INVERSION IN INDIRECT OPTIMAL CONTROL

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Abstract

In this paper we explain how to use inversion (as defined in nonlinear control theory) for indirect optimal control. Given the relative degree r, it is possible to recover r adjoint states and thus to simplify the problem. Explicit proof is given and relies on the triangular structure of the underlying normal form. An example from the literature is treated in the last section.

1 Introduction

Geometric tools have long been used in control theory for feedback linearization [5, 8]. The induced change of variables lead to straightforward resolution of inverse problem, i.e. computation of required inputs for a prescribed behavior of outputs. Optimization of the obtainable trajectories is an important topic, especially for applications, but generally requires numerical solvers. Inversion has lately been used in numerical optimal control and the numerical importance of the relative degree of the output chosen when casting the optimal control problem into a numerical collocation scheme was emphasized in [9]. It yields substantial reduction in required CPU time [7]. The best-case scenario is full feedback linearization (flatness) as considered in [3].

Given the system dynamics and an optimal cost, it was shown how to take advantage of the geometric structure of the dynamics to reduce the dimensionality of a numerical collocation scheme. In general collocation methods, coefficients are used to approximate with basis functions both states and inputs [4]. While it was known since [11] that it is numerically efficient to eliminate the control, it was emphasized in [9] that it is possible to reduce the problem further. Choosing outputs with maximum relative degrees is the key to efficient variable elimination that makes the required number of coefficients smaller. When combined to a nonlinear programming solver, this can induce drastic speed-ups in numerical solving [6].

In this paper we focus on indirect methods (i.e. adjoint methods) as opposed to the previously discussed direct collocation schemes. We show that in this framework it is also possible to take advantage of the geometric structure of the dynamics. More precisely we show that given a SISO system with a n-dimensional state, the relative degree of the system also plays a role in its dual dynamics. We derive an equivalent rewriting of the dynamics of the two-point boundary value problem where we eliminate as many variables as we can. In the case of full feedback linearisability (flatness) the induced equation is a $2n$-degree differential equation in the linearizing output.

The article organizes as follows. In section 1 we detail the problem we address (SISO system with generic terminal and integral cost). In section 2 we explain how to recover the geometric structure of the adjoint equation. Further we detail how to take boundary conditions into account for two special cases: terminal cost with or without terminal constraints. In section 3 we treat a numerical example from the literature.

2 Problem settings

Consider an optimal control problem with dynamics

$$\frac{dx}{dt} = f(x) + g(x)u \ , \ u \in \mathbb{R}, \ x \in \mathbb{R}^n \tag{1}$$

where all vector fields and functions are real-analytic.

It is desired to find a trajectory of (1), i.e. $[0, t_f] \ni t \mapsto (x, u)(t) \in \mathbb{R}^{n+1}$, that minimizes the performance index

$$\min \phi(x(t_f)) + \int_0^{t_f} L(x(t), u(t))dt \tag{2}$$

where L is a smooth nonlinear function

A (SISO) single input single output system

$$\begin{aligned} \frac{dx}{dt} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

is said to have *relative degree* (see for instance [5]) r at point x_0 if $L_g L_f^k h(x) = 0$, in a neighborhood of x_0 and for all $k < r-1$, and $L_g L_f^{r-1} h(x_0) \neq 0$ where $L_f h(x) = \sum_{i=1}^n \frac{\partial h}{\partial x_i} f_i(x)$ is the derivative of h along f. Roughly speaking, r is the number of times one has to differentiate y before u appears.

An important result (see [5]) is as follows. Suppose the system (3) has relative degree r at x^0. Then $r \leq n$. Set

$$\begin{aligned} \phi_1(x) &= h(x) \\ \phi_2(x) &= L_f h(x) \\ &\vdots \\ \phi_r(x) &= L_f^{r-1} h(x) \end{aligned}$$

If r is strictly less than n, it is always possible to find $n - r$ more functions $\phi_{r+1}(x), ..., \phi_n(x)$ such that the mapping

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix}$$

has a Jacobian matrix which is nonsingular at x^0 and therefore qualifies as a local coordinates transformation in a neighborhood of x^0. The value at x^0 of these additional functions can be fixed arbitrarily. Moreover, it is always possible to choose $\phi_{r+1}(x), ..., \phi_n(x)$ in such a way that $L_g \phi_i(x) = 0$, for all $r + 1 \le i \le n$ and all x around x^0.

