On the computation of π-flat outputs for differential-delay systems

Felix Antritter[a], Franck Cazaurang[b], Jean Lévine[c], Johannes Middeke[d]

[a] *Chair for Computer-Algebra in Control, EIT8.3, Universität der Bundeswehr München, Germany*
[b] *Univ. Bordeaux, IMS, UMR 5218,33405 Talence cedex, France*
[c] *CAS-Mathématiques et Systèmes, MINES-ParisTech, 35 rue Saint-Honoré, 77300 Fontainebleau, France*
[d] *Research Institute for Symbolic Computation (RISC), Johannes Kepler University, Altenbergerstraße 52, A-4040 Linz, Austria*

Abstract

We introduce a new definition of π-flatness for linear differential delay systems with time-varying coefficients. We characterize π- and π-0-flat outputs and provide an algorithm to efficiently compute such outputs. We present an academic example of motion planning to discuss the pertinence of the approach.

Keywords:
differential delay systems, differential flatness, π-flatness, Ore polynomials, motion planning, Modules

1. Introduction

Differential flatness, roughly speaking, means that all the variables of an under-determined system of ordinary differential equations can be expressed as functions of a particular output, called flat output, and a finite number of its successive time derivatives ([1, 2, 3], see also [4, 5, 6] and the references therein).

For time-delay systems and more general classes of infinite-dimensional systems, extensions of this concept have been proposed and thoroughly discussed in [7, 8, 9, 10]. In a linear context, relations with the notion of system parameterization [11, 12] and, in the behavioral approach of [13], with *latent variables* of *observable image representations* [14], have been established. Other theoretic approaches have been proposed e.g. in [15, 16]. Interesting control applications of linear differential-delay systems may be found in [7, 9, 10].

Characterizing differential flatness and flat outputs has been an active topic since the beginning of this theory. The interested reader may find a historical perspective of this question in [5, 6]. Constructive algorithms, relying on standard computer algebra environments, may be found e.g. in [17, 18] for nonlinear finite-dimensional systems, or [19, 20] for linear systems over Ore algebras.

For algebraic reasons recalled in Section 4, the notions of π-freeness and π-flatness have been introduced in the context of linear differential-delay systems by [7, 21, 9, 10] and then for linear time-varying differential-delay systems, with polynomial time dependence, by [16, 22].

The results and algorithms proposed in this paper concern time-varying differential-delay systems with meromorphic time-dependence. Our approach may be seen as an extension of [21, 23, 5] to this context.

Our main contributions are (1) a new definition of π- and π-k-flatness, (2) the characterization of π- and π-0-flatness in terms of the *hyper-regularity* of the system matrices, (3) yielding an elementary algorithm, of polynomial time complexity, to compute π- and π-0-flat outputs, based on the row/column reduction of the former matrices.

Let us emphasize that the introduced definitions are directly suitable for motion planning and that our formalism is able to consider state and input variables separately or not. Note that, though the natural algebraic framework used to describe such systems is based on a ring, denoted by O, of polynomials in both the delay and differential operators, which is non principal, the evaluation of our π- and π-0-flatness criteria relies upon computations over a principal ideal ring, denoted by $\overline{O}$, containing O, thus making the computations much simpler.

The paper is organized as follows. A first section briefly presents recalls on operators and signals (Section 2), followed by recalls on matrices over the non commutative differential ring O (Section 3). In the latter section, the notions of hyper-regularity and row and column reduction are introduced and characterized in this non commutative context. Section 4 then deals with differential algebraic notions of systems, π-flatness and π-k-flatness and contains the main results characterizing π-flat and π-0-flat outputs. Algorithms to compute such π-flat and π-0-flat outputs are then deduced. Finally, the proposed methodology is illustrated by an example of motion planning in Section 5.

2. Operators & signal space

2.1. Recalls on Ore polynomials

In this paper we consider linear mixed differential and time-delay operators. We will model them using the so-called Ore polynomials. These are a class of non-commutative polynomials, named after Øystein Ore who was the first to discuss them in [24]. In the following we will give a brief introduction to Ore polynomials, pointing to the appropriate references for details.

Let K be a ring and let $\sigma\colon K \to K$ be an automorphism. An additive map $\vartheta\colon K \to K$ is called a σ-*derivation* if for all a and $b \in K$ the σ-*Leibniz rule* $\vartheta(ab) = \sigma(a)\vartheta(b) + \vartheta(a)b$ holds (compare with [25, Sect. 7.3]). Consider the free K-left module generated by the powers of an indeterminate x. We define the right-multiplication of x by an element of K with the *commutation rule*

$$xa = \sigma(a)x + \vartheta(a) \qquad \text{for all } a \in K.$$

Assuming associativity and distributivity, this rule allows us to compute arbitrary products. It can be shown (see, e. g., [25, Thm. 7.3.1]) that this makes the free module into a ring which we call the ring of *(left) Ore polynomials* in x w. r. t. σ and ϑ. In the literature this is usually denoted by $K[x; \sigma, \vartheta]$.

The *degree* of an Ore polynomial p is defined as the largest exponent n such that x^n has a non-zero coefficient in p. We use $\deg 0 = -\infty$. If K is a domain, then we have the familiar rule $\deg(pq) = \deg p + \deg q$ for all p and $q \in K[x; \sigma, \vartheta]$. If K is a division ring, then it is possible to divide elements in $K[x; \sigma, \vartheta]$ with remainder in a way which is very similar to the

usual polynomial division. See [26] for details. This turns $K[x; \sigma, \vartheta]$ into a (left and right) principal ideal domain and thus into a (left or right) Ore ring (see also [27, Prop. 5.9]). Thus, we can form the field of (left or right) fractions $K(x; \sigma, \vartheta)$. See for example [28] for an extensive introduction on how to properly define the various arithmetic operations for such fractions.

If K is commutative, $\sigma =$ id and $\vartheta = 0$ are the identity and the zero map, respectively, then $K[x; \text{id}, 0]$ is just the usual polynomial ring $K[x]$. Two other important special cases of Ore polynomials are linear differential and delay operators, which we discuss in the following.

Example 1. *Let K be the field of meromorphic functions over the real line (see e.g. [29, p. 42])).*

1. *Assume first that $\vartheta = 0$ and that $\sigma = \delta$ is the* time delay operator *which is defined by $\delta f(t) = f(t - \tau)$ for all $f \in K$ where $\tau > 0$ is a fixed real number. In an abuse of notation, we will denote the ring $K[x; \delta, 0]$ just by $K[\delta]$, i. e., we identify δ with the Ore variable x. The commutation rule for $K[\delta]$ is $\delta f(t) = f(t - \tau)\delta$ for all $f(t) \in K$. We call it the ring of* time delay operators.

2. *Assume now, that $\sigma =$ id. Let $\vartheta = \partial$ be the usual derivation in the sense of calculus. Then, the ring $K[x; \text{id}, \partial]$—which we will just write as $K[\partial]$—has the commutation rule $\partial f(t) = f(t)\partial + \dot{f}(t)$ for all $f(t) \in K$. This is the ring of* differential operators.

Since the maps ∂ and δ commute, we may extend ∂ to $K[\delta]$ by setting $\partial(\delta) = 0$. Thus, the ring $O = K[\delta, \partial]$—or in more complete notation $K[x; \delta, 0][y; \text{id}, \partial]$—is well-defined and has the commutation rules

$$\delta f(t) = f(t - \tau)\delta, \qquad \partial f(t) = f(t)\partial + \dot{f}(t) \qquad \text{and} \qquad \partial \delta = \delta \partial$$

where $f(t) \in K$. We call O the ring of time-delay differential operators. The commutation of ∂ and δ means that this ring is an Ore algebra in the sense of [30, Def. 1.2]. Using the formulæ in [28, Thm. 13], we may extend the action of ∂ on the fractions in $K(\delta)$. Thus, also the ring $\overline{O} = K(\delta)[\partial]$ is a well-defined Ore polynomial ring.

