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Impact of regular perturbations in input constrained optimal control problems

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Summary
This article explores the impact of regular perturbations (ie, small terms) in input constrained optimal control problems for nonlinear systems. In detail, it is shown that perturbation terms of magnitude ε appearing in the dynamics or the cost function lead to a variation of magnitude $K\varepsilon^2$ in the optimal cost. The scale factor K can be estimated from the nominal ($\varepsilon = 0$) solution and the analytic expressions of the perturbations. This result extends existing results that have been established in the absence of input constraints. Technically, the result is proven by means of interior penalties which allow constructing a sequence of suboptimal feasible solutions. Two numerical examples serve as illustration.

KEYWORDS
input constraints, interior methods, optimal control, regular perturbations

1 | INTRODUCTION

In optimal control, one wishes to determine control laws for a given dynamic system optimizing a criterion.[1-3] From theoretical and numerical viewpoints, the number of state variables and the presence of constraints greatly affect the resolution of optimal control problems (OCPs) by increasing its theoretic and numerical complexity. This observation holds for all methods, from dynamic programming,[4] Pontryagin minimum principle (PMP) based methods,[5,6] or direct formulations (eg, collocation methods).[7] Therefore, it is tempting to simplify the equations defining the OCP to ease the difficulty. The simplifications hopefully enable easier and faster determination of the solution, but this comes at the price of suboptimality with respect to the original problem as neither the true dynamics nor the true cost function are accurately accounted for when the simplifications are employed. In this perspective, a central question is to quantitatively evaluate the cost of dealing with simplified equations.

Formally, consider that the equations defining the OCP under consideration are dependent on some parameter ε. In system theory, such small additive terms are called regular perturbations.[8-10] In the absence of any constraints, it has been studied in References 11 and 12 (and references therein) how such perturbations affect the optimality of the solution and the state trajectories. Precisely, see References 11 and 12, if the error in the right-hand side of the dynamics and the cost function between the simplified model and the perturbed model are of magnitude ε, then the error in the optimal state trajectories and the control is bounded in the sense of L_2 norm by a function linear in ε. As a consequence, the induced suboptimality in any Lipschitz cost is bounded by a quadratic function of the form $K\varepsilon^2$.

In real situations, however, OCPs have to include constraints in their formulation.[13-18] These are the cases under consideration in this article. Interestingly, it is possible to connect constrained OCPs to unconstrained ones. Several recent works have proposed to deal with constraints by means of unconstrained representation of the variables, for example, by saturation functions[19-22] or by using a method based on interior penalties.[23,24] The latter method allows one to solve constrained OCPs by generating a convergent sequence of OCPs. By introducing penalties with a weight factor in the cost function, a new unconstrained problem can be defined for which the solution is determined from the usual stationarity conditions. Under some mild assumptions, this solution is then shown to converge to the solution of the initial constrained problem when the weight on the penalty tends to zero. The result is built around the classic ideas of penalty in finite-dimensional optimization.[25]

In this article, we employ this connection and extend the perturbation results to the cases of input constrained OCP. The proposed methodology is grounded on the results of Reference 11 about the robustness of cost, control and state with respect to model errors and the result of Reference 26, which we use to generate a sequence of problems without constraints. By studying the limits of the sequence, we show that, here also, the error in the cost function is bounded by $K\varepsilon^2$ where K is a fixed parameter.

Rather than simply stating the existence of K, we propose a way to estimate K. Importantly, the estimation method solely uses the $\varepsilon = 0$ solution and the perturbed equations. It produces an upper bound on K. This estimate is not sharp, but it is sufficient in many situations to establish that some model details are not worth consideration as the added complexity they induce is not creating sufficient cost improvement.

For illustration, we present a problem of energy management system for a parallel hybrid electric vehicle (HEV). In this problem, it is shown that the benefit of considering the engine temperature dynamics in the minimization of the fuel consumption, as has been considered in References 27-30 is actually very limited.

The article is organized as follows. Section 2 contains the problem statement and sketches the contribution. Section 3 presents preliminary results instrumental in proving the main result in Section 4. For convenience, a practical guide or "cookbook" is proposed in Section 5 summarizing the equations needed for the estimation of the parameter K. Section 6 gives numerical applications of the previous algorithm. A toy example and the HEV application are presented. In Section 7, the use of K as a tool of model design is discussed. Finally, Section 8 gives conclusions and perspectives and it is followed by appendices containing several proofs that have been omitted from the main stream of the article.

2 | PROBLEM FORMULATION AND MAIN RESULT

Consider the following OCP, which we refer to as OCP_ε,

$$\min_{u\in U^{ad}} \left[J_\varepsilon(u) = \int_0^T [L_0(x,u) + \varepsilon L_1(x,u)]\, dt \right], \tag{1}$$

where L_0 and L_1 are C^2 functions, and their first and second derivatives are assumed to be bounded, T is a fixed parameter, $\varepsilon \in [0,1]$ is a parameter scaling error terms (perturbations) in the cost function and the state dynamics defined below in (2), and $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the control variables of the following nonlinear dynamics with given initial conditions X_0

$$\frac{dx}{dt} = f_0(x,u) + \varepsilon f_1(x,u), \quad x(0) = X_0, \tag{2}$$

where f_0 and f_1 are C^2 functions with bounded first and second derivatives. We note Γ a Lipschitz constant for f_0. The control function u is constrained to belong to the set U^{ad} defined by

$$U^{ad} = \{u \in L^\infty[0,T] : \ u_{\min} \le u_i(t) \le u_{\max}, \ \text{ a.e. } \ t \in [0,T], \forall i \in \{1,\dots,m\}\}.$$

As exposed in Reference 26, U^{ad} can be generalized to be the set of integrable functions with values in a compact convex set with a non empty interior, without adding complexity (except for notations) in the computations that follow.

For convenience, we note $\sigma \triangleq [x,u]$. Furthermore, the following assumptions are considered:

Assumption 1 (Existence and uniqueness). For any $\varepsilon \ge 0$, the OCP (1) possesses a unique solution. u^*_ε denotes the corresponding optimal control and x^*_ε is the corresponding solution of the differential equation (2) (for $u = u^*_\varepsilon$).

The Hamiltonian associated with the problem for $\varepsilon = 0$ is

$$H_0(\sigma,p) = L_0(\sigma) + p^T f_0(\sigma).$$

The perturbating Hamiltonian is

$$H_1(\sigma, p) = L_1(\sigma) + p^T f_1(\sigma).$$

For any $\varepsilon \geq 0$, p^*_ε is the adjoint state associated with x^*_ε. For establishing the main result of this article, we formulate two additional assumptions.

Assumption 2 (Convexity condition on H_0). There exists $\beta > 0$ such that

$$\begin{cases} \partial_{uu} H_0(\sigma, p^*_0) \geq \beta I \quad \text{uniformly in} \quad \sigma, \\ (\partial_{xx} H_0 - \partial_{xu} H_0 [\partial_{uu} H_0]^{-1} \partial_{ux} H_0)(\sigma, p^*_0) \geq 0 \quad \text{uniformly in} \quad \sigma. \end{cases}$$

These inequalities are known in the calculus of variations as convexity conditions or strengthened Legendre-Clebsch conditions.[2] Furthermore, an assumption is formulated on the perturbating Hamiltonian. Let us first define some quantities that depend only on the unperturbed problem. We define

$$\begin{aligned} \gamma_1 &= \frac{1}{\beta} \sup_\sigma \left\| \partial_{ux} H_0(\sigma, p^*_0) \right\| \quad \alpha_1(t) = 2\Gamma \frac{e^{2\Gamma(1+\gamma_1)t} - 1}{1 + \gamma_1} \\ d_1 &= \int_0^T \alpha_1(t) dt \quad \alpha_3 = 2\left[2 + \gamma_1^2 d_1\right]. \end{aligned} \tag{3}$$

Assumption 3 (Boundedness of H_1). The perturbating Hamiltonian satisfies

$$\inf_\sigma \left\| \partial_{\sigma\sigma} H_1(\sigma, p^*_0) \right\| \leq \frac{\beta}{2(\alpha_3 + d_1)}. \tag{4}$$

Theorem 1 (Main result). *There exists a positive constant K such that the suboptimality of u^*_0 is upper bounded under the form*

$$\Delta J \triangleq J_\varepsilon(u^*_0) - J_\varepsilon(u^*_\varepsilon) \leq K\varepsilon^2 \quad \forall \varepsilon \in [0, 1]. \tag{5}$$

Remark 1. The quantity K is a linear combination of the squares of bounds on the perturbating terms f_1 and L_1 evaluated along the unperturbed optimal trajectory, and of the squares bounds on the derivatives of these perturbating terms. Also, K tends to the infinity when the convexity constant β tends to 0 like $\frac{1}{\beta}$, and K depends on the bounds of the second derivatives of f_1 and L_1, and of the nominal costate p_0, but not linearly. The bound K increases as the bounded-output (BIBO) behavior of the nominal system increases.

3 | PRELIMINARY RESULTS

To prove Theorem 1, we establish some preliminary technical results. A sequence of unconstrained problems can be considered, which converges to OCP_ε. For this, following Reference 26, a penalty function $P(u)$ is introduced into the cost. This penalty function is used to define the penalized OCP,

$$\min_{u \in U^{ad}} \left[J^r_\varepsilon(u) = \int_0^T [L_0(\sigma) + \varepsilon L_1(\sigma) + rP(u)] dt \right], \quad r > 0. \tag{6}$$

This approach is very general, see Reference 31 and references therein. For each value of $r > 0$, the solution of OCP (6) is determined from *simple stationarity conditions on the Hamiltonian* since the optimum is interior. To exploit this technique, we formulate the following assumption:

Assumption 4 (Penalty properties). The penalty $P(.) :]u_{\min}, u_{\max}[\mapsto \mathbb{R}^+$ satisfies the following conditions:[26]

- the function $P(.)$ is C^1, strictly convex, and non-decreasing,
- the penalty $P(.)$ and its derivative $P'(.)$ grow unbounded as u reaches either $u_{\min}$ or $u_{\max}$.

As was shown in Reference 24, when r goes to zero, under Assumption 4, the optimal value of the modified cost (6) converges to the optimal cost of (1) under input constraints and the penalty term $rP(u)$ goes to zero. Because $P(.)$ takes infinite value outside the domain defining U^{ad} and on its boundary, the solutions lie inside this open domain.

Using the PMP, the two-point boundary value problem (TPBVP) associated with the nominal problem (for $\epsilon = 0$) is given by (2) and

$$-\dot{p}_0^{rT} = \partial_x L_0(\sigma_0^r) + p_0^{rT} \partial_x f_0(\sigma_0^r), \quad p_0^{rT}(T) = 0, \tag{7}$$

$$\partial_u L_0(\sigma_0^r) + r\partial_u P(u_0^r) + p_0^{rT} \partial_u f_0(\sigma_0^r) = 0. \tag{8}$$

Here σ_0^r denotes the optimal state and control for (6) with $\epsilon = 0$, and p_0^r is the related costate. From theorem 4 of Reference 26, one has that as the r parameter approaches 0, then u_0^r and x_0^r approach u_0^* and x_0^* in the L^2 and L^∞ norms, respectively. From (7), it follows that p_0^r approaches p_0^* in the L^∞ norm.

The Hamiltonian associated with the problem (for $\epsilon = 0$) is

$$H_0^r(\sigma, p) = H_0(\sigma, p) + rP(u).$$

In the case $\epsilon > 0$, the Hamiltonian associated with this problem is

$$H_\epsilon^r(\sigma, p) = L_0(\sigma) + \epsilon L_1(\sigma) + p^T \left[f_0(\sigma) + \epsilon f_1(\sigma)\right] + rP(u) = H_0^r(\sigma, p) + \epsilon H_1(\sigma, p), \tag{9}$$

where $H_1(\sigma, p) \triangleq L_1(\sigma) + p^T f_1(\sigma)$ is independent of the penalty function. For any r, we note p_ϵ^r the adjoint state associated with the state x_ϵ^r and u_ϵ^r the optimal control solution of the OCP (6) for $\epsilon \geq 0$. Denote for any x, u, x_ϵ^r, and u_ϵ^r

$$\begin{aligned} &w \triangleq [\sigma\ p], \quad \delta x^r \triangleq x - x_0^r, \quad \delta u^r \triangleq u - u_0^r, \quad \delta\sigma^r \triangleq \sigma - \sigma_0^r, \\ &\delta x_\epsilon^r \triangleq x_\epsilon^r - x_0^r, \quad \delta u_\epsilon^r \triangleq u_\epsilon^r - u_0^r, \quad \delta\sigma_\epsilon^r \triangleq \sigma_\epsilon^r - \sigma_0^r. \end{aligned}$$

To estimate an upper bound on ΔJ, the two following Propositions 1 and 2 are used. These two general results are based on Taylor expansion and differential calculus.

Proposition 1 (Second-order expansion). *For any control u, $J_\epsilon^r(u)$ can be written as*

$$\begin{aligned} J_\epsilon^r(u) = &\int_0^T \left[H_\epsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r\right] dt + \epsilon \int_0^T \left[N^0(t) \cdot \delta u^r + N^1(t) \cdot \delta x^r\right] dt \\ &+ \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\epsilon^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r)(\delta\sigma^r)^2 d\lambda d\mu dt, \end{aligned} \tag{10}$$

where

$$N^0(t) \triangleq \partial_u H_1(\sigma_0^r, p_0^r), \quad N^1(t) \triangleq \partial_x H_1(\sigma_0^r, p_0^r).$$

As the term $\int_0^T \left[H_\epsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r\right] dt$ depends only on the nominal trajectories, it can be seen as a constant term. The second term $\int_0^T \left[N^0(t)\delta u^r + N^1(t)\delta x^r\right] dt$ represents the first-order variation of the cost due to the state and control trajectories variations.

Proof. The proof is based on Taylor expansion and is given in Appendix A. This expansion uses the stationarity condition (8); interiorness is instrumental in the proof. ∎

For any given r, x_0^r is the solution of the differential equation (2) for the control u_0^r:

$$\frac{dX_0^r}{dt} = f_0(X_0^r, u_0^r) + \epsilon f_1(X_0^r, u_0^r), \quad X_0^r(0) = x_0(0), \tag{11}$$

while x_0^r satisfies

$$\frac{dx_0^r}{dt} = f_0(x_0^r, u_0^r), \quad x_0^r(0) = x_0(0). \tag{12}$$

The two trajectories of $X_0^r(t)$ and $X_0^r(t)$ have the same control input u_0^r and the same initial conditions. The following proposition gives an upper bound on $\|X_0^r(t) - x_0^r(t)\|$.

