}{\alpha_\ell}}$, $\lambda = \rho/2$ and $\gamma = \sqrt{\frac{\sigma}{\rho\alpha_\ell}}$. □

5 Conclusion

This paper focused on investigating the Input-to-State Stability (ISS) of Linear Integral Difference Equations, which are a type of difference equations with pointwise and distributed delays. We established that the ISS of these systems is equivalent to the existence of an ISS Lyapunov functional. Drawing from recent literature on necessary conditions for the exponential stability of difference equations, we introduced a quadratic Lyapunov functional that includes the derivative of the delay Lyapunov matrix. The analysis relied on the properties of the fundamental matrix and the delay Lyapunov matrix.

Future works should focus on extending the proposed ISS-Lyapunov analysis to hyperbolic PDEs of balance laws, as it has been shown in [4] that such systems can be expressed as IDEs. For such PDEs, it would then be possible to improve the previous Lyapunov conditions obtained in [7, 8, 21] and derive less conservative Lyapunov functionals, that do not necessarily require the sufficient-only dissipativity condition outlined in [7]. The development of explicit Lyapunov functions would have significant implications for control strategies for PDEs. It would open up the possibility of combining classical backstepping controllers with event-triggered control mechanisms, thereby expanding on the results of [14, 13].

Furthermore, such generic ISS-Lyapunov functionals could be employed to evaluate the existence of robustness margins. For example, similar to the work done for time delay systems, a Lyapunov functional could demonstrate robustness in the presence of stochastic uncertainties or stochastic delays [24, 5].

A Proof of Lemma 12

The proof of this lemma involves lengthy algebraic manipulations on the Dini derivative of the functional v_0 along the system trajectories.

We start our analysis by observing that v_0 involves the derivative of U', which should be understood in the sense of distributions. Hence, the Jump rule [15, 38] formulates this distribution as the sum of Dirac distributions and the distribution associated with U''. Consequently, applying this rule, changes of variable (respectively, $\tau = \xi + \tau_i - \tau_j - \theta$ for a_1, $\tau = \theta_1 - \xi_1 + \xi_2 - \theta_2$ for a_2 and $\tau = \theta + \tau_k + \xi - \mu$ for a_3) and Fubini's theorem, v_0 can be rewritten as

$$v_0(\varphi) = \sum_{i=1}^{3} \sum_{j=1}^{2} a_i^j(\varphi) \, , \quad (\mathrm{A}.1)$$

in which,

$$\begin{aligned} a_1^1(\varphi) = &- \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \sum_{\tau_c \in (\xi+\tau_i-\tau_j, \xi+\tau_i)} \varphi^T(\xi) A_i^T \Delta U'(\tau_c) \\ &\times A_j \varphi(\tau_i - \tau_j - \tau + \xi) d\tau d\xi \, , \end{aligned} \quad (\mathrm{A}.2)$$

$$\begin{aligned} a_2^1(\varphi) = &- \int_{-\tau_M}^{0} \int_{-\tau_M}^{0} \int_{\xi_1}^{0} \sum_{\tau_c \in \mathcal{I}((-\xi_1+\xi_2+\theta_1, -\xi_1+\theta_1))} \varphi^T(\theta_1) \\ &\times F^T(\xi_1) \Delta U'(\tau_c) F(\xi_2) \varphi(-\xi_1 + \xi_2 + \theta_1 - \tau_c) d\theta_1 d\xi_2 d\xi_1 \, , \end{aligned} \quad (\mathrm{A}.3)$$

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$$a_3^1(\varphi) = -\sum_{k=1}^{M}\int_{-\tau_M}^{0}\int_{-\tau_k}^{0}\sum_{\tau_c\in\mathcal{I}((\theta+\tau_k+\xi,\theta+\tau_k))}\varphi^T(\theta) \times A_k^T\Delta U'(\tau_c)F(\xi)\varphi(\theta+\tau_k+\xi-\tau_c)d\theta d\xi\,, \quad (A.4)$$

are terms encompassing the discontinuities of U' and

$$a_1^2(\varphi) = -\sum_{i,j=1}^{M}\int_{-\tau_i}^{0}\int_{\xi+\tau_i-\tau_j}^{\xi+\tau_i}\varphi^T(\xi)A_i^TU''(\tau)A_j \times \varphi(\tau_i-\tau_j-\tau+\xi)d\tau d\xi\,, \quad (A.5)$$

$$a_2^2(\varphi) = -\int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\int_{\xi_1}^{0}\int_{-\xi_1+\xi_2+\theta_1}^{-\xi_1+\theta_1}\varphi^T(\theta_1)F^T(\xi_1) \times U''(\tau)F(\xi_2)\varphi(-\xi_1+\xi_2+\theta_1-\tau)d\tau d\theta_1 d\xi_2 d\xi_1\,, \quad (A.6)$$

$$a_3^2(\varphi) = -\sum_{k=1}^{M}\int_{-\tau_M}^{0}\int_{-\tau_k}^{0}\int_{\theta+\tau_k+\xi}^{\theta+\tau_k}\varphi^T(\theta)A_k^TU''(\tau) \times F(\xi)\varphi(\theta+\tau_k+\xi-\tau)d\tau d\theta d\xi\,. \quad (A.7)$$

are terms corresponding to the standard derivative U''.

For the sake of clarity, we will examine separately the Dini derivatives of a_1^1 and a_1^2 first, then a_2^1 and a_2^2 and finally a_3^1 and a_3^2 in the sequel.