Since δ is an automorphism, by [25, Sect. 7.3] we may form the ring of formal Laurent series $K((\delta))$ in δ with coefficients in K. That is, $K((\delta))$ consists of elements of the form $\sum_{j \geq N} f_j \delta^j$ where $N \in \mathbb{Z}$ and $f_j \in K$ for all $j \geq N$. Moreover, by [25, Prop. 7.3.7] we may embed $K(\delta)$ into $K((\delta))$.

2.2. Signal space

The signal space has to be contained in the domain of the previously defined operators. Such a space is not unique in general and we may choose it in accordance to our control objectives. A signal space suitable for *motion planning tasks* should satisfy at least the following requirements. It should:

1. be closed under the action of any polynomial of differentiation (∂) and delay (δ)

2. be closed under advances defined by inverses of δ-polynomials,

3. contain C^∞ functions with compact support in $\mathbb{R}$.

Therefore we introduce

$$\mathcal{S} = \{f \colon \mathbb{R} \to \mathbb{R} \mid \exists E_f \subseteq \mathbb{R} \text{ discrete} : f \in C^\infty(\mathbb{R} \setminus E_f, \mathbb{R})$$
$$\text{and } \exists t_0 \in \mathbb{R} \ \forall t \in \mathbb{R} : t < t_0 \implies f(t) = 0\}.$$

which is an adaptation of a space introduced in [31, Sec. 3]. Indeed, $\mathcal{S}$ fullfills the first and third property and it is even an O-module.

As far as property 2 is concerned: if $\pi \in K[\delta]$ with $\pi \neq 0$, let us compute $\pi^{-1} f$ for $f \in \mathcal{S}$. To this end we can develop π^{-1} into a formal Laurent series $\pi^{-1} = \sum_{j \geq N} a_j \delta^j$ where $N \in \mathbb{Z}$ and $a_j \in K$ for all $j \geq N$ (see Appendix A). Then for any $f \in \mathcal{S}$ and $t \in \mathbb{R}$ we have

$$(\pi^{-1} f)(t) = \sum_{j \geq N} a_j \delta^j f(t) = \sum_{j \geq N} a_j f(t - j\tau).$$

Since $f(t - j\tau) = 0$ for $j\tau > t - t_0$ all but finitely many of the summands on the right hand side vanish at any given t. We can check that $\pi^{-1} f$ is again in $\mathcal{S}$. Thus, property 2 is satisfied. Moreover, one can show that $\mathcal{S}$ is an $\overline{O}$-module. Note that changing O to $\overline{O}$ in conjunction with $\mathcal{S}$ will yield significant simplifications in the sequel.

The example discussed in Section 5 also shows that $\mathcal{S}$ is relevant for motion planning tasks. Note in addition that $\mathcal{S}$ contains splines and step functions.

3. Matrices & hyper-regularity

We model systems of linear time-delay differential equations using matrices of operators. The set of all $n \times m$ matrices with entries in O is denoted by $O^{n \times m}$. Square matrices in $O^{n \times n}$ which possess a two-sided inverse that is also in $O^{n \times n}$ are called *unimodular*. The set of all unimodular matrices of $O^{n \times n}$ is denoted by $\mathrm{Gl}_n(O)$. We write $\mathbf{1}_n$ for the $n \times n$ identity matrix and $\mathbf{0}_{n \times m}$ for the $n \times m$ zero matrix. In both cases we will omit the indices when they are obvious from the context. We use $O^{1 \times m}$ for the set of row vectors of length m and O^n for the set of column vectors of length n. Given a matrix $M \in O^{n \times m}$ we denote its O-row space by $O^{1 \times n} M$. For a matrix $M = (M_{i,j}) \in O^{n \times m}$ we define $\deg_\partial M = \max\{\deg_\partial M_{i,j} \mid i = 1, \ldots, n, j = 1, \ldots, m\}$. This will also be applied to row or column vectors regarding them as $1 \times m$- or $n \times 1$-matrices, respectively. We write $M_{i,*}$ for the i-th row of M and $M_{*,j}$ for the j-th column of M where $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Let $M \in O^{n \times m}$. Since $K(\delta)$ is a division ring, $\overline{O}$ is a principal ideal domain. This means we can apply [32, Thm. 8.1.1] in order to find unimodular matrices $S \in \mathrm{Gl}_n(\overline{O})$ and $T \in \mathrm{Gl}_m(\overline{O})$ such that $S M T$ is in *Smith-Jacobson form*. This is a diagonal form with the additional property that each diagonal element is a total divisor of the next one.

Definition 1 (Hyper-regularity [5]). *A matrix $M \in O^{n \times m}$ is called* hyper-regular *if the diagonal elements of its Smith-Jacobson form are all 1, i.e., if there are unimodular matrices $S \in \mathrm{Gl}_n(\overline{O})$ and $T \in \mathrm{Gl}_m(\overline{O})$ such that*

$$n \geq m \quad \text{and} \quad S M T = \begin{pmatrix} \mathbf{1}_m \\ \mathbf{0}_{n-m \times m} \end{pmatrix}$$
$$\text{or} \quad n < m \quad \text{and} \quad S M T = (\mathbf{1}_n, \mathbf{0}_{n \times m-n}).$$

Usually, the given method for obtaining a Smith-Jacobson form involves computing a diagonal form by repeatedly taking greatest common (left or right) divisors as a first step. Since the performance of this approach is difficult to predict and most likely exponential due to degree growth, we will give an alternate, and computationally more efficient, characterisation of hyper-regularity below.

Proposition 1 ([19]).   *(i) A matrix $M \in O^{n \times m}$, with $n < m$, is* hyper-regular *if, and only if, it possesses a right-inverse, i.e. $\exists\, T \in \mathrm{Gl}_m(\overline{O})$ such that $MT = (\mathbf{1}_n, \mathbf{0}_{n \times (m-n)})$.*

   (ii) A matrix $M \in O^{n \times m}$, with $n \geq m$, is hyper-regular *if, and only if, it possesses a left-inverse, i.e. $\exists\, S \in \mathrm{Gl}_n(\overline{O})$ such that $SM = \begin{pmatrix} \mathbf{1}_m \\ \mathbf{0}_{(n-m)\times m} \end{pmatrix}$.*

PROOF. We only prove (i). The proof of (ii) follows the same lines and is left to the reader. Let $M \in O^{n \times m}$ be given with $n < m$. M is hyper-regular if, and only if, there are matrices $S \in \mathrm{Gl}_n(\overline{O})$ and $T \in \mathrm{Gl}_m(\overline{O})$ such that $SMT = (\mathbf{1}_n, \mathbf{0}_{n \times m-n})$. Thus, using the identity

$$\begin{pmatrix} \mathbf{1}_n, & \mathbf{0} \end{pmatrix} = S^{-1} \begin{pmatrix} \mathbf{1}_n, & \mathbf{0} \end{pmatrix} \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{m-n} \end{pmatrix}$$

we get

$$\begin{pmatrix} \mathbf{1}_n, & \mathbf{0} \end{pmatrix} = S^{-1}(SMT) \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{m-n} \end{pmatrix} = M\left(T \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{m-n} \end{pmatrix} \right).$$

which proves that M is hyper-regular if, and only if, it has $\left(T \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{m-n} \end{pmatrix} \right)$ as right-inverse.   □

We will apply the technique of row-reduction for the computation of left- or right-inverses. A matrix $M \in \overline{O}^{n \times m}$ is called *row-reduced* (or row-proper) if for all row vectors $v \in \overline{O}^{1 \times n}$ the so-called *predictable degree property*

$$\deg_\partial vM = \max\{\deg_\partial v_j + \deg_\partial M_{j,*} \mid j = 1, \dots, n\}$$

holds. Note, that this definition differs from the usual one as given in [33, Sect. 2] or [34, Sect. 2.2], but is shown to be equivalent in [33, Lem. A.1 (a)] where one can easily check that the proof also works for division rings instead of fields. Similarly, the algorithm outlined in the proof of [33, Thm. 2.2] can be easily transferred to division rings, and we obtain that for every matrix $M \in O^{n \times m}$ there exists a matrix $S \in \mathrm{Gl}_n(\overline{O})$ such that the non-zero rows of SM form a row-reduced submatrix. Using the results in [35] row-reduction is of low polynomial complexity in the size of M and its degree. Directly from the definition we obtain that the rows of a row-reduced matrix must be linearly independent.