Proposition 2. *Consider (11) and (12), the error $\|X_0^r(t) - x_0^r(t)\|$ satisfies*

$$\|X_0^r(t) - x_0^r(t)\| \leq F_1 q(t)\varepsilon, \tag{13}$$

where

$$F_1 = \sup_{t\in[0,T]} \|f_1(\sigma_0^r(t))\|, \quad q(t) = \frac{1}{\Gamma}(e^{\Gamma t} - 1). \tag{14}$$

and Γ is the Lipschitz constant of f_ε.

Proof. The proof is given in Appendix B. ∎

Remark 2. Observe that evaluating the upper bounds given in (14) does not require to solve the perturbed OCP. The first quantity F_1 evaluates the perturbation term on the dynamics along the non perturbed trajectory, and $q(t)$ quantitatively expresses the BIBO behavior of f_0.

4 | PROOF OF THE MAIN RESULT

To prove Theorem 1, we need the following intermediate upper bounds on $x_\varepsilon^r(t) - x_0^r(t)$ and $u_\varepsilon^r(s) - u_0^r(s)$.

Lemma 1. *There exist positive constants c_x and c_u such that, for all $r > 0$ and all penalty functions $P(.)$*

$$\left|x_\varepsilon^r(t) - x_0^r(t)\right|^2 \leq c_x^2\varepsilon^2, \tag{15}$$

$$\int_0^T \left|u_\varepsilon^r(s) - u_0^r(s)\right|^2 ds \leq c_u^2\varepsilon^2. \tag{16}$$

The proof of this lemma is divided into two parts, each of which is summarized in a proposition.

1. First, an upper bound is derived for the quantity

$$M_0 \triangleq J_\varepsilon^r(u_0^r) - \int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r\right] dt. \tag{17}$$

This upper bound is given in Proposition 3.

2. Then, using Assumption 2, we define a new variable

$$z(\lambda, \mu, t) \triangleq \delta u_\varepsilon^r + [\partial_{uu}H_0^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r)]^{-1}\partial_{ux}H_0^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r)\delta x_\varepsilon^r. \tag{18}$$

An upper bound on

$$R \triangleq \int_0^T \int_0^1 \int_0^1 \lambda\|z(\lambda, \mu, t)\|^2 d\lambda d\mu dt, \tag{19}$$

is given in Proposition 4 where the inequalities (15), (16) are derived.

4.1 | Upper bound on M_0 defined in (17)

An upper bound on M_0 is calculated in the following proposition.

Proposition 3. *There exist positive constants c_0 and c_1 such that, for all $r > 0$, for all P,*

$$|M_0| \leq (c_0 F_1^2 + c_1)\varepsilon^2. \tag{20}$$

These are

$$c_0 = \frac{1}{2}\left(\sup_{t\in[0,T]} \partial_{xx}H_0^r(\sigma_0^r, p_0^r) + m\right)\int_0^T q^2(t)dt + \frac{1}{2}\sup_{t\in[0,T]} \partial_{xx}H_1(\sigma_0^r, p_0^r)\int_0^T q^2(t)dt, \tag{21}$$

$$c_1 = \frac{1}{2m}\int_0^T k_1^2(t)dt, \tag{22}$$

where m is a (free) positive constant, q is given in (14), and k_1 is an upper bound on $N^1(t)$. In particular, c_0, c_1, and the upper bound in (20) are independent of $rP(.)$.

Proof. The proof is based on the second-order expansion given by (10). From Proposition 1, the penalized cost function $J_\varepsilon^r(u_0^r)$ can be rewritten in the form

$$\begin{aligned} J_\varepsilon^r(u_0^r) = &\int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r\right]dt + \varepsilon\int_0^T N^1(t)\cdot(X_0^r - x_0^r)dt \\ &+ \int_0^T\int_0^1\int_0^1 \lambda\partial_{xx}H_\varepsilon^r(x_0^r + \lambda\mu(X_0^r - x_0^r), u_0^r, p_0^r)(X_0^r - x_0^r)^2 d\lambda d\mu dt, \end{aligned} \tag{23}$$

where x_0^r and x_0^r are defined in (11) and (12). The quantity M_0, defined in (17) can be written from (23) as

$$M_0 = \varepsilon\int_0^T N^1(t)\cdot(X_0^r - x_0^r)dt + \int_0^T\int_0^1\int_0^1 \lambda\partial_{xx}H_\varepsilon^r(x_0^r + \lambda\mu(X_0^r - x_0^r), u_0^r, p_0^r)(X_0^r - x_0^r)^2 d\lambda d\mu dt. \tag{24}$$

In this expression, the penalty on the control disappears from the calculation because the two state trajectories x_0^r and x_0^r share the same control input (the error in the state trajectories is induced by the perturbation terms in the state dynamics). Since the first derivatives of L_1 and f_1 are bounded by assumption, N^0 and N^1 are bounded:

$$\left|N^1(t)\right| \leq k_1(t), \quad \left|N^0(t)\right| \leq k_2(t). \tag{25}$$

Indeed, the terms $N^1(t)$ and $N^0(t)$ depend only on the nominal trajectories and they can be bounded by functions of time. The upper bound on $N^1(t)\cdot(X_0^r - x_0^r)$ can be written as

$$\varepsilon\int_0^T N^1(t)\cdot(X_0^r - x_0^r)dt \leq \frac{\varepsilon^2}{2m}\int_0^T (N^1(t))^2 dt + \frac{m}{2}\int_0^T (X_0^r - x_0^r)^2 dt,$$

using the following inequality, for any a, b and $m > 0$: $2ab \leq \frac{1}{m}a^2 + mb^2$. Inserting Equation (13) to bound $X_0^r - x_0^r$ yields

$$\begin{aligned} \varepsilon\int_0^T N^1(t)\cdot(X_0^r - x_0^r)dt &\leq \frac{\varepsilon^2}{2m}\int_0^T k_1^2(t)dt + \frac{\varepsilon^2 m}{2}F_1^2\int_0^T q^2(t)dt, \\ &\leq \frac{\varepsilon^2}{2}\left(\frac{1}{m}\int_0^T k_1^2(t)dt + mF_1^2\int_0^T q^2(t)dt\right). \end{aligned}$$

From the decomposition in Equation (9), we have

$$\partial_{xx}H_\varepsilon^r(.) = \partial_{xx}H_0^r(.) + \varepsilon\partial_{xx}H_1(.).$$

As the second derivatives of L_0 and f_0 are assumed to be bounded and the term $\partial_{xx}H_0^r$ is independent of the penalty $P(.)$, we can define

$$\gamma_0 = \sup_{t\in[0,T]} \partial_{xx}H_0^r(.).$$

By using the relation (13), we derive that

$$\left|\int_0^T\int_0^1\int_0^1 \lambda\partial_{xx}H_0^r(x_0^r+\lambda\mu(X_0^r-x_0^r),u_0^r,p_0^r)(X_0^r-x_0^r)^2 d\lambda d\mu dt\right| \le \frac{\varepsilon^2}{2}\gamma_0 F_1^2\int_0^T q^2(t)dt.$$

As ε is in $[0,1]$, $\varepsilon^3\le\varepsilon^2$ and we can write the following upper bound

$$\left|\int_0^T\int_0^1\int_0^1 \varepsilon\lambda\partial_{xx}H_1^r(x_0^r+\lambda\mu(X_0^r-x_0^r),u_0^r,p_0^r)(X_0^r-x_0^r)^2 d\lambda d\mu dt\right| \le \frac{\varepsilon^2}{2}\sup_{t\in[0,T]}\partial_{xx}H_1(.)F_1^2\int_0^T q^2(t)dt.$$

From Equation (24), M_0 is thus bounded by

$$\left|J_\varepsilon^r(u_0^r)-\int_0^T\left[H_\varepsilon^r(\sigma_0^r,p_0^r)-p_0^{rT}\dot{x}_0^r\right]dt\right| \le (c_0F_1^2+c_1)\varepsilon^2,$$

where c_0 and c_1 are given in (21) and (22). They are independent of $rP(.)$. This concludes the proof. ■

Remark 3. Observe that the bound given in (20) involves the square of F_1, and that c_1 is a bound on the square of the perturbed cost and dynamics, following (22)-(25). Their evaluation does not require to solve the perturbed OCP.

4.2 | Upper bound on R defined in (19)

Proposition 4. *There exists a constant c_2, such that, for all $r>0$, for all P,*

$$R\le c_2\varepsilon^2,$$

where c_2 is proportional to the inverse of the convexity parameter β defined in Assumption 2 and proportional to the square of the perturbating terms and their derivatives.

Proof. Essentially, the proof is based on the decomposition suggested in Proposition 1 and the convexity conditions given in Assumption 2. The variable z defined in (18) will be helpful as it allows to deal with diagonal quadratic forms.

Since u_ε^r is the optimal control of the perturbed problem, it satisfies

$$J_\varepsilon^r(u_\varepsilon^r)\le J_\varepsilon^r(u_0^r),$$

which gives

$$J_\varepsilon^r(u_\varepsilon^r)-\int_0^T\left[H_\varepsilon^r(\sigma_0^r,p_0^r)-p_0^{rT}\dot{x}_0^r\right]dt \le J_\varepsilon^r(u_0^r)-\int_0^T\left[H_\varepsilon^r(\sigma_0^r,p_0^r)-p_0^{rT}\dot{x}_0^r\right]dt \le (c_0F_1^2+c_1)\varepsilon^2,$$

that leads to

$$J_\varepsilon^r(u_\varepsilon^r)-\int_0^T\left[H_\varepsilon^r(\sigma_0^r,p_0^r)-p_0^{rT}\dot{x}_0^r\right]dt \le (c_0F_1^2+c_1)\varepsilon^2. \tag{26}$$

By using Proposition 1, $J_\varepsilon^r(u_\varepsilon^r)$ can be written under the form

$$J_\varepsilon^r(u_\varepsilon^r) = \int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r\right] dt + \varepsilon \int_0^T \left[N^0(t)\delta u_\varepsilon^r + N^1(t)\delta x_\varepsilon^r\right] dt$$
$$+ \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt,$$

and, we have

$$J_\varepsilon^r(u_\varepsilon^r) - \int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r\right] dt = \varepsilon \int_0^T \left[N^0(t)\delta u_\varepsilon^r + N^1(t)\delta x_\varepsilon^r\right] dt$$
$$+ \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt.$$

By combining this expression with (26), we obtain

$$(c_0 F_1^2 + c_1)\varepsilon^2 \geq \varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r\right] dt + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt. \tag{27}$$

From the expression of H_ε^r in (9), we have

$$\partial_{\sigma\sigma} H_\varepsilon^r(.) = \partial_{\sigma\sigma} H_0^r(.) + \varepsilon \partial_{\sigma\sigma} H_1(.).$$

To find a bound on $\partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2$, every factor of δu_ε^r in the second-order variation of the cost $J_\varepsilon^r(u_\varepsilon^r)$ is substituted by terms in δx_ε^r and z defined by (18). This allows us to handle a diagonal quadratic form in terms of z and δx_ε^r. The following expression of $\partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2$ holds

$$\partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 = \delta x_\varepsilon^{rT} \partial_{xx} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)\delta x_\varepsilon^r$$
$$+ \delta u_\varepsilon^{rT} \partial_{uu} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)\delta u_\varepsilon^r + 2\delta u_\varepsilon^{rT} \partial_{ux} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)\delta x_\varepsilon^r,$$

which can be written using the variable z as

$$\partial_{\sigma\sigma} H_0^r(.)(\delta\sigma_\varepsilon^r)^2 = z^T \partial_{uu} H_0^r(.) z + \delta x_\varepsilon^{rT} \left[\partial_{xx} H_0^r - \partial_{xu} H_0^r [\partial_{uu} H_0^r]^{-1} \partial_{ux} H_0^r\right](.)\delta x_\varepsilon^r.$$

The term $\partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2$ is written as the sum of terms whose signs are known from the second-order optimality conditions given in Assumption 2,

$$\partial_{\sigma\sigma} H_0^r(.)(\delta\sigma_\varepsilon^r)^2 \geq \beta \|z(\lambda, \mu, t)\|^2.$$

Thus, Equation (27) implies

$$(c_0 F_1^2 + c_1)\varepsilon^2 \geq \varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r\right] dt + \beta R$$
$$+ \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt. \tag{28}$$

We now estimate the error in the state trajectories due to the control input variation and perturbations in the dynamics.

Proof. There exist positive constants (α_3, α_4) and bounded time functions (α_1, α_2) such that

$$\|\delta x_\varepsilon^r(t)\|^2 \leq \alpha_1(t) \int_0^T \int_0^1 \int_0^1 \lambda \|z(\lambda, \mu, t)\|^2 d\lambda d\mu dt + \alpha_2(t) F_1^2 \varepsilon^2, \tag{29}$$

$$\int_0^T \|\delta u_\varepsilon^r(t)\|^2 dt \le \alpha_3 \int_0^T \int_0^1 \int_0^1 \lambda \|z(\lambda,\mu,t)\|^2 d\lambda d\mu dt + \alpha_4 F_1^2 \varepsilon^2, \tag{30}$$

where F_1 is given by

$$F_1 = \sup_{t\in[0,T]} \left\| f_1(\sigma_0^r(t)) \right\|,$$

and the variable z is defined in (18). The expression of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is given in Equations (C4) and (C6). ■

The proof of this lemma is given in Appendix C.