The implication of this result is that there exists a change of coordinates

$$x \mapsto (z_1, z_2, ..., z_r, \eta) \equiv (\zeta, \eta) \in \mathbb{R}^n$$

such that the systems equations may be written under the following normal form

$$\begin{aligned}
\frac{dz_i}{dt} &= z_{i+1} \text{ for } i = 1 \ldots r - 1 & (4) \\
\frac{dz_r}{dt} &= b(\zeta, \eta) + a(\zeta, \eta)u & (5) \\
\frac{d\eta}{dt} &= q(\zeta, \eta) & (6)
\end{aligned}$$

where $a(\zeta, \eta)$ is nonzero for all (ζ, η) in a neighborhood of $(\zeta, \eta) = \phi(x^0)$.

The inverse change of coordinates gives

$$\begin{aligned}
x(t) &= A(\zeta(t), \eta(t)) & (7) \\
u(t) &= B(\zeta(t), \frac{dz_r}{dt}, \eta(t)) & (8)
\end{aligned}$$

In the coordinates (ζ, η), (2) is rewritten as

$$\min \phi(A(\zeta(t_f), \eta(t_f))) + \int_0^{t_f} L(A(\zeta(t), \eta(t)), u(t))dt \quad (9)$$

with dynamics (4,5,6) (when one does not substitute u using (8)).

3 Hamiltonian computations

3.1 The Hamiltonian

Because of the triangular structure of the previous normal form, the Hamiltonian of cost (9) with dynamics (4,5,6) is

$$\begin{aligned}
H(\zeta, \eta, u, \lambda, \mu) &= L(A(\zeta, \eta), u) + \sum_{i=1}^{i=r-1} \lambda_i z_{i+1} \\
&\quad + \lambda_r[b(\zeta, \eta) + a(\zeta, \eta)u] + \mu^T q(\zeta, \eta)
\end{aligned}$$

3.2 Adjoint equations

The necessary conditions of Pontryaguin on the Hamiltonian imply

$$\frac{d\lambda_1}{dt} = -\frac{\partial L}{\partial A}\frac{\partial A}{\partial z_1} - \lambda_r\left[\frac{\partial b}{\partial z_1} + \frac{\partial a}{\partial z_1}u\right] - \mu^T\frac{\partial q}{\partial z_1} \quad (10)$$

$$\frac{d\lambda_i}{dt} = -\frac{\partial L}{\partial A}\frac{\partial A}{\partial z_i} - \lambda_{i-1} - \lambda_r\left[\frac{\partial b}{\partial z_i} + \frac{\partial a}{\partial z_i}u\right] - \mu^T\frac{\partial q}{\partial z_i}$$

$$\text{for } i = 2 \ldots r \quad (11)$$

$$\frac{d\mu_i}{dt} = -\frac{\partial L}{\partial A}\frac{\partial A}{\partial \eta_i} - \lambda_r\left[\frac{\partial b}{\partial \eta_i} + \frac{\partial a}{\partial \eta_i}u\right] - \mu^T\frac{\partial q}{\partial \eta_i} \quad (12)$$

on the optimal trajectory.

3.3 Optimality conditions

Since there is no constraint on the control u, $\frac{\partial H}{\partial u}$ is necessary zero along the optimal trajectory. Observe that this partial derivative does not involve the adjoint states $(\lambda_1, \ldots, \lambda_{r-1}, \mu)$, i. e. , there is a function $G_0(z_1, \ldots, z_r, u, \eta)$ such that

$$\frac{\partial H}{\partial u} = G_0(z_1, \ldots, z_r, u, \eta) + \lambda_r a(\zeta, \eta) = 0 \quad (13)$$

along the optimal trajectory. Inserting (8) into (13) shows that there exists F_0 such that

$$\frac{\partial H}{\partial u} = F_0(z_1, \ldots, z_r, \frac{dz_r}{dt}, \eta, \lambda_r) = 0 \quad (14)$$

along the optimal trajectory.