Row-reducedness is connected to the Popov normal form (see, e. g., [36, Def. 2.2]) which is essentially a row-reduced matrix with additional properties to make it unique. One may prove that for each matrix in $O^{n \times m}$ there exists exactly one matrix Popov form having the same row space. Also, row-reduction may be regarded as a special case of Gröbner basis computation—see [35].

We consider now the case $n \leq m$. Let $M \in O^{n \times m}$, and let $\tilde{S} \in \mathrm{Gl}_n(O)$ be such that

$$\tilde{S}M = \begin{pmatrix} \tilde{M} \\ \mathbf{0} \end{pmatrix}.$$

where $\tilde{M} \in \overline{O}^{k \times m}$ is row-reduced and k is the (left) row-rank of M by [33, Thm. A.2]. Assume first that M is hyper-regular. As discussed above, this means that all unit vectors of $O^{1 \times m}$ are in the row-space of $\tilde{M}$. Thus, we must have $k = m$ and by [33, Lem. A.1 (c)] all rows of $\tilde{M}$ have ∂-degree 0 or below. Thus, $\tilde{M} \in K(\delta)^{m \times m}$. Since all rows of $\tilde{M}$ are linearly independent, we conclude that $\tilde{M}$ has maximal (left) row rank.

Conversely, if row-reduction of M yields a matrix of degree 0 and (left) row-rank m, then clearly M is hyper-regular.

There is also the analogue concept of *column-reduction*. Each matrix may be brought into column-reduced form (up to zero rows) by right multiplication with a unimodular matrix. All the results cited above hold with the appropriate changes. In total, we have proved the following lemma.

Lemma 1 ([19]). *A matrix $M \in O^{n \times m}$ is hyper-regular if and only if*

1. *$n \geq m$ and row-reduction yields a matrix of ∂-degree 0 and left row-rank m*

2. *or, $n \leq m$ and column-reduction yields a matrix of ∂-degree 0 and right column-rank n.*

4. Systems

Consider systems of the kind

$$Ax = Bu, \tag{1}$$

with the pseudo state x of dimension n, input u of dimension m and the matrices $A \in \mathcal{R}^{n \times n}$ and $B \in \mathcal{R}^{n \times m}$, where $\mathcal{R}$ is a ring. For the analysis of such systems, two important objects are considered (see e.g. [37, 38, 39, 40, 7, 13, 11, 16, 34]):

- its *behavior* $\mathcal{B} \triangleq \ker(A, -B)$, where the kernel is taken with respect to the chosen signal space[2], where the components of the variables x and u are supposed to live

- and its *system module* $\mathcal{M} \triangleq \mathcal{R}^{1 \times (n+m)} / \mathcal{R}^{1 \times n}(A, -B)$, where $\mathcal{R}^{1 \times p}$ is the set of row vectors of length p, for every $p \in \mathbb{N}$, with components in $\mathcal{R}$ and where $\mathcal{R}^{1 \times n}(A, -B)$ is the module generated by the rows of the matrix $(A, -B)$.

Linear time-invariant differential systems, i. e. without delay, the ring $\mathcal{R}$ being chosen as $\mathbb{R}[\partial]$ and being commutative, are shown to be *differentially flat* if, and only if, their system module $\mathcal{M}$ over $\mathcal{R}$ is free, and a flat output is, by definition, a basis of the free module $\mathcal{M}$ (see e. g. [2, 5]). This property may serve as a flatness definition for linear differential-delay systems ([7, 21, 10]). Nevertheless, only few systems have a free system module. Moreover, since the considered base ring $\mathcal{R} = O = K[\delta, \partial]$ is not a principal ideal domain, freeness is in general different e. g. from torsion freeness or projective freeness (see e. g. [7]), with important, but otherwise unclear, pratical consequences on the control possibilities of the system. However, for linear *time-invariant* differential-delay systems, namely with $\mathcal{R} = \mathbb{R}[\delta, \partial]$, in virtue of the *localization* property of a commutative Ore algebra, torsion freeness of $\mathcal{M}$ is equivalent to the existence of a so-called *liberation polynomial* $\pi \neq 0$, $\pi \in \mathbb{R}[\delta]$, such that $\pi^{-1}\mathcal{M}$ is free. This property is called

---

[2]this space is not uniquely defined and may be chosen according to the specific application one is interested in, such as motion planning, tracking, etc. See section 2

π-freeness (see again [7]) and a basis of the free module $\pi^{-1}\mathcal{M}$ is called a π-flat output. It can be interpreted as follows: if the system module, finitely generated by m input variables, is torsion free, it admits a basis of the form $\pi^{-1}\{y_1, \ldots, y_m\}$ where $y_i \in \mathcal{R}$, $i = 1, \ldots, m$. Consequently, every system variable can be expressed as a combination of derivatives, delays and advances of $(y_1, \ldots, y_m)$. Note that the action of π^{-1} on a variable z may be interpreted as a formal power series in $\delta^{-k}z = z(\cdot + k\tau)$ for $k \geq -N$, for some $N \in \mathbb{N}$, and thus a combination of advances, or predictions, of z (see Section 2).

To extend this approach to time-varying differential-delay systems without restricting the time dependence of the system coefficients to be polynomial, as in [16, 22] where effective Gröbner bases techniques are used, many new difficulties appear, and in particular the fact that the set made of the powers of a given polynomial is not in general an Ore set. *Therefore, we are lead to propose a different definition of π-flatness, based on an extension of the "practical" flatness definition*, in the spirit of [21, 9], saying that all the system variables are expressible in terms of a flat output, its derivatives, delays and advances in finite number. Moreover we introduce the definition of π-k-flatness, which is an analogue, in our differential-delay context, to the notion of k-flatness (see [41, 42, 43, 44]). This viewpoint is developed in the next two subsections.

4.1. Recalls on the framework

From now on, we take System (1) with $\mathcal{R} = O = K[\partial, \delta]$ as in Section 2.1. Recall that δ, the delay operator, is defined by $\delta f(t) = f(t - \tau)$ for all $t \in \mathbb{R}$, where τ is a given positive real number and $f \in K$, and that $\partial \triangleq \frac{d}{dt}$ is the ordinary time derivative operator. The ground field K is the field of meromorphic functions over the real line and the notation $\overline{O} = K(\delta)[\partial]$ is defined in Section 2.1. Furthermore, we will consider only the signal space $\mathcal{S}$ from Section 2.2.

We assume that the matrix $(A, -B)$ has full (left) row rank.

In this case, System (1) is a differential-delay system, possibly with time-varying coefficients, the coefficients being meromorphic functions of time. An example of such system is provided by the following:

Example 2. *Consider the system*

$$\dot{x}_1(t) = a(t)(x_2(t - 1) - x_2(t - 2))$$
$$\dot{x}_2(t) = u(t - 1),$$

where the pseudo-state $x \triangleq (x_1, x_2)$ belongs to $\mathcal{S}^2$, the control u belongs to $\mathcal{S}$, and the function a is meromorphic with respect to time. Here, the matrices A and B are given by

$$A \triangleq \begin{pmatrix} \partial & -a(\delta - \delta^2) \\ 0 & \partial \end{pmatrix}, \quad B \triangleq \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

and, as announced, they are matrices over the ring $O = K[\delta, \partial]$.