We now proceed with establishing a bound for R defined in (19), which appears in (29) and (30). Consider (3) and (C6). By using Young inequality (holding for any a, b and $m > 0$)

$$2ab \ge -\frac{1}{m}a^2 - mb^2,$$

the term $\varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r(t) + N^1 \delta x_\varepsilon^r(t)\right] dt$ is lower bounded as follows

$$\begin{aligned} \varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r\right] dt &\ge -\int_0^T \left[\frac{\varepsilon^2}{2m}\left\{(N^0(t))^2 + (N^1(t))^2\right\} + \frac{m}{2}\left\{\|\delta x_\varepsilon^r\|^2 + \|\delta u_\varepsilon^r\|^2\right\}\right] dt, \\ &\ge -\frac{\varepsilon^2}{2m}\int_0^T (k_2^2(t) + k_1^2(t))dt - F_1^2 \frac{\varepsilon^2 m}{2}\left(\alpha_4 + \int_0^T \alpha_2(s)ds\right) \\ &\quad - \frac{m}{2}\left[\alpha_3 + \int_0^T \alpha_1(s)ds\right] R. \end{aligned} \tag{31}$$

Inserting (31) into (28) yields, using (3),

$$\begin{aligned} (c_0 F_1^2 + c_1)\varepsilon^2 \ge& -\varepsilon^2\left[\frac{1}{2m}\int_0^T (k_2^2(t) + k_1^2(t))dt + \frac{m}{2}F_1^2(\alpha_4 + d_2)\right] - \frac{m}{2}\left[\alpha_3 + d_1\right] R \\ &+ \beta R + \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt. \end{aligned} \tag{32}$$

The term $\varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt$ gives rise to a term in ε^3 (which can be bounded ε^2 as $\varepsilon \le 1$). We obtain for this last term:

$$\varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(.)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt \ge -\frac{1}{2}\inf_\sigma \left\|\partial_{\sigma\sigma} H_1(\sigma, p_0^r)\right\| \left[F_1^2(\alpha_4 + d_2)\varepsilon^2 + \varepsilon(\alpha_3 + d_1)R\right]. \tag{33}$$

Inequalities (32) and (33) imply that

$$\begin{aligned} &\left[\beta - \frac{m}{2}(\alpha_3 + d_1) - \frac{\varepsilon}{2}\inf_\sigma \left\|\partial_{\sigma\sigma} H_1(\sigma, p_0^r)\right\|(\alpha_3 + d_1)\right] R \\ &\le \left[c_0 + \frac{m}{2}(\alpha_4 + d_2) + \frac{1}{2}\inf_\sigma \left\|\partial_{\sigma\sigma} H_1(\sigma, p_0^r)\right\|(\alpha_4 + d_2)\right] F_1^2 \varepsilon^2 + \left[c_1 + \frac{1}{2m}\int_0^T (k_2^2(t) + k_1^2(t))dt\right]\varepsilon^2, \end{aligned} \tag{34}$$

where $(d_1, d_2, \alpha_3, \alpha_4)$ are defined in (C6). We wish that the factor of R in the left-hand side of (34) be positive. Define γ by

$$\gamma = \beta - \frac{m}{2}(\alpha_3 + d_1) - \frac{\varepsilon}{2}\inf \left\|\partial_{\sigma\sigma} H_1(\sigma, p_0^r)\right\|(\alpha_3 + d_1). \tag{35}$$

We want $\gamma > 0$. To start with, we take

$$m = \frac{\beta}{\alpha_3 + d_1}. \tag{36}$$

We then have

$$\gamma = \frac{\beta}{2} - \frac{\varepsilon}{2} \inf_{\sigma} \left\| \partial_{\sigma\sigma} H_1(\sigma, p_0^r) \right\| (\alpha_3 + d_1). \tag{37}$$

Assumption 3 ensures that $\gamma \geq \frac{\beta}{8}$ for any $\varepsilon \in [0, 1]$ and r close to 0, by convergence of the costate discussed in Section 3. To pursue our analysis of (34), we define

$$s_{2a} = c_0 + \frac{m}{2}(\alpha_4 + d_2) + \frac{1}{2} \inf_{\sigma} \left\| \partial_{\sigma\sigma} H_1(\sigma, p_0^r) \right\| (\alpha_4 + d_2),$$
$$s_{2b} = c_1 + \frac{1}{2m} \int_0^T (k_2^2(t) + k_1^2(t)) dt.$$

Remark 4. Observe that s_{2b} is equal to the sum of c_1, which is, as we observed it before, proportional to the square of the perturbating terms, and of the squares of k_1 and k_2 which, as shown in (25), are proportional to a bound on the derivatives of the perturbating terms. Overall, s_{2b} is bound by the square of bounds on the perturbation terms. Also, it tends to the infinity when β tends to zero like $\frac{1}{\beta}$.

Inequality (34) can be written as

$$\frac{\beta}{8} R \leq (s_{2a} F_1^2 + s_{2b}) \varepsilon^2.$$

This gives

$$R \leq 8 \frac{s_{2a} F_1^2 + s_{2b}}{\beta} \varepsilon^2. \tag{38}$$

This concludes the proof. ∎

Remark 5. This relation shows that the upper bound on R is proportional to the square of the inverse of the convexity parameter β (see the remark on s_{2b}). It is also proportional to the square of bounds of the perturbation terms since we have seen that F_1^2 and s_{2b} are proportional to the square of bounds on the perturbating terms.

From the two inequalities (29), (30), the upper bounds on δx_ε^r and δu_ε^r are of the form

$$\|\delta x_\varepsilon^r(t)\|^2 \leq \left[\alpha_1(t) c_2 + \alpha_2(t) F_1^2\right] \varepsilon^2 \triangleq c_x^2(t) \varepsilon^2,$$
$$\int_0^T \|\delta u_\varepsilon^r(t)\|^2 dt \leq \left[\alpha_3 c_2 + \alpha_4 F_1^2\right] \varepsilon^2 \triangleq c_u^2 \varepsilon^2,$$

and the inequalities (15) and (16) of Lemma 1 are proven.

Remark 6. Observe that the bounds on the state and control errors are a linear combination of the square of F_1, which is proportional to a bound on the amplitude of the perturbed dynamics, and of c_2, which is proportional to the square of the bounds on the derivatives of the perturbating terms, and tends to the infinity when β tends to 0.

4.3 | Upper bound on ΔJ

The final step is to establish the upper bound on ΔJ.

Proof. The upper bound on ΔJ is a consequence of the upper bounds on δx_ε^r, δu_ε^r and R given in (15), (16), and (38), respectively. The term $J_\varepsilon^r(u_\varepsilon^r) - J_\varepsilon^r(u_0^r)$ can be written as

$$J_\varepsilon^r(u_\varepsilon^r) - J_\varepsilon^r(u_0^r) = J_\varepsilon^r(u_\varepsilon^r) - \int_0^T \left[H_\varepsilon^r(w_0^r) - p_0^{rT} \dot{x}_0^r\right] dt - J_\varepsilon^r(u_0^r) + \int_0^T \left[H_\varepsilon^r(w_0^r) - p_0^{rT} \dot{x}_0^r\right] dt,$$

which implies

$$J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \le \left| J_\varepsilon^r(u_\varepsilon^r) - \int_0^T \left[H_\varepsilon^r(w_0^r) - p_0^{rT}\dot{x}_0^r \right] dt \right| + \left| J_\varepsilon^r(u_0^r) - \int_0^T \left[H_\varepsilon^r(w_0^r) - p_0^{rT}\dot{x}_0^r \right] dt \right|,$$
$$\le |M_1| + |M_0|,$$

where M_0 is defined in (17) and M_1 is given by

$$M_1 = J_\varepsilon^r(u_\varepsilon^r) - \int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r \right] dt. \tag{39}$$

The upper bound on M_0 is given in (20). This bound is a linear combination of the square of the perturbing dynamics along the nominal trajectory, and on the square of the derivatives of the perturbation terms.

Using Proposition 1, the cost $J_\varepsilon^r(u_\varepsilon^r)$ can be rewritten as

$$\begin{aligned} J_\varepsilon^r(u_\varepsilon^r) = & \int_0^T \left[H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\dot{x}_0^r \right] dt + \varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r \right] dt \\ & + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt \\ & + \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt, \end{aligned}$$

and thus M_1 defined in (39) can be written as follows

$$\begin{aligned} M_1 = & \varepsilon \int_0^T \left[N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r \right] dt + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt \\ & + \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt. \end{aligned}$$

An upper bound on M_1 can be written as

$$\begin{aligned} M_1 \le & \int_0^T \left[\frac{\varepsilon^2}{2m} \left\{ (N^0(t))^2 + (N^1(t))^2 \right\} + \frac{m}{2} \left\{ \|\delta x_\varepsilon^r\|^2 + \|\delta u_\varepsilon^r\|^2 \right\} \right] dt \\ & + \int_0^T \int_0^1 \int_0^1 \lambda \left[z^T \partial_{uu} H_0^r(.) z + \delta x_\varepsilon^{rT} \left[\partial_{xx} H_0^r - \partial_{xu} H_0^r [\partial_{uu} H_0^r]^{-1} \partial_{ux} H_0^r \right] (.) \delta x_\varepsilon^r \right] d\lambda d\mu dt \\ & + \varepsilon \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r)(\delta\sigma_\varepsilon^r)^2 d\lambda d\mu dt. \end{aligned}$$

By using Equations (15), (16), and (38), an upper bound on M_1 is given by

$$M_1 \le c_3(r)\varepsilon^2,$$

where

$$\begin{aligned} c_3(r) = & \int_0^T \left[\frac{1}{2m}(k_1^2(t) + k_1^2(t)) + \frac{m}{2} c_x^2(t) \right] dt + \frac{m}{2} c_u^2 \\ & + \frac{1}{2} \sup_{s \in [0,T]} \|\partial_{\sigma\sigma} H_1(.)\| \left[F_1^2(\alpha_4 + d_2) + (\alpha_3 + d_1) c_2 \right] + \sup_{s \in [0,T]} \left\| \partial_{uu} H_0^r(.) \right\| c_2 \\ & + \sup_{s \in [0,T]} \left\| \partial_{xx} H_0^r - \partial_{xu} H_0^r [\partial_{uu} H_0^r]^{-1} \partial_{ux} H_0^r \right\| \int_0^T c_x^2(t) dt. \end{aligned}$$

detailing the previous bound, we see that k_1 and k_2 are proportional to the bounds of the derivatives of the perturbating terms; that c_2 is proportional to the square of the bounds on the perturbating terms and of their derivatives, and that it tends to the infinity when β tends to the infinity; that F_1 is a bound of the perturbating dynamics along the nominal trajectory; and that c_x and c_u are bounded by a linear combination of F_1 and c_2. Recalling (27), the upper bound on ΔJ is of the form

$$J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \leq (c_0 F_1^2 + c_1)\varepsilon^2 + \min\left[c_3(r), (c_0 F_1^2 + c_1)\right]\varepsilon^2 \triangleq K\varepsilon^2.$$

In this bound, we have estimated c_3; F_1 is proportional to the perturbating dynamics along the nominal trajectory; and c_1 is a bound on the derivatives of the perturbating terms. As $(c_0 F_1^2 + c_1)\varepsilon^2$ is independent of $rP(.)$ and the input constraints are always satisfied when r goes to zero, the upper bound on $J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r)$ is finite and its limit is bounded by $K\varepsilon^2$. As the penalized cost J_ε^r converges to the optimal value of J_ε under input constraint when r goes to zero (see References 25 and 26), there exists a constant K such that

$$J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon) \leq K\varepsilon^2.$$

The perturbation does not affect the feasibility of the control constraint, the latter being independent of the state trajectories. This remark would not be true in the presence of state constraints since the perturbations affect the state trajectories and may jeopardize the state constraints. This concludes the proof. ■

5 | ESTIMATION OF K

5.1 | Detailed estimate

The purpose of the main result, that is, Theorem 1, is to quantify the suboptimality induced by modeling errors in the presence of control constraints. The value of K can be quantitatively estimated. This estimation is carried out in the five steps of the "cookbook":

1. Step 1: Calculate the nominal trajectories (state, adjoint state, and the nominal control) for $\varepsilon = 0$.
2. Step 2: Estimate the coefficients $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ giving the upper bounds on the state and the control trajectories

$$\|\delta x_\varepsilon^r(t)\|^2 \leq \alpha_1(t)\int_0^T\int_0^1\int_0^1 \lambda\|z(\lambda,\mu,t)\|^2 d\lambda d\mu dt + \alpha_2(t)F_1^2\varepsilon^2,$$
$$\int_0^T \|\delta u_\varepsilon^r(t)\|^2 dt \leq \alpha_3\int_0^T\int_0^1\int_0^1 \lambda\|z(\lambda,\mu,t)\|^2 d\lambda d\mu dt + \alpha_4 F_1^2\varepsilon^2,$$

   where F_1 is the maximum error in the state dynamics, z is defined in (18). This estimation can be achieved using the Lipschitz property (in accordance with Appendix C) or the first-order expansion of the dynamics of $\delta x_\varepsilon^r(t)$.
3. Step 3: Estimate an upper bound on M_0 given by

$$|M_0| \leq (c_0 F_1^2 + c_1)\varepsilon^2 \triangleq c\varepsilon^2,$$

$$c_0 = \frac{1}{2}\left(\sup_{t\in[0,T]}\partial_{xx}H_0(.) + m\right)\int_0^T q^2(t)dt + \frac{1}{2}\sup_{t\in[0,T]}\partial_{xx}H_1(.)\int_0^T q^2(t)dt,$$
$$c_1 = \frac{1}{2m}\int_0^T k_1^2(t)dt,$$

   where m is a positive constant (whose value will be calculated below) and q is given in (14).
4. Step 4: Estimate the upper bound on R given by

$$R \leq \frac{2}{\beta}\left(s_{2a}F_1^2 + s_{2b}\right)\varepsilon^2 = c_2\varepsilon^2,$$

where

$$s_{2a} = c_0 + \left[\frac{m}{2} + \frac{T}{2}\inf_{s\in[0,T]} \|\partial_{\sigma\sigma}H_1(.)\|\right]\left(\alpha_4 + \int_0^T \alpha_2(s)ds\right),$$

$$s_{2b} = c_1 + \frac{1}{2m}\int_0^T (k_2^2(t) + k_1^2(t))dt,$$

and m is given by

$$m = \frac{\beta}{\alpha_3 + \int_0^T \alpha_1(s)ds}.$$

The upper bounds on δx_ε and δu_ε become of the form

$$\|\delta x_\varepsilon^r(t)\|^2 \leq \left[\frac{4}{\beta}\alpha_1(t)\left(s_{2a}F_1^2 + s_{2b}\right) + \alpha_2(t)F_1^2\right]\varepsilon^2 = c_x^2(t)\varepsilon^2,$$

$$\int_0^T \|\delta u_\varepsilon^r(t)\|^2 dt \leq \left[\frac{4}{\beta}\alpha_3\left(s_{2a}F_1^2 + s_{2b}\right) + \alpha_4 F_1^2\right]\varepsilon^2 = c_u^2\varepsilon^2.$$

5. Step 5: Estimate the upper bound on ΔJ of the form $K\varepsilon^2$ where

$$\begin{aligned}
K &= c_0F_1^2 + c_1 + \min\left[c_3,\ c_0F_1^2 + c_1\right],\\
c_3 &= \int_0^T \left[\frac{1}{2m}(k_1^2(t) + k_1^2(t)) + \frac{m}{2}c_x^2(t)\right]dt + \frac{m}{2}c_u^2\\
&\quad + \frac{1}{2}\sup_{s\in[0,T]} \|\partial_{\sigma\sigma}H_1(.)\| \left[F_1^2\left(\alpha_4 + \int_0^T \alpha_2(s)ds\right) + \left(\alpha_3 + \int_0^T \alpha_1(s)ds\right)c_2\right]\\
&\quad + \sup_{s\in[0,T]} \|\partial_{uu}H_0(.)\|\,c_2 + \sup_{s\in[0,T]} \left\|\partial_{xx}H_0 - \partial_{xu}H_0[\partial_{uu}H_0]^{-1}\partial_{ux}H_0\right\| \int_0^T c_x^2(t)dt.
\end{aligned} \tag{40}$$

The upper bounds on the Hamiltonian H_0 and H_1 and their derivatives are calculated on the nominal trajectories (for $\varepsilon = 0$). The obtained upper bound on ΔJ will be conservative. Alternatively, the inequalities used in the calculation of K can be improved and better results for K can be obtained on a case-by-case basis.