Its derivatives are thus also zero along the trajectory. Their structure is given by the following lemma

Lemma 1. *For $i > r$, let $z_i(t) = \frac{d^{r-i}z_r}{dt^{r-i}} = \frac{d^{i-1}z_1}{dt^{i-1}}$. For $i = 1 \ldots r - 1$, the ith time derivative of $\frac{\partial H}{\partial u}$ along the optimal trajectory has the structure*

$$\frac{d^i}{dt^i}\left(\frac{\partial H}{\partial u}\right) = F_i(z_1, \ldots, z_{r+i+1}, \eta, \lambda_{r-i}, \ldots, \lambda_r, \mu) \quad (15)$$

where F_i is affine with respect to $(\lambda_{r-i}, \ldots, \lambda_r, \mu)$ and satisfies

$$\frac{\partial F_i}{\partial \lambda_{r-i}} = (-1)^i a(z_1, ..., z_r, \eta) \quad (16)$$

Proof. Equation (13) gives the desired result for $i = 0$. The proof proceeds then by induction. Let us assume that (15) holds for $i < r - 1$. Taking one more derivative of (15) and using (8) yields equation (17). This last function is of the form (15).

If F_i is affine with respect to $(\lambda_{r-i}, \ldots, \lambda_r, \mu)$, then the previous computations show that F_{i+1} is affine with respect to $(\lambda_{r-i-1}, \ldots, \lambda_r, \mu)$ (H is also affine with respect to the adjoint states). Observe that $\frac{d\lambda_1}{dt}$ does not appear in the computations since $r - i > 1$. Also, λ_{r-i-1} only appears once (for $j = r - i$) and the derivative of the right handside expression with respect to it is $-\frac{\partial F_i}{\partial \lambda_{r-i}}$, which proves (16) for $i + 1$. $\square$

$$
\begin{aligned}
\frac{d^{i+1}}{dt^{i+1}}\left(\frac{\partial H}{\partial u}\right) &= \sum_{j=1}^{r+i+1}\frac{\partial F_i}{\partial z_j}z_{j+1} + \frac{\partial F_i}{\partial \eta}q + \sum_{j=r-i}^{r}\frac{\partial F_i}{\partial \lambda_j}\left[-\frac{\partial L}{\partial A}\frac{\partial A}{\partial z_j} - \lambda_{j-1} - \lambda_r\left[\frac{\partial b}{\partial z_j} + \frac{\partial a}{\partial z_j}u\right] - \mu^T\frac{\partial q}{\partial z_j}\right] \\
&\quad - \frac{\partial F_i}{\partial \mu}(z_1,\ldots,z_{r+i+1},\eta)\left(\frac{\partial H}{\partial \eta}\right)^T (\zeta,\eta,\lambda_r,\mu) \\
&= \sum_{j=1}^{r+i+1}\frac{\partial F_i}{\partial z_j}z_{j+1} + \frac{\partial F_i}{\partial \eta}q \\
&\quad + \sum_{j=r-i}^{j=r}\frac{\partial F_i}{\partial \lambda_j}\left[-\frac{\partial L}{\partial A}\frac{\partial A}{\partial z_j} - \lambda_{j-1} - \lambda_r\left[\frac{\partial b}{\partial z_j} + \frac{\partial a}{\partial z_j}B(z_1,\ldots,z_{r+1},\eta)\right] - \mu^T\frac{\partial q}{\partial z_i}\right] - \frac{\partial F_i}{\partial \mu}\left(\frac{\partial H}{\partial \eta}\right)^T \\
&\equiv F_{i+1}(z_1,\ldots,z_{r+i+2},\eta,\lambda_{r-i-1},\ldots,\lambda_r,\mu) \tag{17}
\end{aligned}
$$

The adjoint states $(\lambda_1,...,\lambda_r)$ corresponding to the invertible part of the normal form (or upper part) can be eliminated from (15) along the optimal trajectory.