4.2. π-flatness

One usual way to consider System (1) is to bring all the variables in one side, i.e. to consider the system

$$(A, -B)\begin{pmatrix} x \\ u \end{pmatrix} = \mathbf{0}$$

7

or, in other words,

$$F\xi = 0 \tag{2}$$

with $F \triangleq (A, -B)$ and $\xi \triangleq \begin{pmatrix} x \\ u \end{pmatrix}$.

Definition 2 (π-flatness). *The system (2) is called π-flat if there exist $\pi \in K[\delta]$ and matrices $P \in O^{m \times (n+m)}$ and $Q \in O^{(n+m) \times m}$ such that*

$$\pi^{-1} Q \mathcal{S}^m = \mathcal{B} = \ker F \qquad and \qquad \pi^{-1} P \, \pi^{-1} Q = \mathbf{1}_m.$$

Equivalently, there exist a polynomial $\pi \in K[\delta]$ and matrices P and Q of suitable dimensions over the ring O, such that $\xi = \pi^{-1} Q y$, with $y = \pi^{-1} P \xi$ for all $\xi \in \mathcal{S}^{n+m}$, where $y \in \mathcal{S}^m$ is a π-flat output. In other words, π-flatness means that all the system variables can be expressed as linear combinations of y and a finite number of its delays, advances and derivatives, in an invertible way, namely the matrix $\pi^{-1} P$ admits the matrix $\pi^{-1} Q$ as right-pseudo-inverse.

If $\pi \in K$, then the system is simply called *flat* and y a flat output.

This definition is thus an extension of the one proposed by [2, 3] for finite-dimensional non-linear ordinary differential systems or by [21] for time-invariant delay-differential systems. Other definitions for time-varying linear differential-delay systems with polynomial time dependence are proposed by Chyzak, Quadrat and Robertz [16].

Remark 1. *Note that Definition 2 is equivalent to the existence of matrices $\bar{P} \in \overline{O}^{m \times (n+m)}$ and $\bar{Q} \in \overline{O}^{(n+m) \times m}$ such that $\bar{Q} \mathcal{S}^m = \mathcal{B}$ and $\bar{P} \bar{Q} = \mathbf{1}_m$. We recover π by computation of the left common denominator of $\bar{P}$ and $\bar{Q}$ [27, Prop. 5.3]. Therefore all computations can be done over $\overline{O}$.*

We have the following proposition:

Proposition 2. *Assume that y is a π-flat output of system (2). Let us set $y = \bar{T} z$ with $\bar{T} \in \mathrm{Gl}_m(\overline{O})$. There exists a polynomial $\kappa \in K[\delta]$ such that z is a κ-flat output of system (2).*

PROOF. By Remark 1, it is sufficient to prove that $\bar{T}^{-1} \bar{P}$ and $\bar{Q} \bar{T}$ are related to the transformed flat output z. Obviously, we have $\bar{T}^{-1} \bar{P} \bar{Q} \bar{T} = \mathbf{1}_m$ and moreover $\bar{Q} \bar{T} \mathcal{S}^m = \bar{Q} \mathcal{S}^m = \mathcal{B}$. With κ being a common denominator of $\bar{T}^{-1} \bar{P}$ and $\bar{Q} \bar{T}$, the claim follows. $\square$

Remark 2. *To every π-flat system there obviously corresponds a polynomial $\pi_0 \in K[\delta]$ of minimal degree such that the system is π_0-flat.*

To characterize π-flat systems, we introduce the following definition:

Definition 3. *We call $\overline{\mathcal{M}} \triangleq \overline{O}^{n+m} / \overline{O}^n F$ the extended system module.*

Theorem 1. *We have the following equivalences:*

(i) System (2) is π-flat;

(ii) The extended system module $\overline{\mathcal{M}}$ is free;

(iii) The matrix F is hyper-regular over $\overline{O}$.

Remark 3. *Note that in the linear time-invariant case Definition 2 is equivalent to the definitions of [7, 16].*

For the proof we need the following lemma:

Lemma 2. *Let Λ be an arbitrary ring and $M \in \Lambda^{p \times q}$, and let $S \in \mathrm{Gl}_p(\Lambda)$ and $T \in \mathrm{Gl}_q(\Lambda)$ be unimodular. Then $\Lambda^{1 \times q}/\Lambda^{1 \times p}M \cong \Lambda^{1 \times q}/\Lambda^{1 \times p}(S M T)$ as left Λ-modules.*

PROOF. The map $\varphi = v \mapsto vT$ is an automorphism of $\Lambda^{1 \times q}$ because T is unimodular. The kernel of the composition $\pi \circ \varphi$ of the projection $\pi\colon \Lambda^{1 \times q} \to \Lambda^{1 \times q}/\Lambda^{1 \times p}S M T$ with φ is $\varphi^{-1}(\Lambda^{1 \times p}S M T) = \Lambda^{1 \times p}S M = \Lambda^{1 \times p}M$ where the second identity follows from the unimodularity of S. Since $\pi \circ \varphi$ is also surjective, by the first isomorphism theorem for modules (see, e. g., [27, Theorem 1.17]) there is an isomorphism $\Lambda^{1 \times q}/\Lambda^{1 \times p}M \to \Lambda^{1 \times q}/\Lambda^{1 \times p}(S M T)$. $\qquad\square$

PROOF (OF THEOREM 1). We prove first that (i) is equivalent to (iii). If the system $F\xi = 0$ is π-flat, there are matrices $\bar{P} \in \overline{O}^{m \times (n+m)}$ and $\bar{Q} \in \overline{O}^{(n+m) \times m}$ with common denominator $\pi \in K[\delta] \setminus \{0\}$ such that $\bar{Q}\mathcal{S}^m = \mathcal{B} = \ker F$ and $\bar{P}\bar{Q} = \mathbf{1}_m$. That means that $\bar{Q}$ is surjective and that the following diagramme is exact.

$$0 \longrightarrow \mathcal{S}^m \underset{\bar{P}}{\overset{\bar{Q}}{\longrightarrow}} \mathcal{S}^{n+m} \underset{\bar{E}}{\overset{F}{\longrightarrow}} \operatorname{im} F \longrightarrow 0$$

Since $\bar{P}\bar{Q} = \mathbf{1}$, by the splitting lemma (see, e. g., [45, Prop. 3.2]) one can show (see [46]) that there is a matrix $\bar{E} \in \overline{O}^{k \times n}$ such that $F\bar{E} = \mathbf{1}_n$ where n is the (left) rank of $\operatorname{im} F$ because the rows of F are linearly independent by assumption. Therefore F is hyper-regular.

Conversely, let F be hyper-regular. Using Lemma 1 we may thus compute $\bar{W} \in \mathrm{Gl}_{n+m}(\overline{O})$ such that $F\bar{W} = (\mathbf{1}_n, \mathbf{0})$. If we let $\bar{Q}$ be the last m columns of $\bar{W}$ and $\bar{P}$ the last m rows of $\bar{W}^{-1}$, then we have $F\bar{Q} = \mathbf{0}$ and $\bar{P}\bar{Q} = \mathbf{1}_m$. Extracting a common denominator π of $\bar{P}$ and $\bar{Q}$, we have proved that $F\xi = 0$ is π-flat.

Next we show that (iii) is equivalent to (ii). Assume first that F is hyper-regular. Since the unit vectors are in the column space of F, by column-reduction we obtain an invertible matrix $\bar{T} \in \mathrm{Gl}_{n+m}(\overline{O})$ such that $F\bar{T} = (\mathbf{1}_n, \mathbf{0})$. By Lemma 2, this means that

$$\overline{\mathcal{M}} = \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}F} \cong \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}F\bar{T}} \cong \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}(\mathbf{1}_n, \mathbf{0})} \cong \overline{O}^{1 \times m}$$

which is free.