5.2 | The big picture

The results established in this article hold for all $\epsilon \in [0, 1]$. To (conservatively) estimate K, the first thing to do is to solve the unperturbed OCP (with the nominal cost L_0 and dynamics f_0). Let K_0 be a bound of the perturbating terms f_1 and L_1 along the trajectory driven by the optimal control of the unperturbed OCP. We need then global estimates of the first and second derivatives of f_1 and L_1. Let K_1 be a global bound on the derivatives of f_1 and L_1. Let K_2 be a bound on the second derivatives of the Hamiltonian, with the costate being the costate for the unperturbed problem. We denote by B an estimate of the bounded-input, BIBO stability of the system $\dot{x} = f_0$ around the nominal trajectory and control. Then, there exists a numerical constant C, which may be conservative due to to our wish to *not* compute the solution of the perturbed problems, essentially because it is more complicated or because f_1 and L_1 are not precisely known, such that

$$K \leq C(1 + K_2)(1 + B)\frac{K_0^2 + K_1^2}{\beta^2}. \tag{41}$$

The constant C depends only on the solution of the unperturbed OCP ($\epsilon = 0$) and does not depend on the perturbation terms f_1 and L_1.

6 | ILLUSTRATIVE EXAMPLES

To illustrate the method presented in Section 5, two examples are considered: a linear quadratic (LQ) (toy) problem under input constraints and an energy management system for HEVs described in more details in References 32 and [33]. The estimation of K is done for each example and its value is compared with the real value calculated from numerical solution of the associated nominal and perturbed problems.

6.1 | LQ problem

Consider the following LQ problem

$$J_\varepsilon(u) = \frac{1}{2}\int_0^T \left(\left(1+\frac{\varepsilon}{6}\right)u^2 + x_1^2\right)dt,$$

where x_1, x_2 and u are the state and the control variables of the following linear system

$$\dot{x}_1 = x_2 - \frac{\varepsilon}{24}x_1, \quad x_1(0) = 4,$$
$$\dot{x}_2 = -\left(1-\frac{\varepsilon}{20}\right)x_2 + u, \quad x_2(0) = 4.$$

The parameter ε models the uncertainties (parameters variation) in the model ($\varepsilon \in [0,1]$). The control u is constrained to belong to the set U^{ad} defined by

$$u_{\min} \le u(t) \le u_{\max}.$$

The Hamiltonian H_ε associated with this OCP is given by

$$H_\varepsilon(x_1,x_2,u,p_1,p_2) = H_0(x_1,x_2,u,p_1,p_2) + \varepsilon\left(-\frac{p_1}{24}x_1 + \frac{p_2}{20}x_2 + \frac{u^2}{12}\right),$$

where H_0 is the Hamiltonian associated with the nominal problem ($\varepsilon = 0$) and it is given by

$$H_0(x_1,x_2,u,p_1,p_2) = \frac{1}{2}(u^2 + x_1^2) + p_1x_2 + p_2(-x_2 + u).$$

The following notations are used:

- The nominal state and costate trajectories for $\varepsilon = 0$: (y_1, y_2, p_1, p_2).
- The solutions of the dynamics equations for the nominal control $u = u_0$ and for $\varepsilon > 0$: (x_1, x_2).
- The optimal state and costate trajectories for $\varepsilon > 0$: $(x_1^*,\ x_2^*,\ p_1^*,\ p_2^*)$.
- The error on the state and the control trajectories $\delta\xi_1 \triangleq x_1 - y_1,\ \delta\xi_2 \triangleq x_2 - y_2,\ \delta x_1 \triangleq x_1^* - y_1,\ \delta x_2 \triangleq x_2^* - y_2,\ \delta u \triangleq u_\varepsilon - u_0$.

6.1.1 | Upper bounds on $\delta\xi_i$

The dynamics of $\delta\xi_1$ and $\delta\xi_2$ are given by

$$\frac{d(\delta\xi_1)}{dt} = \delta\xi_2 - \frac{\varepsilon}{24}\delta\xi_1 - \frac{\varepsilon}{24}y_1, \quad \delta\xi_1(0) = 0, \tag{42}$$

$$\frac{d(\delta\xi_2)}{dt} = -\left(1-\frac{\varepsilon}{20}\right)\delta\xi_2 + \frac{\varepsilon}{20}y_2, \quad \delta\xi_2(0) = 0. \tag{43}$$

The transition matrix Φ of this linear time-invariant system is given by

$$\Phi(t,\tau,\varepsilon) = \begin{bmatrix} \Phi_{11}(t,\tau,\varepsilon) & \Phi_{12}(t,\tau,\varepsilon) \\ \Phi_{21}(t,\tau,\varepsilon) & \Phi_{22}(t,\tau,\varepsilon) \end{bmatrix} = \begin{bmatrix} e^{-\frac{\varepsilon}{24}(t-\tau)} & \frac{120e^{(\frac{\varepsilon}{20}-1)(t-\tau)}-120e^{-\frac{\varepsilon}{24}(t-\tau)}}{11\varepsilon-120} \\ 0 & e^{(\frac{\varepsilon}{20}-1)(t-\tau)} \end{bmatrix}. \tag{44}$$

By using (44) and (42), (43), $\delta\xi_1$ and $\delta\xi_2$ can be bounded as follows

$$\begin{aligned} \|\delta\xi_1(t)\| &\leq \varepsilon \left| \int_0^t \left[-\frac{1}{24} y_1(\tau)\Phi_{11}(t,\tau,0) + \frac{1}{20} y_2(\tau)\Phi_{12}(t,\tau,0) \right] d\tau \right|, \\ \|\delta\xi_2(t)\| &\leq \varepsilon \left| \int_0^t \frac{1}{20} y_2(\tau)\Phi_{22}(t,\tau,1) d\tau \right|. \end{aligned}$$

The two upper bounds on $\delta\xi_1$ and $\delta\xi_2$, which depend only on the nominal trajectories, are of the form

$$\|\delta\xi_1(t)\| \leq \varepsilon\alpha_{21}(t), \quad \|\delta\xi_2(t)\| \leq \varepsilon\alpha_{22}(t), \tag{45}$$

where

$$\alpha_{21}(t) = \left| \int_0^t \left[-\frac{y_1(\tau)}{24}\Phi_{11}(t,\tau,0) + \frac{y_2(\tau)}{20}\Phi_{12}(t,\tau,0) \right] d\tau \right|, \quad \alpha_{22}(t) = \left| \int_0^t \frac{y_2(\tau)}{20}\Phi_{22}(t,\tau,1) d\tau \right|.$$

Note that α_{21} and α_{22} depend only on the nominal trajectories. They are evaluated numerically.

6.1.2 | Upper bounds on δx_i

The dynamics of δx_1 and δx_2 are similar to (42), (43) but contain an input term δu

$$\begin{aligned} \frac{d(\delta x_1)}{dt} &= \delta x_2 - \frac{\varepsilon}{24}\delta x_1 - \frac{\varepsilon}{24} y_1, \quad \delta x_1(0) = 0, \\ \frac{d(\delta x_2)}{dt} &= -\left(1 - \frac{\varepsilon}{20}\right)\delta x_2 + \frac{\varepsilon}{20} y_2 + \delta u, \quad \delta x_2(0) = 0. \end{aligned}$$

By using the transition matrix $\Phi(t,\tau,\varepsilon)$ given in (44), this differential system is solved as

$$\begin{aligned} \delta x_1(t) &= \int_0^t \Phi_{12}(t,\tau,\varepsilon)\delta u(\tau) d\tau + \varepsilon \int_0^t \left[-\frac{1}{24} y_1(\tau)\Phi_{11}(t,\tau,\varepsilon) + \frac{1}{20} y_2(\tau)\Phi_{12}(t,\tau,\varepsilon) \right] d\tau, \\ \delta x_2(t) &= \int_0^t \Phi_{22}(t,\tau,\varepsilon)\delta u(\tau) d\tau + \varepsilon \int_0^t \frac{1}{20} y_2(\tau)\Phi_{22}(t,\tau,\varepsilon) d\tau. \end{aligned}$$

From Cauchy-Schwarz inequality, the upper bounds on $\delta x_1(t)$ and $\delta x_2(t)$ are of the form

$$\begin{aligned} |\delta x_1(t)| &\leq \sqrt{\int_0^t \Phi_{12}^2(t,\tau,0) d\tau} \sqrt{\int_0^t \delta u^2(\tau) d\tau} + \varepsilon\alpha_{21}(t) = \alpha_{11}(t)\sqrt{\int_0^t \delta u^2(\tau) d\tau} + \varepsilon\alpha_{21}(t), \\ |\delta x_2(t)| &\leq \sqrt{\int_0^t \Phi_{22}^2(t,\tau,1) d\tau} \sqrt{\int_0^t \delta u^2(\tau) d\tau} + \varepsilon\alpha_{22}(t) = \alpha_{12}(t)\sqrt{\int_0^t \delta u^2(\tau) d\tau} + \varepsilon\alpha_{22}(t). \end{aligned}$$

In this example, the variable z defined in (18) is equal to δu because $\partial_{ux} H_0 = 0$. The upper bounds on $\delta x_1(t)$ and $\delta x_2(t)$ can be written as

$$\begin{aligned} |\delta x_1(t)| &\leq \alpha_{11}(t)\sqrt{R} + \varepsilon\alpha_{21}(t), \\ |\delta x_2(t)| &\leq \alpha_{12}(t)\sqrt{R} + \varepsilon\alpha_{22}(t), \end{aligned}$$

where

$$\alpha_{11}(t)=\sqrt{\int_0^t \Phi_{12}^2(t,\tau,0)d\tau},\quad \alpha_{12}(t)=\sqrt{\int_0^t \Phi_{22}^2(t,\tau,1)d\tau},$$
$$R=\int_0^T \delta u^2(\tau)d\tau.$$

In the expressions of α_{11} and α_{12}, the asymptotic stability of the system is used to obtain upper bounds independent on ε. To make the connection with the notations used in Step 2 of Section 5, the coefficients α_1 and α_2 are given by

$$\alpha_1(t)=\begin{bmatrix}\alpha_{11}(t)\\ \alpha_{12}(t)\end{bmatrix},\quad \alpha_2(t)=\begin{bmatrix}\alpha_{21}(t)\\ \alpha_{22}(t)\end{bmatrix}.$$

6.1.3 | Upper bound on *R*

The quantity M_0 defined by

$$M_0=J_\varepsilon(u_0)-\int_0^T\left[H_\varepsilon(y_1,y_2,u_0,p_1,p_2)-p_1\dot{y}_1-p_2\dot{y}_2\right]dt,$$

can be written from Proposition 1 under the form

$$M_0=\varepsilon\int_0^T\left[N_{11}(t)\delta\xi_1(t)+N_{12}(t)\delta\xi_2(t)\right]dt+\frac{1}{2}\int_0^T\delta\xi_1^2dt,$$

where $N_{11}(t)=\frac{-p_1(t)}{24}$, $N_{12}(t)=\frac{p_2(t)}{20}$. The numerical values of N_{11} and N_{12} are given by the adjoint state trajectories of the nominal problem. By using the upper bounds in (45), an upper bound on M_0 is

$$|M_0|\le\varepsilon^2\int_0^T\left[\frac{\alpha_{21}^2(t)}{2}+\left|\frac{-p_1(t)}{24}\alpha_{21}(t)+\frac{p_2(t)}{20}\alpha_{22}(t)\right|\right]dt\triangleq c\varepsilon^2. \tag{46}$$

In this upper bound, c depends only on the nominal trajectories. *The estimation of an upper bound on M_0 represents Step 3 in the methodology described in Section 5 to estimate the value of K.*

In the same spirit, M_1 defined by

$$M_1=J_\varepsilon(u_\varepsilon)-\int_0^T\left[H_\varepsilon(y_1,y_2,u_0,p_1,p_2)-p_1\dot{y}_1-p_2\dot{y}_2\right]dt,$$

can be written by using Proposition 1 under the form

$$M_1=\varepsilon\int_0^T\left[N_{11}(t)\delta x_1(t)+N_{12}(t)\delta x_2(t)+N_0(t)\delta u\right]dt+\frac{1}{2}\int_0^T\left(\delta x_1^2+\left(1+\frac{\varepsilon}{6}\right)\delta u^2\right)dt, \tag{47}$$

where $N_0(t)=\frac{u_0(t)}{6}$. As u_ε is the optimal control, and from (46), (47), we derive

$$c\varepsilon^2\ge\varepsilon\int_0^T\left[N_{11}(t)\delta x_1(t)+N_{12}(t)\delta x_2(t)+N_0(t)\delta u\right]dt+\frac{1}{2}\int_0^T\left(\delta x_1^2+\left(1+\frac{\varepsilon}{6}\right)\delta u^2\right)dt.$$

By using Young inequality (holding for any a, b and $m>0$) $2ab\ge-\frac{1}{m}a^2-mb^2$, we obtain

$$c\varepsilon^2\ge-\frac{\varepsilon^2}{2m}\int_0^T\left[N_{11}^2(t)+N_{12}^2(t)+N_0^2(t)\right]dt-\frac{m}{2}\int_0^T\left[\delta x_1^2(t)+\delta x_2^2(t)+\delta u^2(t)\right]dt+\frac{1}{2}\int_0^T\left(\delta x_1^2+\left(1+\frac{\varepsilon}{6}\right)\delta u^2\right)dt,$$

yielding

$$\frac{1}{2}\int_0^T \left(\left(1+\frac{\varepsilon}{6}\right)\delta u^2 + (1-m)\delta x_1^2 - m\delta x_2^2 - m\delta u^2\right)dt \le c\varepsilon^2 + \frac{\varepsilon^2}{2m}\int_0^T \left[N_{11}^2(t) + N_{12}^2(t) + N_0^2(t)\right] dt.$$