A.1 Dini derivative of a_1^1 and a_1^2

First, applying the computations given in [3][Lemma 8] with $h(t) = f(t) + \int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi$, one gets

$$D^+a_1^1(X_{[t]}) = -X^T(t)WX(t) - 2X^T(t)\Delta U'(0)h(t) - 2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}X^T(t+\tau_c-\tau_i)A_i^T\Delta U'(\tau_c)h(t) + h(t)^T\Delta U'(0)h(t)\,. \quad (A.8)$$

Second, let us introduce

$$A : \xi \in [-\tau_M, 0] \mapsto \begin{cases} A_i & \text{if } \xi \in [-\tau_i, \tau_{i-1}), i > 1 \\ A_1 & \text{otherwise} \end{cases} \quad (A.9)$$

$$L_1^2(\theta,\tau,\xi_1,\xi_2) = X^T(\theta)A(\xi_1)^TU''(\tau) \times A(\xi_2)X(\theta-\xi_1+\xi_2-\tau)\,. \quad (A.10)$$

One can then apply Lemma 16 (given below in Appendix C) with $\mathcal{L} = L_1^2$ (using the symmetry property of U'' to prove (C.5)) and in particular (C.7) to conclude that

$$D^+a_1^2(X_{[t]}) = -2\sum_{1\le i,j\le M}\left(\int_{-\tau_j}^{\tau_i-\tau_j} L_1^2(t+\tau-\tau_i+\tau_j,\tau,-\tau_i,-\tau_j)d\tau - \int_{0}^{\tau_i} L_1^2(t+\tau-\tau_i,\tau,-\tau_i,-\tau_j)d\tau\right). \quad (A.11)$$

A.2 Dini derivative of a_2^1 and a_2^2

Let us now focus on a_2^1 and a_2^2 and introduce

$$L_2^1(\theta,\tau,\xi_1,\xi_2) = X^T(\theta)F(\xi_1)^T\Delta U'(\tau)F(\xi_2)X(\theta-\xi_1+\xi_2-\tau)\,, \quad (A.12)$$

$$L_2^2(\theta,\tau,\xi_1,\xi_2) = X^T(\theta)F(\xi_1)^TU''(\tau)F(\xi_2)X(\theta-\xi_1+\xi_2-\tau)\,, \quad (A.13)$$

to apply Lemma 16 with, respectively, $\mathcal{L} = L_2^1$ (using the symmetry property of $\Delta U'$ to prove (C.5)) and $\mathcal{L} = L_2^2$ (using the symmetry property of U'' to prove (C.5)). More precisely, (C.8) yields

$$D^+a_2^1(X_{[t]}) = \int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\Bigg(L_2^1(t+\xi_1,0,\xi_1,\xi_2) - 2\sum_{\tau_c\in\mathcal{I}((\xi_2,\xi_2-\xi_1))}L_2^1(t+\tau_c+\xi_1-\xi_2,\tau_c,\xi_1,\xi_2) + 2\sum_{\tau_c\in\mathcal{I}((\xi_2,0))}L_2^1(t+\xi_1,\tau_c,\xi_1,\xi_2)\Bigg)d\xi_1 d\xi_2\,, \quad (A.14)$$

as $\int_{-\tau_M}^{0}\int_{-\tau_M}^{0} L_2^1(t,\xi_2-\xi_1,\xi_1,\xi_2)d\xi_1 d\xi_2 = 0$ due to the fact that $\Delta U'$ is zero almost everywhere as $\mathcal{I}((-\tau_M,\tau_M))$ is countable from Lemma 8. Besides, (C.6) yields

$$D^+a_2^2(X_{[t]}) = -2\int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\Big(\int_{\xi_2}^{\xi_2-\xi_1}L_2^2(t+\tau+\xi_1-\xi_2,\tau,\xi_1,\xi_2)d\tau - \int_{0}^{-\xi_1}L_2^2(t+\tau+\xi_1,\tau,\xi_1,\xi_2)d\tau\Big)d\xi_2 d\xi_1\,. \quad (A.15)$$

Let us focus on the first term in (A.15) which, using the change of variable $\tau' = \tau - \xi_2$ and applying Fubini's theorem, can be rewritten as

$$-\int_{-\tau_M}^{0}\int_{0}^{-\xi_1}X^T(t+\tau+\xi_1)F^T(\xi_1)\int_{-\tau_M}^{0}U''(\tau+\xi_2) F(\xi_2)d\xi_2 X(t)d\tau d\xi_1\,,$$

and, using the dynamic property of U'' (24), can then be reformulated as

$$\int_{-\tau_M}^{0}\int_{0}^{-\xi_1}X^T(t+\tau+\xi_1)F^T(\xi_1)\sum_{\tau_c\in\mathcal{I}((\tau-\tau_M,\tau))}\Delta U'(\tau_c) \times F(\tau_c-\tau)X(t)d\tau d\xi_1 - \int_{-\tau_M}^{0}\int_{0}^{-\xi_1}X^T(t+\tau+\xi_1)$$

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$$F^T(\xi_1)(U''(\tau) - \sum_{i=1}^{M} U''(\tau - \tau_i)A_i)X(t)d\tau d\xi_1 \,.$$