Conversely, assume that $\overline{\mathcal{M}}$ is free. Using, e. g., the method in [32, Thm. 8.1.1] we may obtain unimodular matrices $\bar{S} \in \mathrm{Gl}_n(\overline{O})$ and $\bar{T} \in \mathrm{Gl}_{n+m}(\overline{O})$ such that $\bar{S}F\bar{T} = (\bar{\Delta}, \mathbf{0})$ where $\bar{\Delta} = \mathsf{diag}(a_1, \ldots, a_n) \in \overline{O}^{n \times n}$ is a diagonal matrix. By Lemma 2,

$$\overline{\mathcal{M}} = \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}F} \cong \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}\bar{S}F\bar{T}} \cong \frac{\overline{O}^{1 \times (n+m)}}{\overline{O}^{1 \times n}(\bar{\Delta}, \mathbf{0})} \cong \bigoplus_{j=1}^{n} \frac{\overline{O}}{\overline{O}a_j} \oplus \overline{O}^{1 \times m}.$$

Since this module is free by assumption we conclude that all a_j must be units, i. e., we may assume w. l. o. g. that $\bar{\Delta} = \mathbf{1}_n$. By Definition 1, F is hyper-regular. $\qquad\square$

9

Algorithm 1 (Computation of a π-flat output).

Input: *A matrix $F \in O^{n\times(n+m)}$ representing the system* (2).

Output: *An Ore polynomial $\pi \in K[\delta]$ together with matrices $P \in O^{m\times(n+m)}$ and $Q \in O^{(n+m)\times m}$ as in Definition 2 or* FAIL *if such matrices do not exist.*

Procedure:

    1. Use column-reduction to check whether F is hyper-regular. If not, then return FAIL.

    2. Else, let $\bar{W} \in \mathrm{Gl}_{n+m}(\overline{O})$ be such that

$$F\bar{W} = \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_{n\times m} \end{pmatrix}.$$

    3. Let

$$\bar{Q} \triangleq \bar{W}\begin{pmatrix} \mathbf{0}_{n\times m} \\ \mathbf{1}_m \end{pmatrix} \quad and \quad \bar{P} \triangleq \begin{pmatrix} \mathbf{0}_{m\times n} & \mathbf{1}_m \end{pmatrix} \bar{W}^{-1}.$$

    4. Let $\pi \in K[\delta]$ be a common denominator of $\bar{P}$ and $\bar{Q}$.

    5. Set $P \triangleq \pi\bar{P} \in O^{m\times(n+m)}$ and $Q \triangleq \pi\bar{Q} \in O^{(n+m)\times m}$.

    6. Return π, P and Q.

Remark 4. *In the above algorithm we exploit the fact that $\bar{W}^{-1}$ can be computed at the same time as $\bar{W}$ by inverting the elementary actions that compose $\bar{W}$.*

4.3. π-0-flatness

In contrast to the considerations of the previous subsection, it is sometimes necessary to keep the state and input variables separate: in the theory of linear time-invariant systems, controllability is equivalent to the existence of Brunovský's canonical form (see e.g. [47, 48]); the interpretation of some of their states as flat output (see e.g. [2]) shows that flat outputs do not need to depend on u, i.e. there exist $P \in \mathbb{R}^{m\times n}$ and $Q \in \mathbb{R}^{n\times m}$ such that $y = Px$, $x = Qy$ and $PQ = \mathbf{1}_m$. This property is called 0-flatness. The fact that u can be expressed as a function of y is, in this case, an immediate consequence of the system equation with $x = Qy$. More generally:

Definition 4. *We say that a system is π-k-flat, with $k \geq 1$, if and only if there exists a π-flat output y such that the maximal degree with respect to ∂ of the matrix $P\begin{pmatrix} 0_{n-m} \\ \mathbf{1}_m \end{pmatrix}$ is equal to $k - 1$.*

We set $k = 0$, by convention, if $P\begin{pmatrix} 0_{n-m} \\ \mathbf{1}_m \end{pmatrix} = 0$, i.e. y does not depend on u.

Note that π-0-flatness is equivalent to the existence of (π, P, Q) as in Definition 2 such that $P = \begin{pmatrix} P_1 & 0_{m,m} \end{pmatrix}$ with $P_1 \in O^{m\times n}$ and $\pi^{-1}P_1\pi^{-1}Q_1 = \mathbf{1}_m$ where $Q_1 \triangleq \begin{pmatrix} \mathbf{1}_n & 0_{n,m} \end{pmatrix} Q$.

A linear flat system is not necessarily 0-flat, as shown by the following elementary example:

Example 3. *Consider the system*

$$\begin{pmatrix} 1 & -\partial \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = 0$$

or

$$x = \dot{u}.$$

It can be easily seen that $y = u$ is a flat output. Hence the system is 1-flat.

Remark 5. *Note that elementary considerations show that 1-flatness is not preserved under the action of the group of unimodular matrices. Indeed, setting* $\begin{pmatrix} x' \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$ *in the previous example, we get the system* $\begin{pmatrix} -\partial & 1 \end{pmatrix} \begin{pmatrix} x' \\ u' \end{pmatrix} = 0$, *or* $-\dot{x}' = u'$, *which admits* $y' = x'$ *as flat output and is thus obviously 0-flat. However, if we restrict the transformation group to be* $\mathrm{Gl}_n(\overline{O}) \otimes \mathrm{Gl}_m(\overline{O})$, *a group preserving the control variables, then 1-flatness is preserved.*

Example 3 and Remark 5 thus show that obtaining a characterization of π-0-flat systems is of interest.

Lemma 3 (Elimination). *If B in (1) is hyper-regular, the system can be decomposed according to*

$$\begin{pmatrix} \tilde{R} \\ \varphi^{-1} F \end{pmatrix} x = \tilde{M} A x = \tilde{M} B u = \begin{pmatrix} \mathbf{1}_m \\ \mathbf{0}_{(n-m)\times m} \end{pmatrix} u. \tag{3}$$

where $F \in O^{(n-m)\times n}$ *and* $\varphi \in K[\delta]$ *with* $\varphi \neq 0$.

Remark 6. *Let* $\bar{F} \in \overline{O}^{p \times q}$. *Let* $\varphi \in K[\delta]$ *be its common denominator. We have* $\bar{F} = \varphi^{-1} F$ *with* $F \in O^{p \times q}$. *Thus,* $\ker \bar{F} = \ker F$.

Theorem 2. *If B is hyper-regular, we have the following equivalences:*

(i) The control system (1) is π-0-flat;

(ii) The extended system module $\overline{O}^{1 \times n} / \overline{O}^{1 \times (n-m)} F$ *is free, with F defined in (3);*

(iii) F is hyper-regular over $\overline{O}$.