By using the upper bounds on δx_1 and δx_2, this implies

$$\begin{aligned}
&\frac{1}{2}\left(1+\frac{\varepsilon}{6} + 2(1-m)\int_0^T \alpha_{11}^2(t)dt - 2m\int_0^T \alpha_{12}^2(t)dt - m\right) R \\
&\quad \le \varepsilon^2 \int_0^T \left[(m-1)\alpha_{21}^2(t) + m\alpha_{22}^2(t)\right] dt + c\varepsilon^2 + \frac{\varepsilon^2}{2m}\int_0^T \left[N_{11}^2(t) + N_{12}^2(t) + N_0^2(t)\right] dt,
\end{aligned}$$

where m is chosen such that

$$1 + 2(1-m)\int_0^T \alpha_{11}^2(t)dt - 2m\int_0^T \alpha_{12}^2(t)dt - m = \frac{1+2\int_0^T \alpha_{11}^2(t)dt}{2}.$$

The upper bound on R is then of the form, for $\varepsilon \ge 0$

$$R \le 2\frac{c\varepsilon^2 + \frac{\varepsilon^2}{2m}\int_0^T \left[N_{11}^2(t) + N_{12}^2(t) + N_0^2(t)\right] dt + \varepsilon^2\int_0^T \left[(m-1)\alpha_{21}^2(t) + m\alpha_{22}^2(t)\right] dt}{1 + 2(1-m)\int_0^T \alpha_{11}^2(t)dt - 2m\int_0^T \alpha_{12}^2(t)dt - m} \triangleq c_2\varepsilon^2,$$

and the upper bounds on $\delta x_1(t)$ and $\delta x_2(t)$ are

$$\begin{aligned}
|\delta x_1(t)| &\le \left(\alpha_{11}(t)\cdot\sqrt{c_2} + \alpha_{21}(t)\right)\varepsilon \triangleq c_{x1}(t)\varepsilon, \\
|\delta x_2(t)| &\le \left(\alpha_{12}(t)\cdot\sqrt{c_2} + \alpha_{22}(t)\right)\varepsilon \triangleq c_{x2}(t)\varepsilon.
\end{aligned}$$

The estimation of an upper bound on R represents Step 4 in the methodology described in Section 5.

6.1.4 | Upper bound on ΔJ

The last step is to find an upper bound on $\Delta J = J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon) > 0$. For this, ΔJ can be written as

$$\Delta J = J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon) \le \underbrace{|M_0|}_{\le c\varepsilon^2} + |M_1|.$$

From (47) and by using the preceding upper bounds, we obtain

$$\begin{aligned}
|M_1| &= \left|\varepsilon\int_0^T [N_{11}(t)\delta x_1(t) + N_{12}(t)\delta x_2(t) + N_0(t)\delta u(t)]\, dt + \frac{1}{2}\int_0^T \left(\delta x_1^2 + \left(1+\frac{\varepsilon}{6}\right)\delta u^2\right) dt\right|, \\
&\le \int_0^T \left[\varepsilon^2 N_{11}(t)c_{x1}(t) + \varepsilon^2 N_{12}(t)c_{x2}(t) + \frac{\varepsilon^2}{2m_1}N_0^2(t)\right] dt + \frac{\varepsilon^2}{2}\int_0^T c_{x1}^2(t)dt \\
&\quad + \frac{1}{2}\left(m_1 + 1 + \frac{\varepsilon}{6}\right)c_2\varepsilon^2,
\end{aligned}$$

where m_1 is

$$m_1 = \sqrt{\frac{\int_0^T N_0^2(t)dt}{c_2}}.$$

TABLE 1 LQ problem parameters

Parameter	$u_{\min}$	$u_{\max}$	T
Value	−1.7	1.7	10

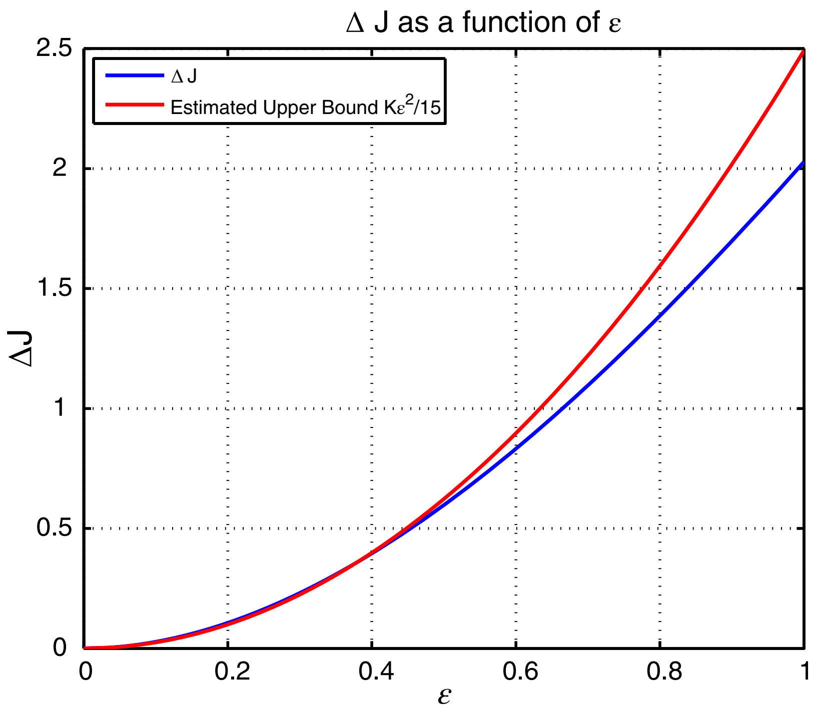

FIGURE 1 $K\varepsilon^2$ for LQ problem [Colour figure can be viewed at wileyonlinelibrary.com]

Finally, the upper bound on ΔJ is $K\varepsilon^2$ where K is given by

$$K = \int_0^T \left[N_{11}(t)c_{x1}(t) + N_{12}(t)c_{x2}(t) + \frac{1}{2m_1}N_0^2(t) + \frac{1}{2}c_{x1}^2(t) \right] dt + \frac{1}{2}\left(m_1 + \frac{7}{6}\right)c_2 + c. \tag{48}$$

The parameter K depends only on the nominal trajectories calculated for $\varepsilon = 0$. The expression of K is similar to the expression given in (40). The difference is in the estimation of the error on the state trajectories: in the general expression, we have used the Lipschitz constant and here we use the transition matrix of the system describing the dynamics of the error on the state trajectories. The obtained value will be less conservative than the general expression in (40).

6.1.5 | Numerical evaluation

The problem parameters are given in Table 1. The two TPBVPs associated with the nominal and the perturbed problems are solved for $\varepsilon \in [0, 1]$ using Matlab routine.[34] The error in the cost function given by $\Delta J = J_\varepsilon(u_0) - J_\varepsilon(u_\varepsilon)$ is evaluated numerically.

The numerical comparison between ΔJ (calculated numerically) and $K\varepsilon^2/15$ (estimated using formula (48)) is shown in Figure 1. The upper bound $K\varepsilon^2/15$ gives a good estimation of the error in the cost and shows the quadratic nature of this error.

The ratio (approx 15) between ΔJ and $K\varepsilon^2$ is due to the conservatism of the calculation method: inequalities manipulation and problem assumptions (global convexity condition in Assumption 2). Additionally, the error in the state $(\delta x_1, \delta x_2)$ and the control variable δu are estimated only from the solution of the nominal problem and they are not exactly calculated. Their estimations are higher than their real values, which will lead to a higher value of K, compared to the real error in the cost ΔJ.

The state trajectories calculated using u_0 and u_1 (for $\varepsilon = 1$) and the control trajectories are given in the plots of Figure 2. These figures show that the perturbation affects the state and the control trajectories.

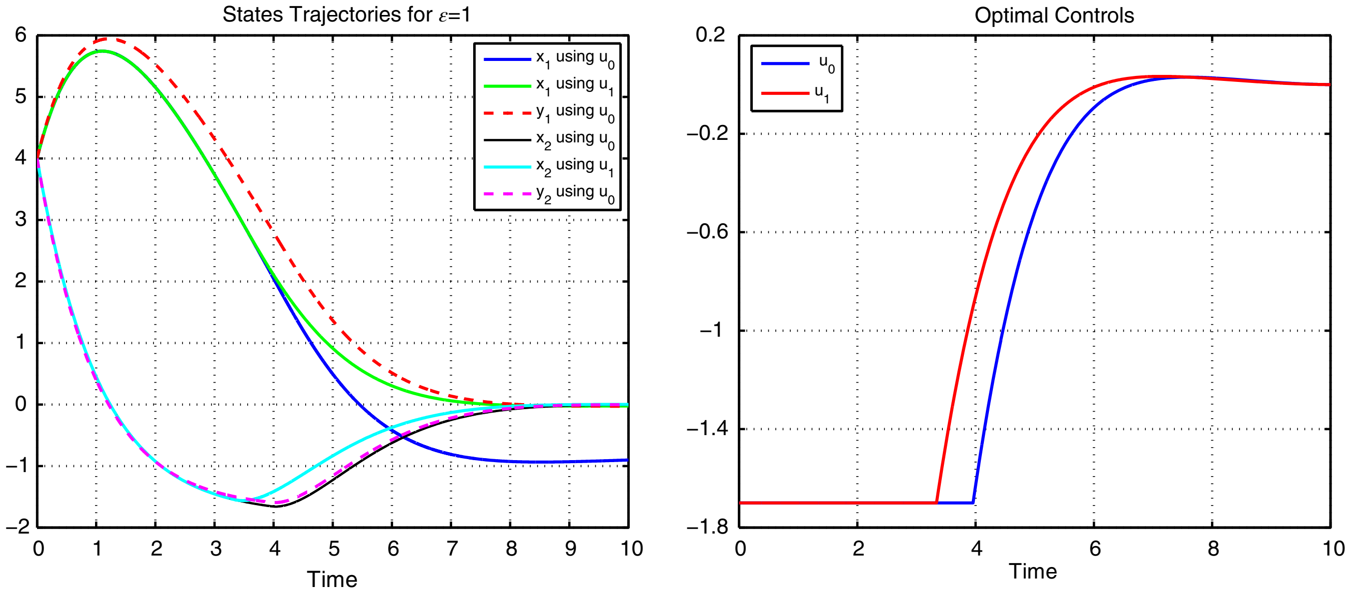

FIGURE 2 State trajectories (left) and optimal controls (right) for $\varepsilon = 1$ in LQ case [Colour figure can be viewed at wileyonlinelibrary.com]

6.2 | Thermal management problem for a parallel HEV

6.2.1 | OCP formulation

The cost function under consideration is the fuel consumption over a fixed time window corresponding to a given driving cycle of duration T

$$J(u) = \int_0^T c(u,t)e(\theta_e)dt,$$

where u is the control variable (the engine torque), θ_e is the engine temperature, and $c(u,t)$ is the fuel consumption rate when the engine is warm. The time variable accounts for the dependence of the consumption on the engine speed, which is a varying set point assumed to be perfectly tracked.

In this model, $e(.)$ is a correction factor of the fuel consumption with respect to the engine temperature θ_e. It is given by the blue curve in Figure 3. The slope of $e(.)$ is parametrized in an affine manner, as shown in Figure 3 according to

$$e(\theta_e,\varepsilon) = \begin{cases} \varepsilon_{\max}\left(1 - \frac{\theta_e}{\theta_w}\right)\varepsilon + 1, & \theta_e \le \theta_w, \\ 1, & \theta_e > \theta_w, \end{cases}$$

where $\varepsilon_{\max} = 0.59$, $\varepsilon \in [0,1]$, and $\theta_w = 70°\text{C}$. When $\varepsilon = 0$ (red curve in Figure 3), the correction factor is constant and equal to 1 (warm engine start) and the engine temperature does not impact the fuel consumption. When $\varepsilon = 1$ (blue curve in Figure 3), the correction factor has maximum sensitivity with respect to θ_e (cold engine start). All the curves between the lower ($\varepsilon = 0$) and the upper ($\varepsilon = 1$) boundaries are mathematical extrapolations with no physical interpretation.

Two (decoupled) dynamics are considered:

- The dynamics of the state of charge (SOC) of the battery, denoted by ξ, is given by

$$\frac{d\xi}{dt} = f(u,t), \quad \xi(0) = \xi_0, \tag{49}$$

  where f is a nonlinear function of its argument. The general expression is given in Reference 27. One operational constraint requires that the final value of ξ should be equal to its initial value

$$\xi(T) = \xi(0).$$

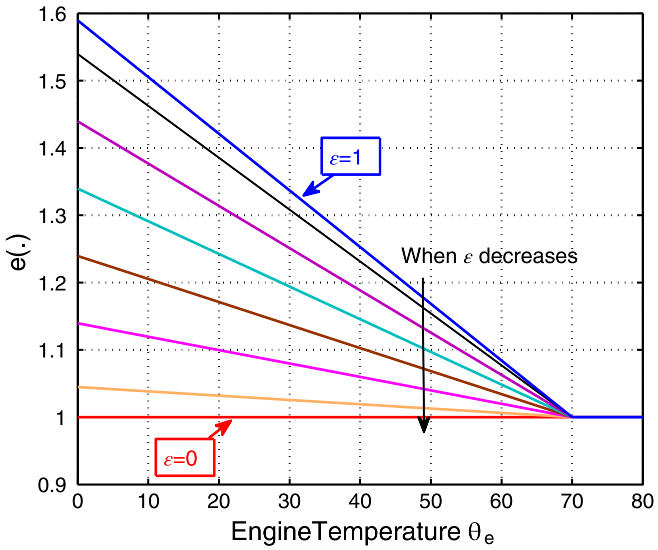

FIGURE 3 Correction factor of the fuel consumption [Colour figure can be viewed at wileyonlinelibrary.com]

- The engine temperature dynamics is given by

$$\frac{d\theta_e}{dt} = g(u, t, \theta_e), \quad \theta_e(0) = \theta_0, \tag{50}$$

where g is a nonlinear function described in Reference 32. The constraints on the control input are given by

$$u_{\min}(t) \leq u(t) \leq u_{\max}(t),$$

where $u_{\min}(t)$ and $u_{\max}(t)$ are determined from the driving conditions and physical limitations of the engine and the electric motor. For more details on the model and the formulation of the optimization problem, one can refer to References 32, [35], and [36]. Generally, the cost function to be minimized is

$$J_\varepsilon(u) = \beta(\xi(T) - \xi(0))^2 + \int_0^T c(u, t)e(\theta_e, \varepsilon)dt,$$

where β is a parameter used here to penalize the final constraint on the SOC. The perturbed and the nominal OCPs, denoted by (OCP_ε) and (OCP_0), respectively, are defined by:

$$(\text{OCP}_\varepsilon)\begin{cases} \min\limits_u \left[J_\varepsilon(u) = \beta(\xi(T) - \xi(0))^2 + \int_0^T c(u, t)e(\theta_e, \varepsilon)dt\right], \\ \frac{d\xi}{dt} = f(u, t), \quad \xi(0) = \xi_0, \\ \frac{d\theta_e}{dt} = g(u, t, \theta_e), \quad \theta_e(0) = \theta_0, \\ u_{\min}(t) \leq u(t) \leq u_{\max}(t), \end{cases}$$

$$(\text{OCP}_0)\begin{cases} \min\limits_u \left[J_0(u) = \beta(\xi(T) - \xi(0))^2 + \int_0^T c(u, t)dt\right], \\ \frac{d\xi}{dt} = f(u, t), \quad \xi(0) = \xi_0, \\ u_{\min}(t) \leq u(t) \leq u_{\max}(t). \end{cases}$$

From an application viewpoint, the problem (OCP_ε) for $\varepsilon = 1$, which is considered as the perturbed problem, is the most desirable problem as it is more representative and more accurate than the problem (OCP_0) considered as the nominal problem. The problem (OCP_ε) is also the most complex and has two states instead of one.