Therefore, we obtain

$$\begin{aligned} D^+a_2^2(X_{[t]}) &= 2\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} X^T(t+\xi_2+\xi_1)F^T(\xi_1) \\ &\sum_{\tau_c\in\mathcal{I}((\xi_2-\tau_M,\xi_2)} \Delta U'(\tau_c)F(\tau_c-\xi_2)X(t)d\xi_2 d\xi_1 \\ &-2\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} X^T(t+\tau+\xi_1)F^T(\xi_1)(U''(\tau)-\sum_{i=1}^{M} \\ &U''(\tau-\tau_i)A_i)X(t)d\tau d\xi_1 + 2\int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} X(t+ \\ &\tau+\xi_1)F^T(\xi_1)U''(\tau)F(\xi_2)X(t+\xi_2)d\tau d\xi_2 d\xi_1 \,, \end{aligned} \tag{A.16}$$

which, applying again Fubini's theorem and a change of variables on the first integral, can finally be rewritten as

$$\begin{aligned} D^+a_2^2(X_{[t]}) &= 2\int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi_2,\xi_2-\xi_1))} \\ &X^T(t+\tau_c+\xi_1-\xi_2)F^T(\xi_1)\Delta U'(\tau_c)F(\xi_2)X(t)d\xi_2 d\xi_1 \\ &-2\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} X^T(t+\tau+\xi_1)F^T(\xi_1)(U''(\tau) \\ &-\sum_{i=1}^{M} U''(\tau-\tau_i)A_i)X(t)d\tau d\xi_1 + 2\int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} \\ &X(t+\tau+\xi_1)F^T(\xi_1)U''(\tau)F(\xi_2)X(t+\xi_2)d\tau d\xi_2 d\xi_1 \,. \end{aligned} \tag{A.17}$$

A.3 Dini derivative of a_3^1 and a_3^2

Let us define

$$L_3^1(\theta,\tau,\xi_1,\xi_2) = X^T(\theta)A(\xi_1)^T\Delta U'(\tau)F(\xi_2) \times X(\theta-\xi_1+\xi_2-\tau)\,, \tag{A.18}$$

$$L_3^2(\theta,\tau,-\tau_k,\xi) = X^T(\theta)A(\xi_1)^T U''(\tau)F(\xi_2) \times X(\theta-\xi_1+\xi_2-\tau)\,, \tag{A.19}$$

Applying (C.4), with $\mathcal{L} = L_3^1$, yields

$$\begin{aligned} &D^+a_3^1(X_{[t]}) = \\ &-\sum_{k=1}^{M}\int_{-\tau_M}^{0}\Bigg(\sum_{\tau_c\in\mathcal{I}((\xi,\xi+\tau_k))} L_3^1(t+\tau_c-\tau_k-\xi,\tau_c,-\tau_k,\xi) \\ &-\sum_{\tau_c\in\mathcal{I}((\xi,0))} L_3^1(t-\tau_k,\tau_c,-\tau_k,\xi) + L_3^1(t,\xi+\tau_k,-\tau_k,\xi) \\ &+\sum_{\tau_c\in\mathcal{I}((\xi+\tau_k,\tau_k))} L_3^1(t,\tau_c,-\tau_k,\xi) - L_3^1(t+\xi_1,0,-\tau_k,\xi) \\ &-\sum_{\tau_c\in\mathcal{I}((0,\tau_k))} L_3^1(t+\tau_c-\tau_k,\tau_c,-\tau_k,\xi)\Bigg)d\xi\,, \end{aligned} \tag{A.20}$$

in which $\int_{-\tau_M}^{0} L_3^1(t,\xi+\tau_k,-\tau_k,\xi)d\xi = 0$ as the integrand is null almost everywhere. Applying the change of variable $\tau_c' = \tau_c - \tau_k$ and equation (21), the fourth term of (A.20) rewrites

$$-2\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi,0))} X^T(t)\Delta U'(\tau_c)F(\xi)X(t+\xi-\tau_c)d\xi\,.$$

Then, applying Eq. (1) to substitute the terms $\sum_{k=1}^{M} A_k X(t-\tau_k)$ by $X(t) - \int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi - f(t)$ in the second and sixth terms of (A.20), we finally obtain

$$\begin{aligned} D^+a_3^1(X_{[t]}) &= \sum_{k=1}^{M}\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi,\tau_k+\xi))} X^T(t+\tau_c-\tau_k-\xi) \\ &A_k^T\Delta U'(\tau_c)F(\xi)X(t)d\xi - \int_{-\tau_M}^{0}(X^T(t+\xi_2)F^T(\xi_2)d\xi_2 \\ &\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi_1,0))} \Delta U'(\tau_c)F(\xi_1)X(t+\xi_1-\tau_c)d\xi_1) \\ &+\sum_{k=1}^{M}\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((0,\tau_k))} X^T(t+\tau_c-\tau_k)A_k^T\Delta U'(\tau_c)F(\xi) \\ &X(t+\xi)d\xi + X^T(t)\Delta U'(0)\int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi \\ &-\int_{-\tau_M}^{0} X^T(t+\xi_1)F^T(\xi_1)d\xi_1\Delta U'(0)\int_{-\tau_M}^{0} F(\xi_2) \\ &X(t+\xi_2)d\xi_2 - f^T(t)\Delta U'(0)\int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi \\ &-f^T(t)\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi,0))} \Delta U'(\tau_c)F(\xi)X(t+\xi-\tau_c)d\xi\,. \end{aligned} \tag{A.21}$$

Let us now focus on a_3^2. Applying (C.3), with $\mathcal{L} = L_3^2$, yields

$$\begin{aligned} &D^+a_3^2(X_{[t]}) = \\ &-\sum_{k=1}^{M}\int_{-\tau_M}^{0}\Bigg(\int_{\xi}^{\xi+\tau_k} L_3^2(t+\tau-\tau_k-\xi,\tau,-\tau_k,\xi_2)d\tau \\ &-\int_{\xi}^{0} L_3^2(t-\tau_k,\tau,-\tau_k,\xi)d\tau + \int_{\xi+\tau_k}^{\tau_k} L_3^2(t,\tau,-\tau_k,\xi)d\tau \\ &-\int_{0}^{\tau_k} L_3^2(t+\tau-\tau_k,\tau,-\tau_k,\xi)d\tau\Bigg)d\xi\,, \end{aligned} \tag{A.22}$$