PROOF. Let F be hyper-regular over $\overline{O}$. From representation (3) combined with Theorem 1, there exists $\bar{P}_1 \in \overline{O}^{m \times n}$ and $\bar{Q}_1 \in \overline{O}^{n \times m}$ such that $F \bar{Q}_1 = \mathbf{0}$ and $\bar{P}_1 \bar{Q}_1 = \mathbf{1}_m$. Thus, $\bar{P} \triangleq \left(\bar{P}_1, \mathbf{0} \right)$ and $\bar{Q} \triangleq \begin{pmatrix} \bar{Q}_1 \\ \tilde{R}\bar{Q}_1 \end{pmatrix}$ satisfy $\bar{P}\bar{Q} = \bar{P}_1 \bar{Q}_1 = \mathbf{1}_m$ and

$$\tilde{M} \begin{pmatrix} A & -B \end{pmatrix} \bar{Q} = \tilde{M} \left(A\bar{Q}_1 - B\tilde{R}\bar{Q}_1 \right) = \begin{pmatrix} \tilde{R} \\ \varphi^{-1} F \end{pmatrix} \bar{Q}_1 - \begin{pmatrix} \mathbf{1}_m \\ 0 \end{pmatrix} \tilde{R} \bar{Q}_1 = \mathbf{0}. \tag{4}$$

Equation (4) implies that $\bar{Q}\mathcal{S}^m \subseteq \mathcal{B}$. To prove the other inclusion, let $\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \in \mathcal{B}$. Then analogously

$$0 = \tilde{M} \begin{pmatrix} A & -B \end{pmatrix} \gamma = \begin{pmatrix} \tilde{R} \\ \varphi^{-1} F \end{pmatrix} \gamma_1 - \begin{pmatrix} \mathbf{1}_m \\ \mathbf{0} \end{pmatrix} \gamma_2.$$

Since $\ker \varphi^{-1} F = \ker F$, we obtain $\gamma_1 \in \ker F$ and $\tilde{R}\gamma_1 = \gamma_2$. Consequently, there exists $\zeta \in \mathcal{S}^m$ such that $\gamma_1 = \bar{Q}_1 \zeta$ and thus also $\gamma_2 = \tilde{R}\bar{Q}_1 \zeta$. Hence, $\gamma \in \bar{Q}\mathcal{S}^m$ and the other inclusion also holds. Thus, extracting a common denominator π such that $\bar{P} = \pi^{-1} P$ and $\bar{Q} = \pi^{-1} Q$ where $P \in O^{m \times n}$ and $Q \in O^{n \times m}$, we see that π, P and Q fulfill Definition 4. Thus, (iii) implies (i).

Conversely, if (i) holds with π, P and Q being as in Definition 4, then with $Q_1 = (\mathbf{1}_n, \mathbf{0})Q$ and $P_1 = P \begin{pmatrix} \mathbf{1}_n \\ \mathbf{0} \end{pmatrix}$, by the same calculation (4) we obtain $\varphi^{-1} F \pi^{-1} Q_1 = \mathbf{0}$ and $\pi^{-1} P_1 \pi^{-1} Q_1 = \mathbf{1}_m$. Thus, $Fx = 0$ is π-flat. Thus, by Theorem 1 we have (iii). □

11

If the matrix B of system (1) is hyper-regular, we can directly check for the existence of a π-0-flat output, as shown in Theorem 2. A suitable algorithm is as follows:

Algorithm 2 (Computation of a π-0-flat output).

Input: *Matrices $A \in O^{n \times n}$ and $B \in O^{n \times m}$ with B hyper-regular, representing (1).*

Output: *If system (1) is π-0-flat, the polynomial $\pi \in K[\delta]$ and the triple (P, Q, R) of the matrices $P \in O^{m \times n}$, $Q \in O^{n \times m}$ and $R \in O^{m \times m}$ from Definition 2.*

Else, if the system defined by A and B is not π-0-flat, then FAIL.

Procedure:

1. *Compute $\tilde{M} \in \mathrm{Gl}_n(\overline{O})$, s.t. $\tilde{M}B = \begin{pmatrix} \mathbf{1}_m \\ \mathbf{0}_{(n-m) \times m} \end{pmatrix}$.*

2. *Write*
$$\tilde{M}A = \begin{pmatrix} \tilde{R} \\ \varphi^{-1}F \end{pmatrix}$$
   where $\tilde{R} \in \overline{O}^{m \times n}$ and $F \in O^{(n-m) \times n}$.

3. *If Algorithm 1 applied to F returns π_1, P_1 and Q_1, then*

   (a) *Set $\bar{P} = (\pi_1^{-1} P_1, \mathbf{0})$ and*
$$\bar{Q} = \begin{pmatrix} \pi_1^{-1} Q_1 \\ \tilde{R}\pi_1^{-1} Q_1 \end{pmatrix}.$$

   (b) *Let π be a common denominator of $\bar{P}$ and $\bar{Q}$ and set $P = \pi\bar{P} \in O^{m \times n}$ and $Q = \pi\bar{Q} \in O^{n \times m}$.*

   (c) *Return π, P and Q.*

4. *Else, return* FAIL

If the matrix B of system (1) is not hyper-regular, we can check for π-flatness by applying Algorithm 1.

Remark 7. *Note that Algorithms 1 and 2 do not necessarily yield a polynomial π of minimal degree in δ. Especially, it is not guaranteed, that we obtain $\pi \in K$ if the system is flat.*

5. Example

To illustrate the results of the preceding sections and the usefulness of the concept for the feedforward controller design for linear time-delay differential systems, we demonstrate all steps for the system of Example 2. Note that all the necessary computations can be done with a package for the computer algebra system *Maple*, which has been presented in [19]. The package can be obtained from the first author upon request.

The matrix B is hyper-regular and following Algorithm 2 we compute $\tilde{M}$ such that $\tilde{M}B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
We get
$$\tilde{M} = \begin{pmatrix} 0 & \delta^{-1} \\ 1 & 0 \end{pmatrix}.$$

12

Thus:

$$\begin{pmatrix} \tilde{R} \\ F \end{pmatrix} = \tilde{M}A = \begin{pmatrix} 0 & \delta^{-1}\partial \\ \partial & a(\delta^2 - \delta) \end{pmatrix}$$

We now follow Algorithm 1 and compute $\bar{W}$ such that $F\bar{W} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ to

$$\bar{W} = \begin{pmatrix} 0 & 1 \\ (\delta^2 - \delta)^{-1}\frac{1}{a} & -(\delta^2 - \delta)^{-1}\frac{1}{a}\partial \end{pmatrix}.$$

The inverse is

$$\bar{W}^{-1} = \begin{pmatrix} \partial & a(\delta^2 - \delta) \\ 1 & 0 \end{pmatrix}.$$

Subsuming the remaining computations of Algorithm 1 we directly obtain the following matrices in step 3 a) of Algorithm 2, namely

$$\bar{P} = \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \bar{W}^{-1} \quad 0 \right) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \tag{5}$$

and

$$\bar{Q} = \begin{pmatrix} \mathbf{1}_2 \\ \tilde{R} \end{pmatrix} \bar{W} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -(\delta^2 - \delta)^{-1}\frac{1}{a}\partial \\ -\left(\delta^3 - \delta^2\right)^{-1}\left(\frac{1}{a}\partial^2 - \frac{\dot{a}}{a^2}\partial\right) \end{pmatrix}. \tag{6}$$

We readily get $\pi = \delta^3 - \delta^2$.

From (5), a π-0-flat output is given by $y = x_1$. This can also be deduced from the first entry in $\bar{Q}$. From (6) we deduce that $x_2 = -(\delta^2 - \delta)^{-1}\left(\frac{1}{a}\dot{y}\right)$ and $u = -\left(\delta^3 - \delta^2\right)^{-1}\left(\frac{1}{a}\ddot{y} - \frac{\dot{a}}{a^2}\dot{y}\right)$.

We now use these formulas to generate feedforward trajectories corresponding to the following motion planning problem: we are looking for a trajectory of (x_1, x_2, u) starting from $x(0) = (x_1(0), x_2(0)) = (0\ 0)$ and arriving at the final state $x(2) = (1\ 0)$ at time $t = 2$, while x_1 has to be equal to zero before $t = 0$ and to 1 after $t = 2$.

In order to compute explicit feedforward state and input trajectories without integrating the system differential-delay equations, we compute a reference trajectory for the flat output y, which is not constrained by any differential-delay equation thanks to the freeness of the extended system module, and then deduce the state and input trajectories by $\begin{pmatrix} x \\ u \end{pmatrix} = \bar{Q}y$ with $\bar{Q}$ defined by (6). We take $a(t) = t + 3$ and thus $\dot{a}(t) = 1$. Furthermore, we take $\tau = 1$, i.e. $\delta a(t) = a(t - 1)$.