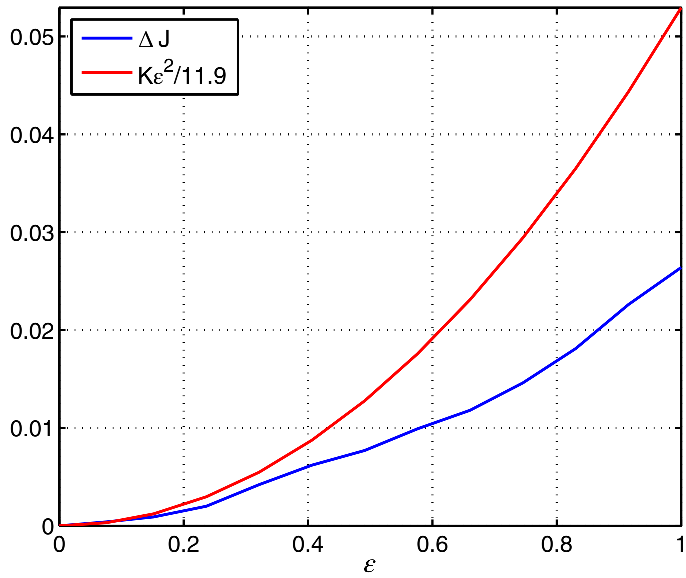

FIGURE 4 Comparison between $\frac{K\epsilon^2}{11.9}$ and ΔJ for the thermal management problem [Colour figure can be viewed at wileyonlinelibrary.com]

6.2.2 | Numerical evaluation

The details of the estimation of K are given in Appendix D. The two problems (OCP_0) and (OCP_ϵ) for $\epsilon \in [0, 1]$ are solved. The induced suboptimality ΔJ is evaluated numerically.

The numerical evaluation of $K\epsilon^2/11.9$ is shown in Figure 4 where ΔJ (calculated numerically) is compared with $K\epsilon^2/11.9$ and K is given by Equation (D7). The error is indeed of quadratic nature. For higher values of ϵ, ΔJ remains below the quadratic conservative estimation of K. The theorem indicates that the error in the optimal cost between the solutions of the two problems (OCP_0) and (OCP_ϵ) can not be more than 11%. Numerical studies show that is less than 1% of the total cost of approx 5 L/100km.

7 | A PRIORI ESTIMATE OF THE ROBUSTNESS WITH RESPECT TO MODELING SIMPLIFICATIONS IN AN OCP

In the previous section, the objective was to quantify the error in the optimal cost due to the presence of modeling errors (represented by $\epsilon \in [0, 1]$). This quantification is given by estimating K from the nominal trajectories. The numerical results presented earlier show that the estimated K is always higher than its real value (the ratio is between 10 and 20 for the considered examples).

Conversely, this value of K can be used to analyze the robustness of the nominal control strategy (calculated for $\epsilon = 0$) by finding an upper bound on ϵ such the error on the optimal cost is bounded by a predefined acceptable limit. The obtained bound on ϵ will be conservative since the estimation of K is conservative. The robustness analysis of the nominal control strategy is addressed by the following question:

> What is the value of ϵ that would lead to a given maximum desired relative error (δ_{max}) on the optimal cost?

To answer this question, a bound on the relative error is defined by

$$\delta_1(\epsilon) = 100\frac{K\epsilon^2}{J_\epsilon(u_0)}$$

is used. This quantity can be estimated numerically, as it depends only on the nominal control u_0 and ϵ. Then, the maximum value of ϵ satisfying $\delta_1(\epsilon) \leq \delta_{\max}$ can be calculated. The obtained value will be conservative (less than its real maximum value), since the estimated value of K is always higher than its real value. To illustrate this approach, we consider the following question for the LQ problem of Section 6.1:

TABLE 2 Maximum values of ε for $\delta_{\max} = 2\%$ (LQ example in Section 6.1)

	$w = 1$	$w = 10$	$w = 20$
ε	0.75	0.24	0.17

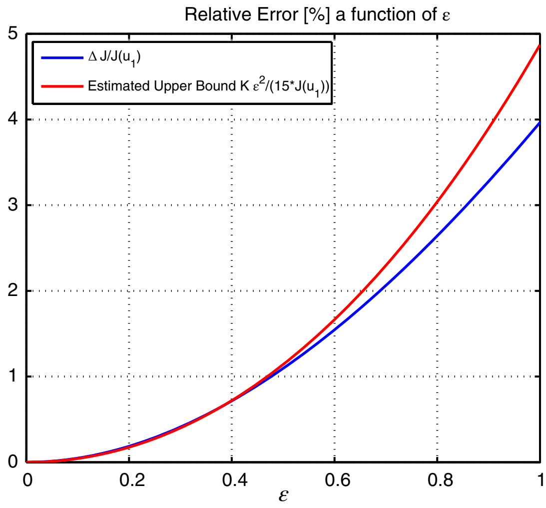

FIGURE 5 Relative error in the optimal cost [Colour figure can be viewed at wileyonlinelibrary.com]

Find the maximum value of ε leading to $\delta_{max} = 2\%$ of the optimal cost.

The obtained values of ε are summarized in Table 2 (note that w is the ratio between the estimated and the real value of K). A value of $w = 10$ is consistent with the conservatism observed in Section 6.1.

From the numerical results presented in Figure 5, the relative error in the optimal cost for $\varepsilon = 0.18$ is 0.13% and for $\varepsilon = 0.55$ is 1.45% (which are less than 2%). The results in Table 2 show that it is possible to estimate a conservative (safe) upper bound on the modeling uncertainties leading to a desired maximum relative error on the optimal cost.

8 | CONCLUSIONS

In this article, the impact of regular perturbation in input constrained OCP for nonlinear systems has been addressed. We show that the error on the cost function value is bounded by a quadratic function of the form $K\varepsilon^2$ for $\varepsilon \in [0, 1]$. The estimation of K from the solution of the simplified OCP allow induced sub-optimality to be quantified a priori. The estimated values of K are conservative as demonstrated in the illustrative examples. The result can be used as follows:

1. Solve the simplified version ($\varepsilon = 0$) of the OCP.
2. Estimate K from the previously obtained solution u_0.
3. Compute $\varepsilon_{\max}$ such that

$$100\frac{K\varepsilon_{\max}^2}{J_1(u_0)} \leq \delta_{\max}\ [\%]$$

   where $\delta_{\max}$ denotes an arbitrary performance index.
4. If $\varepsilon_{\max}$ seems reasonable (it scales the complex terms in the model), then a recommendation is to consider $\varepsilon = 0$ in all cases.

A natural but more difficult extension of this work would be to study the impact of regular perturbation in the presence of state constraints because the perturbation in the dynamics may lead to the violation of the state constraints. Some perturbation sensitivity results have been addressed in Reference 37. The idea is to find a trade-off between the optimality of the solution and the satisfaction of the state constraints.

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REFERENCES

1. Athans M, Falb P-L. *Optimal Control: An Introduction to the Theory and Its Applications*. Dover Publications: Dover, DE; 2006.
2. Bryson A-E, Ho Y-C. *Applied Optimal Control*. Waltham, MA: Ginn and Company; 1969.
3. Lewis F-L, Vrabie D, Syrmos V-L. *Optimal Control*. Hoboken, NJ: John Wiley and Sons; 2012.
4. Bertsekas D. *Dynamic Programming and Optimal Control*. Nashua, New Hampshire: Athena Scientific; 2012.
5. Pontryagin L-S, Boltyanskii V-G, Gamkrelidze R-V, Mishchenko E-F. *The Mathematical Theory of Optimal Processes*. New York, NY/London, UK: Interscience Publishers John Wiley & Sons Inc; 1962.
6. Roberts S, Shipman J. *Two-Point Boundary Value Problems: Shooting Methods*. New York, NY: American Elsevier Publications; 1972.
7. Hargraves C-R, Paris S-W. Direct trajectory optimization using nonlinear programming and collocation. *J Guidance Control Dyn*. 1987;10:338-342.
8. Guckenheimer J, Holmes P. *Averaging and Perturbation from a Geometric Viewpoint*. New York, NY: Springer; 1983.
9. Kokotovic PV. Singular perturbations in optimal control. *Rocky Mountain J Math*. 1976;4:767-774.
10. Murdock JA. *Perturbations: Theory and Methods*. Philadelphia, Pennsylvania: SIAM; 1999:83-133.
11. Bensoussan A. *Perturbation Methods in Optimal Control*. Hoboken, NJ: Wiley; 1988.
12. Dontchev A-L. Perturbations, approximations and sensitivity analysis of optimal control systems. *Lect Notes Control Inf Sci*. 1983; 52:1-153.
13. Eshghi S, Patil RM. Optimal battery pricing and energy management for microgrids. *Am Control Conf (ACC)*. 2015;4994-5001.
14. Fujimoto K, Horiuchi T, Sugie T. Optimal control of Hamiltonian systems with input constraints via iterative learning. Paper presented at: Proceedings of the Proceedings. 42nd IEEE Conference on Decision and Control; Vol 5, 2003.
15. Gissing J, Themann P, Baltzer S, Lichius T, Eckstein L. Optimal control of series plug-in hybrid electric vehicles considering the cabin heat demand. *IEEE Trans Control Syst Tech*. 2016;24(3):1126-1133.
16. Passenbrunner TE, Sassano M, del Re L. Optimal control with input constraints applied to internal combustion engine test benches. Paper presented at: Proceedings of the 9th IFAC Symposium on Nonlinear Control Systems, NOLCOS; 2013.
17. Pesch HJ. A practical guide to the solution of real-life optimal control problems. *Control Cybern*. 1994;23(1):7-60.
18. Subchan S, Zbikowski R. *Computational Optimal Control*. Hoboken, NJ: John Wiley & Sons Ltd; 2009.
19. Graichen K. *Feedforward Control Design for Finite Time Transition Problems of Nonlinear Systems with Input and Output Constraints* (PhD thesis). University of Stuttgart; 2006.
20. Graichen K, Kugi A, Petit N, Chaplais F. Handling constraints in optimal control with saturation functions and system extension. *Syst Control Lett*. 2010;59:671-679.
21. Graichen K, Petit N. Constructive methods for initialization and handling mixed state-input constraints in optimal control. *AIAA J Guidance Control Dyn*. 2008;31:1334-1343.
22. Kapernick B, Graichen K. Nonlinear model predictive control based on constraint transformation. *Opt Control Appl Methods*. 2016;37(4):807-828.
23. Malisani P. *Dynamic Control of Energy in Buildings Using Constrained Optimal Control by Interior Penalty* (PhD thesis). Ecole Nationale Supérieure des Mines de Paris; 2012.
24. Malisani P, Chaplais F, Petit N. A constructive interior penalty method for non linear optimal control problems with state and input constraints. Paper presented at: Proceedings of the American Control Conference; 2012; 2669-2676.
25. Fiacco A-V, McCormick G-P. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. Wiley: New York, NY; 1968.
26. Malisani P, Chaplais F, Petit N. An interior penalty method for optimal control problems with state and input constraints of non-linear systems. *Optim Control Appl Methods*. 2016;37(1):3-33.
27. Merz F, Sciarretta A, Dabadie J-C, and Serrao L. On the optimal thermal management of hybrid-electric vehicles with heat recovery systems. *Oil Gas Sci Tech*, 67(4):601–612, 2012.
28. Michel P, Charlet A, Colin G, Chamaillard Y, Nouillant C, Bloch G. Energy management of HEV to optimize fuel consumption and pollutant emissions. Paper presented at: Proceedings of the International Symposium on Advanced Vehicule Control; 2012.
29. Rousseau G, Sinoquet D, Sciarretta A, Milhau Y. Design optimisation and optimal control for hybrid vehicles. Paper presented at: Proceedings of the International Conference on Engineering Optimization; 2008.
30. Serrao L, Sciarretta A, Grondin O, et al. Open issues in supervisory control of hybrid electric vehicles: a unified approach using optimal control methods. Paper presented at: Proceedings of the RHEVE, International Scientific Conference on Hybrid and Electric Vehicles; 2011.
31. Bonnans JF, Guilbaud, T. Using logarithmic penalties in the shooting algorithm for optimal control problems. Research Report INRIA; 2001.
32. Maamria D, Chaplais F, Petit N, Sciarretta A. Numerical optimal control as a method to evaluate the benefit of thermal management in hybrid electric vehicles. Paper presented at: Paper presented at: Proceedings of the IFAC World Congress; 2014: 4807-4812.
33. Maamria D, Chaplais F, Petit N, Sciarretta A. On the impact of model simplification in input constrained optimal control: application to HEV energy-thermal management. Paper presented at: Proceedings of the 53rd IEEE Conference on Decision and Control; 2014:2529-2535.
34. Shampine L, Kierzenka J, Reichelt M. Solving boundary value problems for ordinary differential equations in Matlab with bvp4c. Tutorial Notes; 2000:1-27.