Using simple changes of variables, the symmetry property of U'', and substituting the terms $\sum_{k=1}^{M} A_k X(t-\tau_k)$

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by $X(t)-\int_{-\tau_M}^{0} F(\xi)X(t+\xi)d\xi - f(t)$ in the second term of equation (A.22), we finally obtain

$$
\begin{aligned}
D^+a_3^2(X_{[t]}) = &-\sum_{k=1}^{M}\int_{-\tau_M}^{0}\int_{\xi}^{\xi+\tau_k} X^T(t+\tau-\tau_k-\xi) \\
&A_k^T U''(\tau)F(\xi)X(t)d\tau d\xi + \int_{-\tau_M}^{0}\int_{0}^{-\xi} X^T(t)U''^T(\tau) \\
&F(\xi)X(t+\xi+\tau)d\tau d\xi - \int_{-\tau_M}^{0}\int_{-\tau_M}^{0}\int_{0}^{-\xi_1} X^T(t+\xi_2) \\
&F^T(\xi_2)U''^T(\tau)F(\xi_1)X(t+\xi_1+\tau)d\tau d\xi_2 d\xi_1 - \int_{-\tau_M}^{0}\int_{0}^{-\xi} \\
&X^T(t+\tau+\xi)F^T(\xi)\sum_{k=1}^{M} U''(\tau-\tau_k)A_kX(t)d\tau d\xi + \sum_{k=1}^{M} \\
&\int_{-\tau_M}^{0}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k)A_k^T U''(\tau)F(\xi)X(t+\xi)d\tau d\xi \\
&- f^T(t)\int_{-\tau_M}^{0}\int_{\xi}^{0} U''(\tau)F(\xi)X(t+\xi-\tau)d\tau d\xi\,.
\end{aligned} \tag{A.23}
$$

A.4 Final computation of D^+v_0

Gathering Eq. (A.8), (A.11), (A.14), (A.17), (A.21), (A.23), one obtains

$$
\begin{aligned}
D^+v_0(X_{[t]}) = &-X^T(t)WX(t) \\
&-\sum_{i,j=1}^{M}\Bigg[2\int_{-\tau_j}^{\tau_i-\tau_j} X^T(t+\tau-\tau_i+\tau_j)A_i^T U''(\tau)A_jX(t)d\tau \\
&-2\int_{-\tau_j}^{0} X^T(t-\tau_i)A_i^T U''(\tau)A_jX(t-\tau_j-\tau)d\tau\Bigg] \\
&-2\sum_{k=1}^{M}\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi,\tau_k+\xi))} X^T(t+\tau_c-\tau_k-\xi)A_k^T\Delta U'(\tau_c) \\
&\times F(\xi)X(t)d\xi - 2\sum_{k=1}^{M}\int_{-\tau_M}^{0}\int_{\xi}^{\xi+\tau_k} X^T(t+\tau-\tau_k-\xi) \\
&\times A_k^T U''(\tau)F(\xi)X(t)d\tau d\xi - 2X^T(t)\Delta U'(0)f(t) \\
&+ f^T(t)\Delta U'(0)f(t) + 2\sum_{k=1}^{M}\int_{-\tau_M}^{0}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k) \\
&\times A_k^T U''(\tau)F(\xi)X(t+\xi)d\tau d\xi \\
&-2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_c-\tau_i)A_i^T\Delta U'(\tau_c)f(t) \\
&-2f^T(t)\int_{-\tau_M}^{0}\sum_{\tau_c\in\mathcal{I}((\xi,0))}\Delta U'(\tau_c)F(\xi)X(+\xi-\tau_c)d\xi \\
&-2f^T(t)\int_{-\tau_M}^{0}\int_{\xi}^{0} U''(\tau)F(\xi)X(t+\xi-\tau)d\tau d\xi\,.
\end{aligned} \tag{A.24}
$$

Applying the change of variable $\tau' = \tau+\tau_j$, the second term of Eq. (A.24) rewrites as

$$
-2\sum_{i,j=1}^{M}\int_{0}^{\tau_i} X^T(t+\tau-\tau_i)A_i^T U''(\tau-\tau_j)A_jX(t)d\tau\,.
$$

Besides, applying the change of variable $\tau' = -\tau$ and using the symmetry property of U'', the third term of Eq. (A.24) can be reformulated as

$$
2\sum_{i,j=1}^{M}\int_{0}^{\tau_j} X^T(t-\tau_i)A_i^T U''^T(\tau)A_jX(t+\tau-\tau_j)\,.
$$

Furthermore, applying the change of variable $\tau' = \tau-\xi$, and the dynamic property (24) of U'', the fifth term of Eq. (A.24) rewrites

$$
\begin{aligned}
&-2\sum_{k=1}^{M}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k)A_k^T[U''(\tau) - \sum_{l=1}^{M} U''(\tau-\tau_l) \\
&A_l - \sum_{\tau_c\in\mathcal{I}((\tau-\tau_M,\tau))}\Delta U'(\tau_c)F(\tau_c-\tau)]X(t)d\tau\,.
\end{aligned}
$$

Finally, using equation (1), the sixth term of Eq. (A.24) rewrites

$$
\begin{aligned}
&2\sum_{k=1}^{M}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k)A_k^T U''(\tau)[X(t) - \sum_{l=1}^{M} A_lX(t- \\
&\tau_l)]d\tau - 2\sum_{k=1}^{M}\int_{0}^{\tau_k} X^T(t+\tau-\tau_k)A_k^T U''(\tau)d\tau f(t)\,.
\end{aligned}
$$

Substituting these terms, the desired result (31) follows, with a final use of Fubini's theorem.

