

# Robustness of optimal control when subject to control constraints

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CAS - January 6 2022

# Outline

At an *interior extremum* point,  
the variation of a function is locally small.

We elaborate on that to show that  
optimal control is robust with respect  
to model perturbations.

# First and second order conditions (Bryson and Ho, 1969)

We wish to minimize the following cost:

$$J(u) = \varphi(x(t_f), t_f) + \int_0^{t_f} L(x, u, t) dt \quad (1)$$

with dynamics:

$$\frac{dx}{dt} = f(x, u, t), \quad x(0) = x_0$$

# Introducing the « Lagrangian »

In the cost equation (1) we introduce a « Lagrangian »  
to take into account the dynamics:

$$J = \varphi(x(t_f), t_f) + \int_0^{t_f} \left\{ L(x, u, t) + \lambda(t)^T \left( f(x, u, t) - \frac{dx}{dt} \right) \right\} dt .$$

Define the Hamiltonian  $H \triangleq L + \lambda^T f$ ,

and integrate  $\lambda \frac{dx}{dt}$  by parts. The cost becomes

$$J = \varphi(x(t_f), t_f) - \lambda^T(t_f)x(t_f) + \lambda^T(0)x(0) + \int_0^{t_f} \left\{ H(x, u, \lambda, t) + \frac{d\lambda^T}{dt} x \right\} dt$$

# Taylor expansion at order 2

Consider a small variation  $\delta x$  on the state and  $\delta u$  on the control.

For the moment, these two variations are independent.

A second order Taylor expansion of the cost yields:

# Taylor expansion at order 2

$$\begin{aligned} \delta J = & \left[ \frac{\partial \varphi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \cdot \delta x(t_f) + \lambda^T(0) \cdot \delta x(0) + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \frac{d\lambda^T}{dt} \right) \cdot \delta x(t) + \frac{\partial H}{\partial u} \cdot \delta u(t) \right] dt \\ & + \frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt . \end{aligned}$$

# Taylor expansion at order 2

Final condition

$$\delta J = \left[ \frac{\partial \varphi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \cdot \delta x(t_f) + \lambda^T(0) \cdot \delta x(0) + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \frac{d\lambda^T}{dt} \right) \cdot \delta x(t) + \frac{\partial H}{\partial u} \cdot \delta u(t) \right] dt$$
$$+ \frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt.$$

# Taylor expansion at order 2

Final condition

Zero

$$\begin{aligned}
 \delta J = & \left[ \frac{\partial \varphi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \cdot \delta x(t_f) + \lambda^T(0) \cdot \delta x(0) + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \frac{d\lambda^T}{dt} \right) \cdot \delta x(t) + \frac{\partial H}{\partial u} \cdot \delta u(t) \right] dt \\
 & + \frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt .
 \end{aligned}$$



# Taylor expansion at order 2

Final condition

Zero

Adjoint equation

$$\begin{aligned}
 \delta J = & \left[ \frac{\partial \varphi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \cdot \delta x(t_f) + \lambda^T(0) \cdot \delta x(0) + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \frac{d\lambda^T}{dt} \right) \cdot \delta x(t) + \frac{\partial H}{\partial u} \cdot \delta u(t) \right] dt \\
 & + \frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt .
 \end{aligned}$$

# Taylor expansion at order 2

Final condition

Zero

Adjoint equation

$$\begin{aligned}
 \delta J = & \left[ \frac{\partial \varphi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \cdot \delta x(t_f) + \lambda^T(0) \cdot \delta x(0) + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial x} + \frac{d\lambda^T}{dt} \right) \cdot \delta x(t) + \frac{\partial H}{\partial u} \cdot \delta u(t) \right] dt \\
 & + \frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt .
 \end{aligned}$$

If  $\delta u$  is unconstrained, setting the first order to zero implies  $\frac{\partial H}{\partial u} = 0$

# Rewriting $\delta J$

$$\frac{1}{2} \delta x(t_f)^T \cdot \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \cdot \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \begin{bmatrix} \delta x(t)^T & \delta u(t)^T \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta u(t) \end{bmatrix} dt \quad (2)$$

for any variations  $\delta x(t)$  and  $\delta u(t)$   
such that first order terms of  $\delta J$  vanish.

# Second order variation

Imagine that, for reasons not detailed here, the state is modified by a quantity  $\delta x$  and the adjoint state by a quantity  $\delta \lambda$ . We wish to study the loss of optimality due to these variations.