In order to obtain an explicit expression for $x_2 = -(\delta^2 - \delta)^{-1}\left(\frac{1}{a}\dot{y}\right)$, we compute the series expansion

$$\left(\delta^2 - \delta\right)^{-1} = \sum_{j \geq -1} -\delta^j.$$

Taking into account that x_1 is required to be constant for $t < 0$ and $t > 2$, the desired trajectory y_d for $y = x_1$ will be constant at these points of time and thus $\dot{y}_d = \ddot{y}_d = 0$ for $t < 0$ and $t > 2$. This yields

$$x_{2,d} = \left(\sum_{j \geq -1} \delta^j\right) \frac{1}{a} \dot{y}_d = X_{2,d}(t)\chi_{[-1,0[}(t) + (X_{2,d}(t) + X_{2,d}(t+1))\chi_{[0,\infty[}(t+1), \tag{7}$$

where $X_{2,d}(t) = \frac{1}{a(t-\lfloor t \rfloor)}\dot{y}_d(t - \lfloor t \rfloor)$.

We make the ansatz

$$y_d(t) = (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5)\chi_{[0,2[}(t) + \chi_{[2,\infty[}(t)$$

the coefficients a_i, $i = 0, \ldots, 5$ being such that $y(0) = x_1(0) = 0$, $y(2) = x_1(2) = 1$, $\dot{y}(0) = 0$, $\dot{y}(2) = 0$, $\ddot{y}(0) = 0$ and $\ddot{y}(2) = 0$. We readily get:

$$y_d(t) = \left(-\frac{45}{4}t^2 + \frac{35}{4}t^3 - \frac{3}{4}t^5\right)\chi_{[0,2[}(t) + \chi_{[2,\infty[}(t). \tag{8}$$

We now insert this expression in (7) to obtain the corresponding feedforward control. Since

$$\pi^{-1} = \left(\delta^3 - \delta^2\right)^{-1} = \sum_{j \geq -2} \delta^j \tag{9}$$

we get

$$u_d(t) = U_d(t)\chi_{[-2,-1[}(t) + (U_d(t) + U_d(t+1))\chi_{[-1,\infty[}(t), \tag{10}$$

where $U_d(t) = -\frac{1}{a(t-\lfloor t \rfloor)}\ddot{y}_d(t - \lfloor t \rfloor) + \frac{1}{a^2(t-\lfloor t \rfloor)}\dot{y}_d(t - \lfloor t \rfloor)$, which completes the solution of this motion planning problem. The resulting trajectories are shown in Figure 1. Though the trajectory for y is relatively smooth, it is generated by an input u which is quite irregular and complicated. We also notice that u has to start 2τ before the beginning of the y-trajectory in order to satisfy the above requirements, which corresponds to the order of the Laurent series associated to π^{-1}.
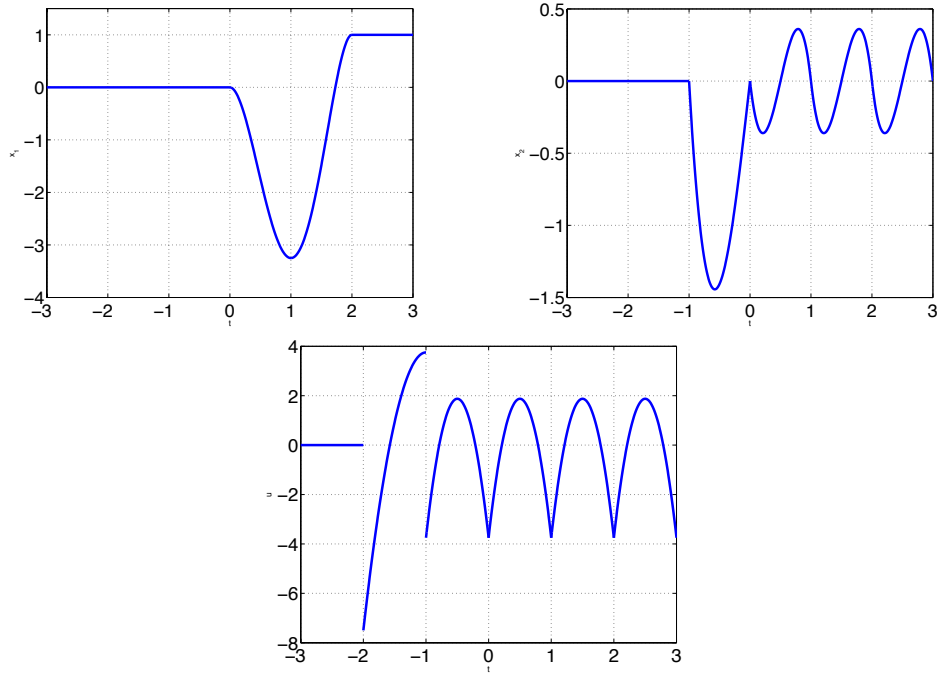

Figure 1: Graphs for $x_1 = y$, x_2 and u

14

6. Conclusion

In this paper we have introduced a new definition of π-flatness and the important particular case of π-0-flatness. These notions have been characterized by means of the properties of matrices over Ore polynomial rings, such as hyper-regularity or row- and column-reducedness. This characterization has yielded an efficient and simple algorithm to test the existence of π- and π-0-flat outputs and compute them, via an algorithm that is implemented in the computer algebra system *Maple* [19]. We have illustrated the pertinence of this approach by an academic example.

Note that the description of all possible π-0-flat outputs can be achieved; it will be the subject of a forthcoming paper. This will lead to a method to determine the minimal order of the polynomial π.

Another potential direction to extend this approach may be to investigate other possible signal spaces.

Finally, the question of increasing the efficiency of the proposed algorithm by using quotient free computations may also be addressed.

References

[1] P. Martin, Contribution à l'étude des systèmes différentiellement plats, Ph.D. thesis, Ecole Nationale Supérieure des Mines de Paris, Paris, France (1992).

[2] M. Fliess, J. Lévine, P. Martin, P. Rouchon, Flatness and defect of non-linear systems: introduction theory and examples, Int. Journal of Control 61 (6) (1995) 1327–1361.

[3] M. Fliess, J. Lévine, P. Martin, P. Rouchon, A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems, IEEE Trans. Automatic Control 44 (5) (1999) 922–937.

[4] H. J. Sira-Ramirez, S. K. Agrawal, Differentially Flat Systems, CRC Press, 2004.

[5] J. Lévine, Analysis and Control of Nonlinear Systems: A Flatness-based Approach, Mathematical Engineering, Springer, 2009.

[6] J. Lévine, On necessary and sufficient conditions for differential flatness, Applicable Algebra in Engineering, Communication and Computing 22 (1) (2011) 47–90.

[7] H. Mounier, Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques, Ph.D. thesis, University of Paris XI (1995).

[8] M. Fliess, H. Mounier, P. Rouchon, J. Rudolph, Systèmes linéaires sur les opérateurs de Mikusiński et commande d'une poutre flexible, in: ESAIM Proc. Conf. Elasticité, viscolélasticité et contrôle optimal, 8èmes entretiens du centre Jacques Cartier, Lyon, France, 1996, pp. 157–168.

[9] N. Petit, Systèmes à retards. Platitude en génie des procédés et contrôle de certaines équations des ondes, Ph.D. thesis, Ecole des Mines de Paris, Paris, France (2000).

[10] J. Rudolph, Flatness Based Control of Distributed Parameter Systems, Shaker Verlag, Aachen, Germany, 2003.