B Proof of Lemma 13

As the proof of this result involves computations which are very similar to the ones presented in the proof of Lemma 12, we will only give a sketch.

In order to reformulate $\bar{v}_0$ and $\tilde{v}_0$, let us first define, for $i\in\{1,2,3\}$ and $j\in\{1,2\}$,

$$
\overline{L}_i^j(t,\theta,\tau,\xi_1,\xi_2) = L_i^j(t+\theta,\tau,\xi_1,\xi_2)e^{\frac{\rho}{2}(2\theta-\tau-\xi_1+\xi_2)}, \tag{B.1}
$$

$$
\tilde{L}_i^j(t,\theta,\tau,\xi_1,\xi_2) = \overline{L}_i^j(t,\theta,\tau,\xi_1,\xi_2) - L_i^j(t+\theta,\tau,\xi_1,\xi_2)\,, \tag{B.2}
$$

in which L_i^j have been defined in (A.10), (A.12), (A.13), (A.18) and (A.19), respectively, and $L_1^2(\theta,\tau,\xi_1,\xi_2) = X^T(t)A^T(\xi_1)$ $\Delta U'(\tau)A(\xi_2)X(\theta-\xi_1+\xi_2-\tau)$.

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We first rewrite $\tilde{v}_0$ and $\bar{v}_0$ into three terms

$$\tilde{v}_0 = \tilde{a}_1 + \tilde{a}_2 + 2\tilde{a}_3\,, \quad \bar{v}_0 = \bar{a}_1 + \bar{a}_2 + 2\bar{a}_3 \tag{B.3}$$

and, from the Jump Rule, rewrite each $\tilde{a}_i$ and $\bar{a}_i$ into two terms as, for any $i \in \{1, 2, 3\}$,

$$\tilde{a}_i(\varphi) = \tilde{a}_i^1(\varphi) + \tilde{a}_i^2(\varphi)\,, \quad \bar{a}_i(\varphi) = \bar{a}_i^1(\varphi) + \bar{a}_i^2(\varphi)\,, \tag{B.4}$$

where $\tilde{a}_i^1, \bar{a}_i^1$ are terms encompassing the discontinuities of U' and $\tilde{a}_i^2, \bar{a}_i^2$ are terms corresponding to the standard derivative U'', which can be written as

$$\tilde{a}_1^1(X_{[t]}) = -\sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \sum_{\tau_c \in (\xi+\tau_i-\tau_j, \xi+\tau_i)} \tilde{L}_1^1(t, \xi, \tau_c, -\tau_i, -\tau_j) d\xi\,, \tag{B.5}$$

$$\tilde{a}_2^1(X_{[t]}) = -\int_{-\tau_M}^{0} \int_{-\tau_M}^{0} \int_{\xi_1}^{0} \tag{B.6}$$
$$\sum_{\tau_c \in \mathcal{I}((-\xi_1+\xi_2+\theta_1, -\xi_1+\theta_1))} \tilde{L}_2^1(t, \theta_1, \tau_c, \xi_1, \xi_2) d\theta_1 d\xi_2 d\xi_1\,,$$

$$\tilde{a}_3^1(X_{[t]}) = -\sum_{k=1}^{M} \int_{-\tau_M}^{0} \int_{-\tau_k}^{0} \tag{B.7}$$
$$\sum_{\tau_c \in \mathcal{I}((\theta+\tau_k+\xi, \theta+\tau_k))} \tilde{L}_3^1(t, \theta, \tau_c, -\tau_k, \xi) d\theta d\xi\,, \tag{B.8}$$

$$\tilde{a}_1^2(X_{[t]}) = -\sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \int_{\xi+\tau_i-\tau_j}^{\xi+\tau_i} \tilde{L}_1^2(t, \xi, \tau, -\tau_i, -\tau_j) d\tau d\xi\,, \tag{B.9}$$

$$\tilde{a}_2^2(X_{[t]}) = -\int_{-\tau_M}^{0} \int_{-\tau_M}^{0} \int_{\xi_1}^{0} \tag{B.10}$$
$$\int_{-\xi_1+\xi_2+\theta_1}^{-\xi_1+\theta_1} \tilde{L}_2^2(t, \theta_1, \tau, \xi_1, \xi_2) d\tau d\theta_1 d\xi_2 d\xi_1\,,$$

$$\tilde{a}_3^2(X_{[t]}) = -\sum_{k=1}^{M} \int_{-\tau_M}^{0} \int_{-\tau_k}^{0} \int_{\theta+\tau_k+\xi}^{\theta+\tau_k} \tilde{L}_3^2(t, \theta, \tau_c, -\tau_k, \xi) d\tau d\theta d\xi\,, \tag{B.11}$$

and similar formulas for the functions $\bar{a}_i^j$ but with $\overline{L}_i^j$ in lieu of $\tilde{L}_i^j$ ($i = 1, 2, 3$, $j = 1, 2$).