Lemma 2. *For $i = 0\ldots r-1$, there exists a function W_i such that*

$$\lambda_{r-i} = W_i(z_1,\ldots,z_{r+i+1},\eta,\mu) \tag{18}$$

along the optimal trajectory at times where $a(z_1,\ldots,z_r)$ is not zero. For concision we shall note $\lambda = W(z_1,...,z_{2r},\eta,\mu)$.

Proof. Gathering the conditions $H_u = 0$ and $\frac{d^i}{dt^i}\left(\frac{\partial H}{\partial u}\right) = 0$ for $i = 1,...,r-1$, one get the following system of r equations

$$F_0(z_1,\ldots,\frac{d^{r+1}z_1}{dt^{r+1}},\eta,\lambda_r) = 0$$

$$F_1(z_1,\ldots,\frac{d^{r+2}z}{dt^{r+2}},\eta,\lambda_{r-1},\lambda_r,\mu) = 0$$

$$\vdots$$

$$F_{r-1}(z_1,\ldots,\frac{d^{2r}z}{dt^{2r}},\eta,\lambda_1,\ldots,\lambda_r,\mu) = 0$$

The system $(F_i = 0)$ is linear and triangular with respect to the λ_i. Up to the sign, the diagonal term is $\frac{\partial F_0}{\partial \lambda_r} = a(z_1,\ldots,z_r)$. Hence the system $(F_i = 0)$ has a unique solution with respect to the λ_i. For $i = 0$, equation (18) is easily drawn from (13). Consider now $F_i = 0$ for $i > 0$. From (15) we see that λ_{r-i} is a function of $(z_1,\ldots,z_{r+i+1},\lambda_{r-i+1},\ldots,\lambda_r)$; for $j < i$, we assume recursively that λ_{r-j} is a function W_j of $(z_1,\ldots,z_{r+j+1})$, with $r+j+1 < r+i+1$. Hence λ_{r-i} is a function W_i of $(z_1,\ldots,z_{r+j+1})$. □

For $i = r-1$, we have

$$\lambda_1(t) = W_{r-1}(z_1(t),\ldots,z_{2r}(t),\eta,\mu) \tag{19}$$

Differentiating this equation and using the adjoint equation on λ_1 leads to

Proposition (Main result). *The optimal output z_1 satisfies the following differential equation of order $(2r)$:*

$$
\begin{aligned}
\sum_{i=1}^{i=2r}\frac{\partial W_{r-1}}{\partial z_i}z_{i+1} + \frac{\partial W_{r-1}}{\partial \eta}q - \frac{\partial W_{r-1}}{\partial \mu}\left(\frac{\partial H}{\partial \eta}\right)^T &= \\
-\mu^T\frac{\partial q}{\partial z_1} - \frac{\partial L}{\partial A}\frac{\partial A}{\partial z_1} & \\
- W_0(z_1,\ldots,z_{r+1},\eta)\left[\frac{\partial b}{\partial z_1} + \frac{\partial a}{\partial z_1}B(z_1,\ldots,z_{r+1},\eta)\right] &
\end{aligned} \tag{20}
$$

with $z_i(t) = \frac{d^{i-1}z_1}{dt^{i-1}}$.

Conversely, assume that z_1 satisfies (20); let η a solution of (6) and μ a solution of

$$
\begin{aligned}
\frac{d\mu_i}{dt} &= -\frac{\partial H}{\partial \eta_i} - \frac{\partial L}{\partial A}\frac{\partial A}{\partial \eta_i} \\
&\quad - W_0(z_1,\ldots,z_{r+1},\eta)\left[\frac{\partial b}{\partial \eta_i} + \frac{\partial a}{\partial \eta_i}u\right] - \mu^T\frac{\partial q}{\partial \eta_i}
\end{aligned} \tag{21}
$$

with

$$u = \frac{z_{r+1} - b(z_1,\ldots,z_r)}{a(z_1,\ldots,z_r)} \tag{22}$$

Define $\Lambda = W(z_1,\ldots z_{2r},\eta,\mu)$ and $\zeta = (z_1,\ldots z_r)$. If $\lambda = (\lambda_1,\ldots \lambda_r)$ satisfies (10,11) and $\partial H/\partial u(\zeta,\eta,u,\lambda,\mu) = 0$, then ζ satisfies (4,5), $\lambda = \Lambda$ and μ satisfies (12).