[11] J. F. Pommaret, A. Quadrat, Localization and parametrization of linear multidimensional control systems, Systems & Control Letters 37 (1999) 247–269.

[12] J. F. Pommaret, Partial Differential Control Theory, Kluwer academic publishers, 2001.

[13] J. W. Polderman, J. C. Willems, Introduction to Mathematical Systems Theory: A Behavorial Approach, Vol. 26 of Texts in Applied Mathematics, Springer Verlag, 1998.

[14] H. Trentelman, On flat systems behaviors and observable image representations, Systems & Control Letters 21 (2004) 51–55.

[15] P. Rocha, J. C. Willems, Behavioral controllability of delay-differential systems, SIAM Journal on Control and Optimization (1997) 254–264.

[16] F. Chyzak, A. Quadrat, D. Robertz, Effective algorithms for parametrizing linear control systems over Ore algebras, Appl. Algebra Eng., Commun. Comput. 16 (5) (2005) 319–376.

[17] F. Antritter, J. Lévine, Towards a computer algebraic algorithm for flat output determination, in: ISSAC'08, Hagenberg, Austria, 2008, pp. 7–14.

[18] F. Antritter, G. Verhoeven, On the symbolic computation of flat outputs for differentially flat systems, in: NOLCOS 2010, Bologna, Italy, 2010, pp. 667–682.

[19] F. Antritter, J. Middeke, A toolbox for the analysis of linear systems with delays, in: CDC/ECC 2011, Orlando, USA, 2011, pp. 1950–1955.

[20] F. Chyzak, A. Quadrat, D. Robertz, OreModules: a symbolic package for the study of multidimensional linear systems, in: Proc. of MTNS'2004, Leuven, Belgium, 2004.

[21] N. Petit, Y. Creff, P. Rouchon, δ-freeness of a class of linear systems, in: Proc. of the European Control Conference, Brussels, Belgium, paper N. 501, 1997, pp. 1069–1075.

[22] F. Chyzak, A. Quadrat, D. Robertz, OreModules: A symbolic package for the study of multidimensional linear systems, in: J. Chiasson, J.-J. Loiseau (Eds.), Applications of Time-Delay Systems, Vol. 352 of Lecture Notes in Control and Information Sciences, Springer, 2007, pp. 233–264.

[23] J. Lévine, D. V. Nguyen, Flat output characterization for linear systems using polynomial matrices, Systems & Control Letters 48 (1) (2003) 69–75.

[24] Ø. Ore, Theory of non-commutative polynomials, Annals of Mathematics 34 (1933) 480 – 508.

[25] P. M. Cohn, Further Algebra and Applications, Springer, Berlin Heidelberg New York, 2003.

[26] M. Bronstein, M. Petkovšek, An introduction to pseudo-linear algebra, Theoretical Computer Science 157 (1996) 3–33.

[27] P. M. Cohn, An introduction to Ring Theory, Springer, Berlin Heidelberg New York, 2000.

[28] J. Ježek, Non-commutative rings of fractions in algebraical approach to control theory, Kybernetika 32 (1) (1996) 81–94.

[29] H. Cartan, Théorie Élémentalre des Fonctions Analytiques d'Une ou Plusieurs Variables Complexes, Hermann, Paris, 1961.

[30] F. Chyzak, B. Salvy, Non-commutative elimination in Ore algebras proves multivariate identities, Journal of Symbolic Computation 26 (2) (1998) 187–227.
    URL `citeseer.ist.psu.edu/chyzak97noncommutative.html`

[31] E. Zerz, An algebraic analysis approach to linear time-varying systems, IMA Journal of Mathematical Control and Information 1 (23) (2006) 113–126.

[32] P. M. Cohn, Free Rings and Their Relations, 2nd Edition, Academic Press, London, 1985.

[33] B. Beckermann, H. Cheng, G. Labahn, Fraction-free row reduction of matrices of Ore polynomials, Journal of Symbolic Computation 41 (2006) 513–543.

[34] E. Zerz, State representations of time-varying linear systems, in: H. A. Park, G. Regensburger (Eds.), Gröbner bases in control theory and signal processing, Radon Series on Computational and Applied Mathematics, de Gruyter, Berlin, 2007, pp. 235–251.

[35] J. Middeke, A computational view on normal forms of matrices of Ore polynomials, Ph.D. thesis, JKU Linz (2011).

[36] P. Davies, H. Cheng, G. Labahn, Computing Popov form of general Ore polynomial matrices, in: Milestones in Computer Algebra (MICA) 2008, 2008, pp. 149–156.

[37] M. Fliess, Some basic structural properties of generalized linear systems, Systems & Control Letters 15 (1990) 391–396.

[38] U. Oberst, Multidimensional constant linear systems, Acta Applicandae Mathematicae 20 (1-2) (1990) 1–175.

[39] J. C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Transactions on Automatic Control 36 (1991) 259–294.

[40] M. Fliess, A remark on Willems' trajectory characterization of linear controllability, Systems & Control Letters 19 (1992) 43–45.

[41] P. Martin, R. M. Murray, P. Rouchon, Flat systems, in: In Proc. of the 4th European Control Conf, 1997, pp. 211–264.

[42] J.-B. Pomet, On dynamic feedback linearization of four-dimensional affine control systems with two inputs, ESAIM-COCV 2 (1997) 151–230, published online by Cambridge University Press, DOI:10.1051/cocv:1997107.

[43] M. Rathinam, R. Murray, Configuration flatness of Lagrangian systems underactuated by one control, SIAM J. Control and Optimization 36 (1) (1998) 164–179.

[44] P. S. Pereira da Silva, Flatness of nonlinear control systems : a Cartan-Kähler approach, in: Proc. MTNS 2000, Perpignan, France, 2000, pp. 1–10.

[45] S. Lang, Algebra, revised 3rd Edition, no. 211 in Graduate texts in mathematics, Springer, New York Berlin Heidelberg, 2002.

[46] J. Middeke, J. Lévine, Description of all π-flat outputs of linear differential-delay systems with time-varying coefficients, Linear Algebra and its ApplicationsSoon to be submitted.

[47] P. Brunovský, A classification of linear controllable systems, Kybernetica 6 (1970) 176–178.

[48] T. Kailath, Linear Systems, Prentice-Hall Information and System Science, Prentice Hall, 1979.

Appendix A. Inversion of a δ-Polynomial

Let $\pi = a_n\delta^n + \ldots + a_k\delta^k \in K[\delta]$ where $k < n$ and $a_k, \ldots, a_n \in K$. Assume that $a_k \neq 0$, i.e. that the order of π is k. The inverse π^{-1} in $K(\!(\delta)\!)$ may be computed in the following way: First,

16

using the fact that

$$\pi^{-1} = \delta^{-k}(a_n\delta^{n-k} + \ldots + a_k)^{-1}$$

we may assume w. l. o. g. that $k = 0$. We make an ansatz $\sum_{j\geq 0} c_j\delta^j$ for π^{-1} and compute

$$1 = \pi \sum_{j\geq 0} c_j\delta^j = \sum_{i=0}^{n}\sum_{j\geq 0} a_i\delta^i(c_j)\delta^{i+j} = \sum_{\ell\geq 0}\Big(\sum_{i=0}^{\min\{\ell,n\}} a_i\delta^i(c_{\ell-i})\Big)\delta^\ell.$$

Comparing coefficients and using $a_0 \neq 0$, we can compute

$$1 = a_0 c_0 \iff c_0 = a_0^{-1}$$

and

$$0 = \sum_{i=0}^{\min\{\ell,n\}} a_i\delta^i(c_{\ell-i}) \iff c_\ell = -a_0^{-1}\sum_{i=0}^{\min\{\ell,n\}-1} a_i\delta^i(c_{\ell-i})$$

for all $\ell \geq 1$. Note, that the left hand side depends only on those c_i which are already computed.