In the following, we first focus on the Dini derivative of $\tilde{v}_0$ in Subsection B.1 and, from it, we obtain the desired inequality on the one of $\bar{v}_0$ in Subsection B.2

B.1 Dini derivative of $\tilde{v}_0$

Observe that, for $i = 1, 2, 3$ and $j = 1, 2$,

$$\frac{\partial \tilde{L}_i^j}{\partial t}(t, \theta, \tau, \xi_1, \xi_2) = \frac{\partial \tilde{L}_i^j}{\partial \xi}(t, \theta, \tau, \xi_1, \xi_2) - \rho \bar{L}_i^j(t, \xi, \tau)\,. \tag{B.12}$$

Hence, taking the time derivative of $\tilde{a}_1^1$, one obtains

$$\begin{aligned} D^+\tilde{a}_1^1(X_{[t]}) = &-\sum_{1 \le i,j \le M} \Big(-\tilde{L}_1^1(t-\tau_i, 0, -\tau_i, -\tau_j) \\ &- \sum_{\tau_c \in \mathcal{I}((-\tau_j, 0))} \tilde{L}_1^1(t-\tau_i, \tau_c, -\tau_i, -\tau_j) \\ &+ \sum_{\tau_c \in \mathcal{I}((\tau_i-\tau_j, \tau_i))} \tilde{L}_1^1(t, \tau_c, -\tau_i, -\tau_j) \\ &- \sum_{\tau_c \in \mathcal{I}((0, \tau_i))} \tilde{L}_1^1(t+\tau_c-\tau_i, \tau_c, -\tau_i, -\tau_j) \\ &+ \sum_{\tau_c \in \mathcal{I}((-\tau_j, \tau_i-\tau_j))} \tilde{L}_1^1(t+\tau_c-\tau_i+\tau_j, \tau_c, -\tau_i, -\tau_j) \Big) \\ &+ \tilde{L}_1^1(t, \tau_i-\tau_j, -\tau_i, -\tau_j) - \rho \bar{a}_1^1(X_{[t]})\,. \end{aligned} \tag{B.13}$$

Using Young's and Cauchy Schwarz's inequalities, we have

$$\begin{aligned} &\sum_{\tau_c \in \mathcal{I}((0, \tau_i))} \tilde{L}_1^1(t+\tau_c-\tau_i, \tau_c, -\tau_i, -\tau_j) \le (1 - \mathrm{e}^{-\rho\tau_M}) \\ &\times \|A\|^2 \|X(t-\tau_j)\| \sum_{\tau_c \in \mathcal{I}((0, \tau_i))} \|\Delta U'(\tau_c)\| \, \|X(t+\tau_c-\tau_i)\| \\ &\le \quad \frac{1 - \mathrm{e}^{-\rho\tau_M}}{2} \Big(K_0^1 \|X(t-\tau_j)\|^2 \\ &\qquad + \|A\|^2 \sum_{\tau_c \in \mathcal{I}((0, \tau_i))} \|\Delta U'(\tau_c)\| \|X(t+\tau_c-\tau_i)\|^2 \Big) \end{aligned}$$

where $K_0^1 = \|A\|^2 \sum_{\tau_c \in \mathcal{I}((-\tau_M, \tau_M))} \|\Delta U'(\tau_c)\|$ (which is well defined due to Lemma 8) and $\|A\| = \max_{1 \le i \le M} \|A_i\|$. Performing analogous computations to deal with the other terms of equation (B.13), we can define a sequence of coefficients $\tilde{d}_q$ such that the series $\sum_{q \ge 0} \tilde{d}_q$ converges and an increasing sequence of delays $\tilde{\tau}_q$ with $\bar{\tau}_0 = 0$ such that

$$\begin{aligned} D^+\tilde{a}_1^1(X_{[t]}) \le K_1^1(1 - e^{-\rho\tau_M}) \sum_{q \ge 0} \tilde{d}_q \|X(t-\tilde{\tau}_q\|^2 \\ - \rho \bar{a}_1^1(X_{[t]})\,, \end{aligned} \tag{B.14}$$

for a certain $K_1^1 > 0$, with all parameters independent of ρ.

From similar computations for $\tilde{a}_1^2$, one concludes that there exists a constant $\tilde{K}_1 > 0$ such that

$$\begin{aligned} D^+\tilde{a}_1(X_{[t]}) \le \tilde{K}_1(1 - e^{-\rho\tau_M}) \Big(\sum_{q \ge 0} \tilde{d}_q \|X(t-\tilde{\tau}_q\|^2 \\ + \|X_{[t]}\|^2_{L^2_{\tau_M}} \Big) - \rho \bar{a}_1(X_{[t]})\,. \end{aligned} \tag{B.15}$$

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Similar computations for $\tilde{a}_2^1, \tilde{a}_2^2, \tilde{a}_3^1$ and $\tilde{a}_3^2$ give the existence of a positive sequence $(\tilde{d}_q)$ such that the series $\sum_{q\geq 0} \tilde{d}_q$ converges, an increasing sequence of delay $\tilde{\tau}_q$, satisfying $\tilde{\tau}_0 = 0$ and a constant $\tilde{K}_3 \geq 0$ such that

$$D^+\tilde{v}_0(X_{[t]}) \leq \tilde{K}_3(1-e^{-\rho\tau_M})\bigg(\sum_{q\geq 0}\tilde{d}_q\|X(t-\tilde{\tau}_q)\|^2 + \|X_{[t]}\|^2_{L^2_{\tau_M}}\bigg) - \rho\bar{v}_0(X_{[t]})\,. \tag{B.16}$$

B.2 Dini derivative of $\bar{v}_0$ – Conclusion of the proof of Lemma 13

By definition, we have $D^+\bar{v}_0(X_{[t]}) = D^+\tilde{v}_0(X_{[t]}) + D^+v_0(X_{[t]})$ in which the expression of $D^+v_0(X_{[t]})$ is given in Lemma 12 and $D^+\tilde{v}_0(X_{[t]})$ satisfies the inequality (B.16).