Proof. The left handside is the time derivative of (19). The right handside is the insertion of (8) and of (18) for $i = 0$ into (10).

The converse part essentially shows that λ can be drawn from z_1 and its derivatives and, of course η and μ. It is closely related to the direct part. From the definition (22) of u, it follows that ζ satisfies (4,5). Since $\partial H/\partial u = 0$, then λ_r satisfies (13), and hence, $\lambda_r = W_0(z_1,\ldots,z_{r+1},\eta) = \Lambda_r$. Together with (21), it follows that μ satisfies (12). Since $\partial H/\partial u = 0$, it follows from lemmas 1 and 2 that $\lambda = W(z_1,\ldots,z_{2r},\eta,\mu) = \Lambda$. □

In other words the $1 + 2(n - r)$ unknowns to be found are z_1, η, and μ. Then the Λ as defined above are appropriate candidates for the adjoint states corresponding to $(z_1, ... z_r)$ (the only adjoint states that are missing).

3.4 Endpoint conditions

With the proposed terminal cost $\phi(x(t_f)) = \phi(A(z_1, ..., z_r, \eta))$, the terminal conditions on the adjoint states are then

$$\lambda_i(t_f) = \frac{\partial}{\partial z_i} \phi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f))), \text{ for } i = 1, ..., r$$

That can be rewritten in terms of conditions upon the derivatives of the flat output

$$W_i(z_1(t_f), ..., z_{r+i+1}(t_f), \eta(t_f), \mu(t_f)) = \frac{\partial}{\partial z_{r-i}} \phi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f))),$$
$$\text{for } i = 1, ..., r$$

3.5 Endpoint equality constraints

Following Bryson and Ho's [2] presentation of optimal control problems we explain how to treat a special case of practical interest. Our proposed approach applies in this situation where extra adjoint states appear.

Let us consider the original problem (1), (2) with the additional constraint

$$\psi(x(t_f)) = 0. \tag{23}$$

Where ψ is a smooth function from $\mathbb{R}^n \to \mathbb{R}^q$, $0 < q < n$. Introducing q extra multipliers $\pi \in \mathbb{R}^q$ we get the terminal conditions

$$\lambda_i(t_f) = \frac{\partial}{\partial z_i} \phi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f)))$$
$$+ \pi^T \frac{\partial}{\partial z_i} \psi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f))) \tag{24}$$

for $i = 1, ..., r$ along with the q terminal constraints (23). Under the assumption that one can extract a non singular $q \times q$ matrix out of the Jacobian of ψ (which is a classical assumption to make (23) solvable), it is possible to isolate an invertible set of q equations in the q unknowns π from the set of equations (24). This means that these multipliers also write in terms of z_1 and its successive derivatives, η and μ. Let $(i_1, ..., i_q)$ be a set of indexes such that the $q \times q$ matrix extracted from the Jacobian of ψ is invertible, let M_q denote this matrix extracted from

$$\left(\frac{\partial}{\partial z} \psi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f))) \right)$$

This matrix depends on $z_1(t_f), ..., z_r(t_f), \eta(t_f)$. Let $\bar{W}$ denote the vector (W_{i_k}), for $k = 1, ..., q$ and $\bar{z}$ the vector (z_{i_k}), for

$k = 1, ..., q$. Then one can solve

$$\pi = M_q^{-1} \Big(\bar{W}(z_1(t_f), ..., z_r(t_f), \eta(t_f), \mu(t_f))^T$$
$$- \left(\frac{\partial}{\partial \bar{z}} \right) \phi(A(z_1(t_f), ..., z_r(t_f), \eta(t_f))) \Big)$$

4 Example

To illustrate the previous result we derive here the optimality conditions for the forced van der Pol Problem.