35. Van Berkel K, Klemm W, Hofman T, Vroemen B, Steinbuch M. Optimal energy management for a mechanical-hybrid vehicle with cold start conditions. *European Control Conf*. 2013;452-457.
36. Van Berkel K, Klemm W, Hofman T, Vroemen B, Steinbuch M. Optimal control of a mechanical hybrid powertrain with cold start conditions. *IEEE Trans Veh Tech*. 2014;63:1555-1566.
37. J-F. Bonnans and A. Hermant. Stability and sensitivity analysis for optimal control problems with a first-order state constraint and application to continuation methods. *ESAIM COCV*, 14:825–863, 2008.
38. Khalil H. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall; 2002.

SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of this article.

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APPENDICES

For convenience, we use the following notations in the appendices:

$$L_\varepsilon(\sigma) \triangleq L_0(\sigma) + \varepsilon L_1(\sigma), \quad f_\varepsilon(\sigma) \triangleq f_0(\sigma) + \varepsilon f_1(\sigma),$$

where ε is the scaling parameter for the perturbation term, as defined in (1).

APPENDIX A. PROOF OF PROPOSITION 1

The following proof can be found in Reference [11] and is briefly recalled here. It mainly uses the stationarity condition on the control variables.

Proof. The proof is essentially the same as in Reference [11]. For any smooth function F of a variable y, its Taylor expansion can be written as

$$F(y) = F(y_0) + \partial_y F(y_0)(y - y_0) + \int_0^1 \int_0^1 \lambda \partial_{yy} F(y_0 + \lambda\mu(y - y_0))(y - y_0)^2 d\lambda d\mu. \tag{A1}$$

Using this expansion, $J_\varepsilon^r(u)$ can be written as

$$\begin{aligned} J_\varepsilon^r(u) = & \int_0^T [L_\varepsilon(\sigma_0^r) + \partial_x L_\varepsilon(\sigma_0^r)\delta x^r + \partial_u L_\varepsilon(\sigma_0^r)\delta u^r]dt + r\int_0^T [P(u_0^r) + \partial_u P(u_0^r)\delta u^r]dt \\ & + \int_0^T \int_0^1 \int_0^1 \lambda \partial_{\sigma\sigma} L_\varepsilon(\sigma_0^r + \lambda\mu\delta\sigma^r)(\delta\sigma^r)^2 d\lambda d\mu dt \\ & + r\int_0^T \int_0^1 \int_0^1 \lambda \partial_{uu} P(u_0^r + \lambda\mu\delta u^r)(\delta u^r)^2 d\lambda d\mu dt. \end{aligned} \tag{A2}$$

Note

$$S \triangleq \partial_x L_\varepsilon(\sigma_0^r)\delta x^r + \partial_u L_\varepsilon(\sigma_0^r)\delta u^r + r\partial_u P(u_0^r)\delta u^r.$$

Using Equation (7) giving the adjoint state and the stationarity condition in (8), S may be rewritten as

$$S = [-\dot{p}_0^{rT} - p_0^{rT}\partial_x f_\varepsilon(\sigma_0^r) + \varepsilon\partial_x L_1(\sigma_0^r) + \varepsilon p_0^{rT}\partial_x f_1(\sigma_0^r)]\delta x^r$$

$$+ [-p_0^{rT}\partial_u f_\varepsilon(\sigma_0^r) + \varepsilon\partial_u L_1(\sigma_0^r) + \varepsilon p_0^{rT}\partial_u f_1(\sigma_0^r)]\delta u^r.$$

By integration, one gets

$$\int_0^T S(t)dt = -\int_0^T \dot{p}_0^{rT}\delta x^r dt - \int_0^T p_0^{rT}\partial_\sigma f_\varepsilon(\sigma_0^r)\delta\sigma^r dt$$
$$+ \varepsilon\int_0^T \left[(\partial_x L_1(\sigma_0^r) + p_0^{rT}\partial_x f_1(\sigma_0^r))\delta x^r + (\partial_u L_1(\sigma_0^r) + p_0^{rT}\partial_u f_1(\sigma_0^r))\delta u^r\right] dt,$$

which, using integration by parts, can be rewritten as

$$\int_0^T S(t)dt = -\left[\underbrace{p_0^{rT}(T)}_{=0}\delta x^r(T) - p_0^r\underbrace{\delta x^r(0)}_{=0} - \int_0^T p_0^{rT}(\dot{x}^r - \dot{x}_0^r)dt\right] - \int_0^T p_0^{rT}\partial_\sigma f_\varepsilon(\sigma_0^r)\delta\sigma^r dt$$
$$+ \varepsilon\int_0^T \left[(\partial_x L_1(\sigma_0^r) + p_0^{rT}\partial_x f_1(\sigma_0^r))\delta x^r + (\partial_u L_1(\sigma_0^r) + p_0^{rT}\partial_u f_1(\sigma_0^r))\delta u^r\right] dt,$$

then

$$\int_0^T S(t)dt = \varepsilon\int_0^T \partial_\sigma H_1(\sigma_0^r, p_0^r)\delta\sigma^r dt + \int_0^T p_0^{rT}(\dot{x}^r - \dot{x}_0^r - \partial_\sigma f_\varepsilon(\sigma_0^r)\delta\sigma^r)dt.$$

From (A1), the term $\dot{x}^r - \dot{x}_0^r - \partial_\sigma f_\varepsilon(\sigma_0^r)\delta\sigma^r$ can be written as

$$\dot{x}^r - \dot{x}_0^r - \partial_\sigma f_\varepsilon(\sigma_0^r)\delta\sigma^r = \varepsilon f_1(\sigma_0^r) + \int_0^1\int_0^1 \lambda\partial_{\sigma\sigma}f_\varepsilon(\sigma_0^r + \lambda\mu\delta\sigma^r)(\delta\sigma^r)^2 d\lambda d\mu. \tag{A3}$$

Using this last equation, the expression of S becomes of the form

$$\int_0^T S(t)dt = \varepsilon\int_0^T \partial_\sigma H_1(\sigma_0^r, p_0^r)\delta\sigma^r dt + \varepsilon\int_0^T p_0^{rT}f_1(\sigma_0^r(t))dt$$
$$+ \int_0^T\int_0^1\int_0^1 \lambda p_0^{rT}\cdot\partial_{\sigma\sigma}f_\varepsilon(\sigma_0^r + \lambda\mu\delta\sigma^r)(\delta\sigma^r)^2 d\lambda d\mu dt. \tag{A4}$$

Recalling that, from the definition of H_ε^r, the term $L_\varepsilon(\sigma_0^r) + rP(u_0^r)$ can be written

$$L_\varepsilon(\sigma_0^r) + rP(u_0^r) = H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\frac{dx_0^r}{dt} - \varepsilon p_0^{rT}f_1(\sigma_0^r),$$
$$= H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}f_\varepsilon(\sigma_0^r). \tag{A5}$$

Replacing (A3), (A4), (A5) in the expansion (A2), one gets

$$J_\varepsilon^r(u) = \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT}\frac{dx_0^r}{dt} - \varepsilon p_0^{rT}f_1(\sigma_0^r)]dt + \varepsilon\int_0^T \partial_\sigma H_1(\sigma_0^r, p_0^r)\delta\sigma^r dt$$
$$+ \varepsilon\int_0^T p_0^{rT}f_1(\sigma_0^r(t))dt + \int_0^T\int_0^1\int_0^1 \lambda p_0^{rT}\cdot\partial_{\sigma\sigma}f_\varepsilon(\sigma_0^r + \lambda\mu\delta\sigma^r)(\delta\sigma^r)^2 d\lambda d\mu dt$$
$$+ \int_0^T\int_0^1\int_0^1 \lambda\left[\partial_{\sigma\sigma}H_\varepsilon^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r) - p_0^{rT}\partial_{\sigma\sigma}f_\varepsilon(\sigma_0^r + \lambda\mu\delta\sigma^r)\right](\delta\sigma^r)^2 d\lambda d\mu dt.$$

The terms $\varepsilon p_0^{rT}f_1(\sigma_0^r)$ and $p_0^{rT}\partial_{\sigma\sigma}f_\varepsilon(.)$ appear in the expression of $J_\varepsilon^r(u)$ with positive and negative signs and they cancel. The formula (10) is proven. Interestingly, in formula (10), the penalty disappears from the first order variation. ∎

APPENDIX B. PROOF OF PROPOSITION 2

The proof is based on Gronwall's lemma.[38]

Proof. The solutions $X_0^r(t)$ and $X_0^r(t)$, for the initial condition $x(0)$, are given by

$$X_0^r(t) = x(0) + \int_0^t f_\varepsilon(X_0^r(\tau), u_0^r(\tau))d\tau,$$
$$x_0^r(t) = x(0) + \int_0^t f_0(x_0^r(\tau), u_0^r(\tau))d\tau.$$

Subtracting the two equations and taking norms yield

$$\left\|X_0^r(t) - x_0^r(t)\right\| \leq \int_0^t \left\|f_\varepsilon(X_0^r(\tau), u_0^r(\tau)) - f_\varepsilon(x_0^r(\tau), u_0^r(\tau))\right\| d\tau + \varepsilon \int_0^t \left\|f_1(x_0^r(\tau), u_0^r(\tau))\right\| d\tau.$$

Note that $X_0^r(t)$ and $X_0^r(t)$ have the same control input u_0^r and the same initial conditions. As f_ε is Γ-Lipschitz and f_1 is bounded, the upper bound on $X_0^r - x_0^r$ implies

$$\left\|X_0^r(t) - x_0^r(t)\right\| \leq \Gamma \int_0^t \left\|X_0^r(\tau) - x_0^r(\tau)\right\| + \varepsilon F_1 t,$$

for some positive constant F_1 defined by

$$F_1 = \sup_{t\in[0,T]} \left\|f_1(\sigma_0^r(t))\right\|.$$

Using Gronwall's lemma,[38] the upper bound on $\left\|X_0^r(t) - x_0^r(t)\right\|$ is given by

$$\left\|X_0^r(t) - x_0^r(t)\right\| \leq \varepsilon F_1 \int_0^t e^{\Gamma(t-\tau)} d\tau.$$

This concludes the proof. ■

APPENDIX C. PROOF OF LEMMA 2

A constructive proof of Lemma 2 is as follows.

Proof. The dynamic of the error on the state trajectories δx_ε^r can be written as

$$\frac{d(\delta x_\varepsilon^r)}{dt} = f_\varepsilon(\sigma_\varepsilon^r) - f_\varepsilon(\sigma_0^r) + \varepsilon f_1(\sigma_0^r).$$

As $\delta x_\varepsilon^r(0) = 0$, we can write

$$\delta x_\varepsilon^r(t) = \int_0^t \left[f_\varepsilon(\sigma_\varepsilon^r) - f_\varepsilon(\sigma_0^r)\right] dt + \varepsilon \int_0^t f_1(\sigma_0^r)dt.$$

Since f_ε is Γ-Lipschitz, this formula yields

$$\|\delta x_\varepsilon^r(t)\| \leq \Gamma \int_0^t \left[\|\delta x_\varepsilon^r(t)\| + \|\delta u_\varepsilon^r(t)\|\right] dt + \varepsilon \left\|\int_0^t f_1(\sigma_0^r)dt\right\|. \tag{C1}$$

From the expression of z in Equation (18), δu_ε^r can be written as

$$\begin{aligned}\delta u_\varepsilon^r &= z - [\partial_{uu} H_0^r(.)]^{-1} \partial_{ux} H_0^r(.) \delta x_\varepsilon^r, \\ &\triangleq z - W(.) \delta x_\varepsilon^r.\end{aligned}$$

As the term $[\partial_{uu} H_0^r(.)]^{-1}$ is bounded by $\frac{1}{\beta}$ (from Assumption 2) and $\partial_{ux} H_0^r(.)$ [1]is bounded independently of $rP(.)$, the bound on $W(.)$, denoted by γ_1, is independent of $rP(.)$ and of the perturbations f_1 and L_1. We have

$$\gamma_1 = \sup_{t\in[0,T]} \|W(.)\|,$$

and we can write the upper bound on δu_ε^r as follows

$$\|\delta u_\varepsilon^r\| \le \|z(\lambda, \mu, t)\| + \gamma_1 \|\delta x_\varepsilon^r\|. \tag{C2}$$

By replacing this inequality in Equation (C1) and using the fact that f_1 is bounded, the upper bound on δx_ε^r implies

$$\|\delta x_\varepsilon^r(t)\| \le \Gamma(1+\gamma_1) \int_0^t \|\delta x_\varepsilon^r(t)\| \, dt + \Gamma \int_0^t \|z(\lambda, \mu, s)\| \, ds + \varepsilon F_1 t.$$

Using Gronwall's lemma,[38] the upper bound on $\delta x_\varepsilon^r(t)$ is of the form

$$\|\delta x_\varepsilon^r(t)\| \le \Gamma \int_0^t e^{\Gamma(1+\gamma_1)(t-s)} \|z(\lambda, \mu, s)\| \, ds + \varepsilon F_1 \int_0^t e^{\Gamma(1+\gamma_1)(t-s)} ds. \tag{C3}$$

From Cauchy-Schwarz inequality applied to the first term of (C3), the upper bound on $\delta x_\varepsilon^r(t)$ can be written as

$$\|\delta x_\varepsilon^r(t)\| \le \Gamma \sqrt{\int_0^t e^{2\Gamma(1+\gamma_1)(t-s)} ds} \sqrt{\int_0^t \|z(\lambda, \mu, s)\|^2 ds} + \frac{\varepsilon F_1}{\Gamma(1+\gamma_1)} (e^{\Gamma(1+\gamma_1)t} - 1).$$

As $(x+y)^2 \le 2x^2 + 2y^2$ and $\int_0^t \|z(\lambda, \mu, \tau)\|^2 d\tau \le \int_0^T \|z(\lambda, \mu, \tau)\|^2 d\tau$, we can write the following inequality

$$\|\delta x_\varepsilon^r(t)\|^2 \le \left[\Gamma \frac{e^{2\Gamma(1+\gamma_1)t} - 1}{1+\gamma_1}\right] \int_0^T \|z(\lambda, \mu, s)\|^2 ds + 2\varepsilon^2 F_1^2 \left[\frac{e^{\Gamma(1+\gamma_1)t} - 1}{\Gamma(1+\gamma_1)}\right]^2.$$

To express the upper bound on $\delta x_\varepsilon^r(t)$ as a function of R, the two sides of this inequality are multiplied by λ and integrated twice with respect to λ and μ

$$\int_0^1 \int_0^1 \lambda \|\delta x_\varepsilon^r(t)\|^2 d\lambda d\mu \le \left[\Gamma \frac{e^{2\Gamma(1+\gamma_1)t} - 1}{1+\gamma_1}\right] R + \varepsilon^2 F_1^2 \left[\frac{e^{\Gamma(1+\gamma_1)t} - 1}{\Gamma(1+\gamma_1)}\right]^2,$$

where R is given by

$$R = \int_0^T \int_0^1 \int_0^1 \lambda \|z(\lambda, \mu, t)\|^2 d\lambda d\mu dt.$$

As δx_ε^r is independent of λ and μ, the upper bound on $\delta x_\varepsilon^r(t)$ can be written as

$$\|\delta x_\varepsilon^r(t)\|^2 \le 2 \left[\Gamma \frac{e^{2\Gamma(1+\gamma_1)t} - 1}{1+\gamma_1}\right] R + 2\varepsilon^2 F_1^2 \left[\frac{e^{\Gamma(1+\gamma_1)t} - 1}{\Gamma(1+\gamma_1)}\right]^2.$$

[1] $\partial_{ux} H_0^r(\sigma) = \partial_{ux} L_0(\sigma) + p^T \partial_{ux} f_0(\sigma)$ as $\partial_{ux} P(u) = 0$.