Besides, observe that, for any $\varepsilon, \varepsilon' > 0$, Cauchy-Schwarz's and Young's inequalities yield

$$X^T(t)\Delta U'(0)f(t) \leq \frac{\varepsilon}{2}\|X(t)\|^2 + \frac{1}{2\varepsilon}\|\Delta U'(0)|^2\|f(t)\|^2\,, \tag{B.17}$$

$$f^T(t)\Delta U'(0)f(t) \leq \|\Delta U'(0)\|\|f(t)\|^2\,, \tag{B.18}$$

$$\begin{aligned}&2\sum_{i=1}^M\sum_{\tau_c((0,\tau_i))}\|X^T(t+\tau_c-\tau_i)\|\|A_i\|\|\Delta U'(\tau_c)\|\|f(t)\| \\ &\quad\leq \frac{1}{\varepsilon}\alpha_1\|f(t)\|^2 + \varepsilon\|A\|^2\sum_{i=1}^M\sum_{\tau_c((0,\tau_i))}\|X^T(t+\tau_c-\tau_i)\|^2\|\Delta U'(\tau_c)\|^2,\end{aligned} \tag{B.19}$$

$$\begin{aligned}&2f^T(t)\int_{-\tau_M}^0\sum_{\tau_c\in\mathcal{I}((\xi,0))}\Delta U'(\tau_c)F(\xi)X(t+\xi-\tau_c)d\xi \\ &\quad\leq \frac{\alpha_2}{\varepsilon'}\|f(t)\|^2 + \varepsilon'\|X_{[t]}\|^2_{L^2_{\tau_M}}\|F\|^2_\infty\sum_{\tau_c\in\mathcal{I}((-\tau_M,0))}\|\Delta U'(\tau_c)\|,\end{aligned} \tag{B.20}$$

$$\begin{aligned}&2f^T(t)\int_{-\tau_M}^0\int_\xi^0 U''(\tau)F(\xi)X(t+\xi-\tau)d\tau d\xi \\ &\quad\leq \frac{\alpha_3}{\varepsilon'}\|f(t)\|^2 + \varepsilon'\|X_{[t]}\|^2_{L^2_{\tau_M}}\|F\|^2_\infty\|U''\|_\infty,\end{aligned} \tag{B.21}$$

$$\begin{aligned}&2\sum_{k=1}^M\int_0^{\tau_k}X^T(t+\tau-\tau_k)A_k^TU''(\tau)d\tau f(t) \\ &\quad\leq \frac{\alpha_4}{\varepsilon'}\|f(t)\|^2 + \varepsilon'\|X_{[t]}\|^2_{L^2_{\tau_M}}M\|A\|^2\|U''\|_\infty,\end{aligned} \tag{B.22}$$

where $\alpha_1 = \sum_{i=1}^M\sum_{\tau_c((0,\tau_i))}\|\Delta U'(\tau_c)\|$, $\alpha_2 = \tau_M \sum_{\tau_c\in\mathcal{I}((-\tau_M,\tau_M))}\|\Delta U'(\tau_c)\|$, $\alpha_3 = \tau_M\int_{-\tau_M}^{\tau_M}\|U''(\tau)\|$ and $\alpha_4 = M\int_{-\tau_M}^{\tau_M}\|U''(\tau)\|d\tau$.

Gathering (B.16), (B.17), (B.18), (B.19), (B.20), (B.21), and (B.22) finally gives the desired result.

C Intermediate results

Lemma 16 *Consider $\mathcal{L} : (\theta,\tau,\xi_1,\xi_2) \in [-\tau_M,+\infty) \times [-\tau_M,\tau_M]\times[-\tau_M,0]^2 \to \mathbb{R}$ and define, for $(t,\xi_1,\xi_2) \in \mathbb{R}_+\times\times[-\tau_M,0]^2$,*

$$I(t,\xi_1,\xi_2) = \int_{\xi_1}^0\int_{\theta+\xi_2-\xi_1}^{\theta-\xi_1}\mathcal{L}(t+\theta,\tau,\xi_1,\xi_2)d\tau d\theta\,, \tag{C.1}$$

$$J(t,\xi_1,\xi_2) = \int_{\xi_1}^0\sum_{\tau_c\in\mathcal{I}((\theta+\xi_2-\xi_1,\theta-\xi_1))}\mathcal{L}(t+\theta,\tau_c,\xi_1,\xi_2)d\theta\,, \tag{C.2}$$

in which $\mathcal{I}(t_0,t_1)$ is the set of discontinuities of the derivative U' of the Lyapunov matrix defined in Section 3.2. Then, it holds,

$$\begin{aligned}D^+I(t,\xi_1,\xi_2) &= \int_{\xi_2}^{\xi_2-\xi_1}\mathcal{L}(t+\tau+\xi_1-\xi_2,\tau,\xi_1,\xi_2)d\tau \\ &- \int_{\xi_2}^0\mathcal{L}(t+\xi_1,\tau,\xi_1,\xi_2)d\tau + \int_{\xi_2-\xi_1}^{-\xi_1}\mathcal{L}(t,\tau,\xi_1,\xi_2)d\tau \\ &- \int_0^{-\xi_1}\mathcal{L}(t+\tau+\xi_1,\tau,\xi_1,\xi_2)d\tau,\end{aligned} \tag{C.3}$$

$$\begin{aligned}D^+J(t,\xi_1,\xi_2) &= \sum_{\tau_c\in\mathcal{I}((\xi_2,\xi_2-\xi_1))}\mathcal{L}(t+\tau_c+\xi_1-\xi_2,\tau_c,\xi_1,\xi_2) \\ &- \sum_{\tau_c\in\mathcal{I}((\xi_2,0))}\mathcal{L}(t+\xi_1,\tau_c,\xi_1,\xi_2) + \mathcal{L}(t,\xi_2-\xi_1,\xi_1,\xi_2) \\ &+ \sum_{\tau_c\in\mathcal{I}((\xi_2-\xi_1,-\xi_1))}\mathcal{L}(t,\tau_c,\xi_1,\xi_2) - \mathcal{L}(t+\xi_1,0,\xi_1,\xi_2) \\ &- \sum_{\tau_c\in\mathcal{I}((0,-\xi_1))}\mathcal{L}(t+\tau_c+\xi_1,\tau_c,\xi_1,\xi_2)\,,\end{aligned} \tag{C.4}$$