The dynamics of this system discussed in [7] and [10] is given by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u$$

It was used in the last two cited references as a benchmark. The optimal control problem is to minimize the quadratic cost function

$$J = \int_0^5 (x_1^2(s) + x_2^2(s) + u^2(s)) ds$$

subject to the dynamics given above and the endpoint constraints

$$x_1(0) = 1, x_2(0) = 0, x_2(5) - x_1(5) = 1$$

The dynamics is flat and $z_1 = x_1$ is a flat output (ideal case). Following the procedure presented in the previous section we get the Hamiltonian

$$H = \frac{1}{2}(x_1^2 + x_2^2 + u^2) + \lambda_1 x_2 + \lambda_2(-x_1 + (1 - x_1^2)x_2 + u)$$

The adjoint equations are

$$\dot{\lambda}_1 = -H_{x_1} = -(x_1 - 2\lambda_2 x_1 x_2) \tag{25}$$
$$\dot{\lambda}_2 = -H_{x_2} = -(x_2 + \lambda_1 + \lambda_2(1 - x_1^2))$$

Solving the two equations $H_u = 0$ and $\frac{dH_u}{dt} = 0$ for the adjoint states we get

$$\lambda_1 = x_1^{(3)} + \dot{x}_1 + 2x_1\dot{x}_1^2 - (1 - x_1^2)\ddot{x}_1$$
$$- \dot{x}_1 - \left(-x_1^{(2)} - x_1 + (1 - x_1^2)\dot{x}_1 \right)(1 - x_1^2) \tag{26}$$
$$\lambda_2 = -x_1^{(2)} - x_1 + (1 - x_1^2)\dot{x}_1 \tag{27}$$

Now one just has to substitute the expression for λ_1 and λ_2 into the differential equation (25) to get the fourth order differential equation to be satisfied by the flat output on the optimal trajectory

$$x_1^{(4)} + 2x_1^3 + 6x_1\dot{x}_1\ddot{x}_1 +$$
$$\left(2x_1\dot{x}_1 - (1 - x_1^2)\ddot{x}_1 \right)(1 - x_1^2) + 2x_1 + \ddot{x}_1 = 0$$

Finally one gets the boundary conditions

$$x_1(0) = 1$$
$$\dot{x}_1(0) = 0$$
$$\dot{x}_1(5) - x_1(5) = 0$$
$$x_1^{(3)}(5) + \dot{x}_1(5) + 2x_1(5)\dot{x}_1^2(5) - (1 - x_1^2(5))\ddot{x}_1(5) - \dot{x}_1(5)$$
$$-(-x_1^{(2)}(5) - x_1(5) + (1 - x_1^2(5))\dot{x}_1(5))(1 - x_1^2(5))$$
$$= x_1^{(2)}(5) + x_1(5) - (1 - x_1^2(5))\dot{x}_1(5)$$

the last relation being just the endpoint condition $\lambda_1(5) + \lambda_2(5) = 0$ expressed in terms of the flat output.

Solving this two point boundary value problem can be done with any appropriate numerical solver, e.g. Matlab `bvp4c`. Results give solutions comparable to the one obtained in [7] and [10], the obtained cost is 1.68588.

The adjoint states λ_1 and λ_2 also writes in terms of the flat output along the optimal trajectory as given by (26) and (27).

Conclusion

In summary, what we have shown is the following: if the system has relative degree r, then r of the adjoint states can be explicitly recovered. To do so one just has to rewrite the problem in its normal form and then write that H_u and its first $r-1$ time derivatives are zero along the optimal trajectories.

As stated in the main result of the paper, finding an optimal trajectory is equivalent to finding time functions $z_1(t) \in \mathbb{R}$, $\eta(t) \in \mathbb{R}^{n-r}$ and $\mu(t) \in \mathbb{R}^{n-r}$ that satisfy (20), (6) and (21). We have fewer unknowns than the classical n states and n adjoint states. We are now left with $1 + 2(n - r)$ unknowns. The ideal case is (as for collocation methods) the fully invertible case (flat case) where $r = n$, as treated in the example in the previous section. In any case we believe this structure could be exploited in implementation of shooting techniques (see [1] for an overview of shooting methods) as it involves fewer unknowns. Numerical relevance of this approach is currently under investigation.

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