By defining

$$\alpha_1(t) \triangleq 2\Gamma\frac{e^{2\Gamma(1+\gamma_1)t}-1}{1+\gamma_1}, \quad \alpha_2(t) \triangleq 2\left[\frac{e^{\Gamma(1+\gamma_1)t}-1}{\Gamma(1+\gamma_1)}\right]^2, \tag{C4}$$

the upper bound on $\delta x_\varepsilon^r(t)$ in (29) is proven.

Using $(x+y)^2 \leq 2x^2 + 2y^2$, (C2) gives

$$\|\delta u_\varepsilon^r\|^2 \leq 2\|z(\lambda,\mu,t)\|^2 + 2\gamma_1^2\|\delta x_\varepsilon^r\|^2,$$

yielding

$$\int_0^T \|\delta u_\varepsilon^r\|^2 dt \leq 2\int_0^T \|z(\lambda,\mu,t)\|^2 dt + 2\gamma_1^2\int_0^T \|\delta x_\varepsilon^r\|^2 dt. \tag{C5}$$

Multiplying by λ and integrating twice with respect to λ and μ, Equation (C5) implies

$$\frac{1}{2}\int_0^T \|\delta u_\varepsilon^r\|^2 dt \leq 2R + \gamma_1^2\int_0^T \|\delta x_\varepsilon^r\|^2 dt.$$

By replacing the upper bound on $\|\delta x_\varepsilon^r\|^2$ given by (29) in this equation, the relationship (30) is proven with (3) and

$$d_2 \triangleq \int_0^T \alpha_2(s)ds, \quad \alpha_4 \triangleq 2\gamma_1^2 d_2, \tag{C6}$$

which are numbers independent from the perturbation terms f_1 and L_1. This concludes the proof. ■

APPENDIX D. THERMAL MANAGEMENT PROBLEM FOR HEV

The Hamiltonian associated with the perturbed problem (OCP_ε) is

$$H_\varepsilon(\theta_e, u, \lambda, \mu, t) = e(\theta_e, \varepsilon)c(u,t) + \lambda f(u,t) + \mu g(\theta_e, u, t),$$

where λ and μ are the adjoint states associated, respectively, with the SOC and θ_e. From the optimality conditions, the associated TPBVP to the perturbed problem is

$$\begin{cases} e(\theta_1,\varepsilon)\partial_u c(u_1^*,t) + p_1\partial_u f(u_1^*,t) + \mu_1\partial_u g(\theta_1,u_1^*,t) = 0, \\ \dot{\lambda}_1 = 0, \quad p_1(T) = 2\beta(\xi_1(T) - \xi_1(0)), \\ -\dot{\mu}_1 = c(u_1^*,t)\partial_\theta e(\theta_1,\varepsilon) + \mu_1\partial_\theta g(\theta_1,u_1^*,t), \quad \mu_1(T) = 0, \end{cases}$$

where (ξ_1, θ_1) are solutions of (49), (50) for the control input u_1^*. For the nominal problem, the associated TPBVP is of the form

$$\begin{cases} \partial_u c(u_0^*,t) + \lambda_0\partial_u f(u_0^*,t) = 0, \\ \dot{\lambda}_0 = 0, \quad \lambda_0(T) = 2\beta(\xi_0(T) - \xi_0(0)), \end{cases}$$

where (ξ_0, θ_0) are solutions of (49), (50) for the control input u_0^*. The following notations will be used

$$\delta\xi = \xi_1 - \xi_0, \quad \delta\theta = \theta_1 - \theta_0, \quad \delta u = u_1^* - u_0^*.$$

As the perturbation terms are only present in the cost function, the errors on the state trajectories depend only on the error on the control variable δu and they can be written in the form

$$|\delta\xi(t)|^2 \le c_\xi^2(t)\int_0^T |\delta u(\tau)|^2 d\tau, \quad |\delta\theta(t)|^2 \le c_\theta^2(t)\int_0^T |\delta u(\tau)|^2 d\tau,$$

where c_ξ and c_θ are functions of time and the nominal control u_0^* (the function f_1 in the general case is null). In this example, the variable z defined in (18) is equal to δu as $\partial_{ux}H_0 = 0$.

Using Proposition 1, the optimal cost $J_\varepsilon(u_1^*)$ can be written as

$$\begin{aligned} J_\varepsilon(u_1^*) = J_\varepsilon(u_0^*) + \varepsilon\int_0^T \left[\left(1-\frac{\theta_0}{\theta_w}\right)\partial_u c(u_0^*,t)\cdot\delta u(t) - \frac{c(u_0^*,t)}{\theta_w}\cdot\delta\theta\right]dt + \beta\cdot\delta\xi(T)^2 \\ + \int_0^T\int_0^1\int_0^1 \rho\partial_{\sigma\sigma}H_1(\sigma_0+\rho k(\sigma_1-\sigma_0), \quad \lambda_0, 0, t)(\sigma_1-\sigma_0)^2 d\rho dk dt, \end{aligned} \tag{D1}$$

where $\sigma = [\theta, u]$. As u_1^* is the optimal control for the perturbed problem, it satisfies

$$J_\varepsilon(u_1^*) \le J_\varepsilon(u_0^*).$$

From Equation (D1), we can write

$$\begin{aligned} \varepsilon\int_0^T \left[\left(1-\frac{\theta_0}{\theta_w}\right)\partial_u c(u_0^*,t)\cdot\delta u(t) - \frac{c(u_0^*,t)}{\theta_w}\cdot\delta\theta\right]dt + \beta\cdot\delta\xi(T)^2 \\ + \int_0^T\int_0^1\int_0^1 \rho\partial_{\sigma\sigma}H_1(\sigma_0+\rho k(\sigma_1-\sigma_0), \lambda_0, 0, t)(\sigma_1-\sigma_0)^2 d\rho dk dt \le 0. \end{aligned} \tag{D2}$$

Consider the notations

$$S_1(t) = \left(1-\frac{\theta_0}{\theta_w}\right)\partial_u c(u_0^*,t), \quad S_2(t) = \frac{c(u_0^*,t)}{\theta_w}, \quad S_3(\theta_e,u,t) = \left(1-\frac{\theta_e}{\theta_w}\right)c(u,t).$$

The quantities S_1 and S_2 are calculated numerically from the nominal trajectories. From the definition of H_ε, we can write

$$H_\varepsilon(\theta_e, u, \lambda_0, 0, t) = H_0(u, \lambda_0, t) + \varepsilon\left(1-\frac{\theta_e}{\theta_w}\right)c(u,t),$$

where H_0 is the Hamiltonian associated with the nominal problem. Equation (D2) becomes of the form

$$\begin{aligned} \varepsilon\int_0^T S_1(t)\delta u(t)dt + \beta\delta\xi^2(T) + \int_0^T\int_0^1\int_0^1 \rho\partial_{uu}H_0(u_0+\rho k\delta u, \lambda_0, t)\delta u^2(t)d\rho dk dt \\ + \varepsilon\int_0^T\int_0^1\int_0^1 \rho\partial_{\sigma\sigma}S_3(\sigma_0+\rho k(\sigma_1-\sigma_0), t)(\sigma_1-\sigma_0)^2 d\rho dk dt \le \varepsilon\int_0^T S_2(t)\delta\theta(t)dt. \end{aligned}$$

The part $\varepsilon\int_0^T\int_0^1\int_0^1 \rho\partial_{\sigma\sigma}S_3(\sigma_0+\rho k(\sigma_1-\sigma_0), t)(\sigma_1-\sigma_0)^2 d\rho dk dt$ leads to a term in ε^3 (as ε is less than 1, we have $\varepsilon^3 \le \varepsilon^2$). We can write from the previous equation that

$$\begin{aligned} \varepsilon\int_0^T S_1(t)\delta u(t)dt + \beta\delta\xi^2(T) + \int_0^T\int_0^1\int_0^1 \rho\partial_{uu}H_0(u_0+\rho k\delta u, \lambda_0, t)\delta u^2(t)d\rho dk dt \\ \le \varepsilon\int_0^T S_2(t)\delta\theta(t)dt. \end{aligned} \tag{D3}$$

Assume that there exists a positive constant γ such that

$$\partial_{uu}H_0(u, \lambda_0, t) \ge \gamma I, \quad \text{uniformly in } u. \tag{D4}$$

From the condition (D4), we derive

$$\int_0^T \int_0^1 \int_0^1 \rho \partial_{uu} H_0(u_0 + \rho k \delta u, \lambda_0, t) \delta u(t)^2 d\rho dk dt \geq \frac{\gamma}{2} \int_0^T \delta u(t)^2 dt.$$

Using the inequalities holding for any x, y and $\alpha > 0$

$$-\frac{x^2}{2\alpha^2} - \frac{\alpha^2 y^2}{2} \leq xy \leq \frac{x^2}{2\alpha^2} + \frac{\alpha^2 y^2}{2},$$

Equation (D3) can be written as

$$\begin{aligned} &-\frac{\varepsilon^2}{2\alpha^2} \int_0^T S_1^2(t)dt - \frac{\alpha^2}{2} \int_0^T \delta u^2(t)dt + \beta \delta \xi^2(T) + \frac{\gamma}{2} \int_0^T \delta u^2(t)dt \\ &\leq \frac{\varepsilon^2}{2\alpha^2} \int_0^T S_2^2(t)dt + \frac{\alpha^2}{2} \int_0^T \delta \theta^2(t)dt. \end{aligned} \tag{D5}$$

Using the upper bounds on $\delta\xi(t)$ and $\delta\theta(t)$, Equation (D5) becomes of the form

$$\left[\frac{\gamma}{2} + \beta c_\xi^2(T) - \frac{\alpha^2}{2}\left[1 + \int_0^T c_\theta^2(t)dt\right]\right] \int_0^T \delta u^2(t)dt \leq \frac{\varepsilon^2}{2\alpha^2} \int_0^T (S_1^2(t) + S_2^2(t))dt. \tag{D6}$$

The parameter α is chosen such that

$$\frac{\gamma}{2} + \beta c_\xi^2(T) - \frac{\alpha^2}{2}\left[1 + \int_0^T c_\theta^2(t)dt\right] = \frac{\gamma}{4} + \frac{1}{2}\beta c_\xi^2(T) \triangleq q,$$

and we get

$$\alpha = \sqrt{\frac{\frac{\gamma}{2} + \beta c_\xi^2(T)}{1 + \int_0^T c_\theta^2(t)dt}}.$$

The parameter α is well defined. From Equation (D6), one derives that

$$\int_0^T \delta u^2(t)dt \leq \frac{\varepsilon^2}{2q\alpha^2} \int_0^T (S_1^2(t) + S_2^2(t))dt \triangleq c_u^2 \varepsilon^2,$$

and the upper bounds on the state trajectories error become of the form

$$\delta\xi^2(T) \leq c_\xi^2 c_u^2 \varepsilon^2, \quad \delta\theta^2(t) \leq c_\theta^2(t) c_u^2 \varepsilon^2.$$

The final step is to find an upper bound of ΔJ. From the expression of $J_\varepsilon(u_1^*)$ given in (D1), we can write

$$\begin{aligned} \Delta J &= \left| \varepsilon \int_0^T [S_1(t)\delta u - S_2(t)\delta\theta]dt + \int_0^T \int_0^1 \int_0^1 \rho \partial_{uu} H_0(., \lambda_0, t) \delta u^2 d\rho dk dt + \beta \delta\xi(T)^2 \right|, \\ &\leq \left[\frac{1}{2\alpha_1} \int_0^T (S_1^2(t) + S_2^2(t))dt + \frac{\alpha_1}{2} c_u^2 \left(1 + \int_0^T c_\theta^2(t)dt\right) + \frac{1}{2} \sup_{[0T]} \partial_{uu} H_0 c_u^2 + \beta c_\xi^2 c_u^2 \right] \varepsilon^2, \end{aligned}$$

where α_1 is determined to minimize the term

$$\frac{1}{2\alpha_1} \int_0^T (S_1^2(t) + S_2^2(t))dt + \frac{\alpha_1}{2} c_u^2 (1 + \int_0^T c_\theta^2(t)dt),$$

and it is given by

$$\alpha_1 = \sqrt{\frac{\int_0^T (S_1^2(t) + S_2^2(t))dt}{c_u^2 + c_u^2 \int_0^T c_\theta^2(t)dt}}.$$

The upper bound on ΔJ is $K\varepsilon^2$ where the formula of K is

$$K = \frac{1}{2\alpha_1} \int_0^T (S_1^2(t) + S_2^2(t))dt + \frac{\alpha_1}{2} c_u^2 + \frac{\alpha_1}{2} c_u^2 \int_0^T c_\theta^2(t)dt + \frac{1}{2} \sup \partial_{uu} H_0(.) c_u^2 + \beta c_\xi^2 c_u^2. \tag{D7}$$

This expression of K is similar to the expression given in (40). The difference is in the estimation of the error on the state trajectories where we use the transition matrix of the system describing the dynamics of the error on the state trajectories.