Besides, if $\mathcal{L}$ is such that, for all $(\theta,\tau,\xi_1,\xi_2)$,

$$\mathcal{L}(\theta+\xi_2-\xi_1-\tau,-\tau,\xi_2,\xi_1) = \mathcal{L}(\theta,\tau,\xi_1,\xi_2), \tag{C.5}$$

then, it holds

$$\begin{aligned}&D^+\left(\int_{-\tau_M}^0\int_{-\tau_M}^0 I(t,\xi_1,\xi_2)d\xi_1 d\xi_2\right) \\ &\quad= 2\int_{-\tau_M}^0\int_{-\tau_M}^0\bigg(\int_{\xi_2}^{\xi_2-\xi_1}\mathcal{L}(t+\tau+\xi_1-\xi_2,\tau,\xi_1,\xi_2)d\tau \\ &\quad- \int_0^{-\xi_1}\mathcal{L}(t+\tau+\xi_1,\tau,\xi_1,\xi_2)d\tau\bigg)d\xi_1 d\xi_2,\end{aligned} \tag{C.6}$$

$$D^+\sum_{1\leq i,j\leq M} I(t,-\tau_i,-\tau_j)$$

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$$
\begin{aligned}
&= 2 \sum_{1 \le i,j \le M} \left(\int_{-\tau_j}^{\tau_i - \tau_j} \mathcal{L}(t+\tau-\tau_i+\tau_j, \tau, -\tau_i, -\tau_j) d\tau \right. \\
&\left. - \int_0^{\tau_i} \mathcal{L}(t+\tau-\tau_i, \tau, -\tau_i, -\tau_j) d\tau \right), \qquad \text{(C.7)}
\end{aligned}
$$

$$
\begin{aligned}
D^+ &\left(\int_{-\tau_M}^0 \int_{-\tau_M}^0 J(t,\xi_1,\xi_2) d\xi_1 d\xi_2 \right) \\
&= \int_{-\tau_M}^0 \int_{-\tau_M}^0 \Bigg(2 \sum_{\tau_c \in \mathcal{I}((\xi_2, \xi_2-\xi_1))} \mathcal{L}(t+\tau_c+\xi_1-\xi_2, \tau_c, \xi_1, \xi_2) \\
&+ \mathcal{L}(t, \xi_2-\xi_1, \xi_1, \xi_2) - \mathcal{L}(t+\xi_1, 0, \xi_1, \xi_2) \\
&- 2 \sum_{\tau_c \in \mathcal{I}((\xi_2, 0))} \mathcal{L}(t+\xi_1, \tau_c, \xi_1, \xi_2) \Bigg) d\xi_1 d\xi_2 \, . \qquad \text{(C.8)}
\end{aligned}
$$

Proof: Applying Fubini's theorem, it holds

$$
\begin{aligned}
I(t,\xi_1,\xi_2) &= \int_{\xi_2}^{\xi_2-\xi_1} \int_{\xi_1}^{\tau+\xi_1-\xi_2} \mathcal{L}(t+\theta, \tau, \xi_1, \xi_2) d\theta d\tau \\
&+ \int_{\xi_2-\xi_1}^0 \int_{\xi_1}^0 \mathcal{L}(t+\theta, \tau, \xi_1, \xi_2) d\theta d\tau \\
&+ \int_0^{-\xi_1} \int_{\tau+\xi_1}^0 \mathcal{L}(t+\theta, \tau, \xi_1, \xi_2) d\theta d\tau, \qquad \text{(C.9)}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
J(t,\xi_1,\xi_2) &= \sum_{\tau_c \in \mathcal{I}((\xi_2, \xi_2-\xi_1))} \int_{\xi_1}^{\tau_c+\xi_1-\xi_2} \mathcal{L}(t+\theta, \tau_c, \xi_1, \xi_2) d\theta \\
&+ \int_{\xi_1}^0 \mathcal{L}(t+\theta, \xi_2-\xi_1, \xi_1, \xi_2) d\theta \qquad \text{(C.10)} \\
&+ \sum_{\tau_c \in \mathcal{I}((\xi_2-\xi_1, 0))} \int_{\xi_1}^0 \mathcal{L}(t+\theta, \tau_c, \xi_1, \xi_2) d\theta \\
&+ \int_{\xi_1}^0 \mathcal{L}(t+\theta, 0, \xi_1, \xi_2) d\theta \\
&+ \sum_{\tau_c \in \mathcal{I}((0, -\xi_1))} \int_{\tau_c+\xi_1}^0 \mathcal{L}(t+\theta, \tau_c, \xi_1, \xi_2) d\theta.
\end{aligned}
$$

Taking a time-derivative and integrating with respect to θ, one obtains (C.3) and (C.4), respectively.

Then, let us integrate twice (C.3) to obtain (C.6). One obtains four terms, in which the third and second ones can be rewritten as the first and fourth ones, respectively, using the change of variable $\tau' = -\tau$, the property (C.5) of $\mathcal{L}$, the change of variable $(\xi_1', \xi_2') = (\xi_2, \xi_1)$ and Fubini's theorem. This gives (C.6). The exact same arguments applied to (C.3) lead to (C.8). Equation (C.7) also follows from very similar lines, with $(\xi_1, \xi_2) = (-\tau_i, -\tau_j)$.

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