To do so, it is sufficient to study the variation of the cost for controls which satisfy the necessary stationarity condition

$$\frac{\partial H}{\partial u} = 0.$$

# Variation of the stationarity condition

Using concise notations for the partial derivatives,  
we must have

$$H_{ux}\delta x + H_{uu}\delta u + H_{u,\lambda}\delta\lambda = 0$$

that is,

$$H_{ux}\delta x + H_{uu}\delta u + f_u^T\delta\lambda = 0$$

If  $H_{uu}$  is invertible, we obtain the feedback equation

$$\delta u^* = -H_{uu}^{-1} (H_{ux} \cdot \delta x + f_u^T \cdot \delta\lambda) \quad (3)$$

# Sufficient condition for optimality

Inserting (3) into the local variation (2) of  $\delta J$ ,  $\delta J$  can be written as:

$$\delta J = \frac{1}{2} \delta x(t_f)^T \frac{\partial^2 \varphi}{\partial x^2}(x(t_f), t_f) \delta x(t_f) + \frac{1}{2} \int_0^{t_f} \left[ \delta x(t)^T (H_{xx} - H_{xu} H_{uu}^{-1} H_{ux}) \delta x(t) + \delta \lambda(t)^T f_u H_{uu}^{-1} f_u^T \delta \lambda(t) \right] dt.$$

We see that  $\delta J$  is nonnegative for any  $\delta x, \delta \lambda$  if the final cost matrix is nonnegative and if

$$\begin{aligned} H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} &\geq 0 \\ H_{uu} &> 0 \end{aligned}$$

( $H_{uu}$  must be invertible)

These conditions are called the strong **Legendre-Clebsch** conditions.

# Sufficient condition for optimality

For static optimization problems,  
we know that convexity is essential to guarantee optimality.

The generalization of this  
to optimal control problems appears to be  
the Legendre–Clebsch conditions.

# Regular perturbations in optimal control (Bensoussan 1988)

We introduce perturbations in the *unconstrained problem* (1) as follows

$$\min_u \left[ J_\varepsilon(u) = \int_0^T [L_0(x, u) + \varepsilon L_1(x, u)] dt \right] \quad (4.1)$$

with the perturbed dynamics

$$\frac{dx}{dt} = f_0(x, u) + \varepsilon f_1(x, u), \quad x(0) = X_0 \quad (4.2)$$

This perturbation is characterized by its *magnitude*, which means that all the functions and their derivatives are assumed to be bounded.

To simplify things, we shall assume that, for  $0 \leq \varepsilon < \varepsilon_1$ , the control problems have a unique solution  $u^*(\varepsilon)$ .



# Regular perturbations in optimal control (Bensoussan 1988)

Theorem 3: assuming the strong **Legendre–Clebsch** conditions, there exists some  $K \geq 0$  such that, for  $\varepsilon$  in a neighborhood of 0, the following bound holds

$$0 \leq J_\varepsilon(u^*(0)) - \inf_u J_\varepsilon(u) \leq K\varepsilon^2$$

The Legendre–Clebsch conditions are assumed to hold uniformly in  $(x, u)$  and for the optimal adjoint state of  $J_0$ .

Convergence of the control and state follows.

# The benefit of optimization

Standard calculus easily shows that the cost is Lipschitz with respect to  $\varepsilon$ .

The benefit of using optimal control is to obtain an error bound that is proportional to  $\varepsilon^2$ .

# Control constraints

The previous results rely heavily on the stationarity condition

$$\frac{\partial H}{\partial u} = 0$$

In practice we have bounded controls, that is, we impose

$$u \in U^{ad}$$

where  $U^{ad}$  is a compact convex set with non-empty interior.

# Pontryagin minimum principle

At the boundary of  $U^{ad}$ , the variations  $\delta u$  are constrained.  
As a consequence, the proof of the stationarity condition  
on the Hamiltonian  $H$  is invalid.

# Pontryagin minimum principle

Instead, we have a *constrained* minimization condition on  $H$ :

$u(t)$  minimizes  $H(\lambda(t), x(t), u, t)$  for  $u \in U^{ad}$ .

The previous variational computations are invalid.

# Pontryagin minimum principle

In the following,  
we shall manage the abstract set  $U^{ad}$  numerically  
by using a *gauge function*.

# Gauge function of $U^{ad}$

We assume that 0 belongs to the interior of  $U^{ad}$ .

The *gauge function*  $G_{U^{ad}}$  of  $U^{ad}$  is defined by

$$G_{U^{ad}}(u) = \inf \{ \lambda \geq 0 \text{ s.t. } u \in \lambda U^{ad} \}$$

This gauge is a norm iff  $U^{ad}$  is symmetric with respect to 0.

$u$  is a (resp.) interior, boundary or exterior point de  $U^{ad}$  if and only if

$$G_{U^{ad}}(u) < 1, G_{U^{ad}}(u) = 1 \text{ or } G_{U^{ad}}(u) > 1.$$

If  $U^{ad}$  is defined by  $c(u) \leq 0$ ,  $c$  convex, its gauge is defined by

$$G_{U^{ad}}(u) = \inf \left\{ \lambda \geq 0 \text{ s.t. } c \left( \frac{u}{\lambda} \right) \leq 0, \forall i \right\}.$$

# Example

The gauge function for  $U^{ad} = [-1,1]$  is defined by

$$\begin{aligned} G_{U^{ad}}(u) &= \inf \left\{ \lambda \geq 0 \text{ s.t. } u \in \lambda U^{ad} \right\} = \inf \left\{ \lambda \geq 0 \text{ s.t. } \frac{|u|}{\lambda} \leq 1 \right\} \\ &= \inf \left\{ \lambda \geq 0 \text{ s.t. } |u| \leq \lambda \right\} = |u| \end{aligned}$$

that is,

$$G_{[-1,1]}(u) = |u|$$



# Penalty for a static constraint $u \in U^{ad}$

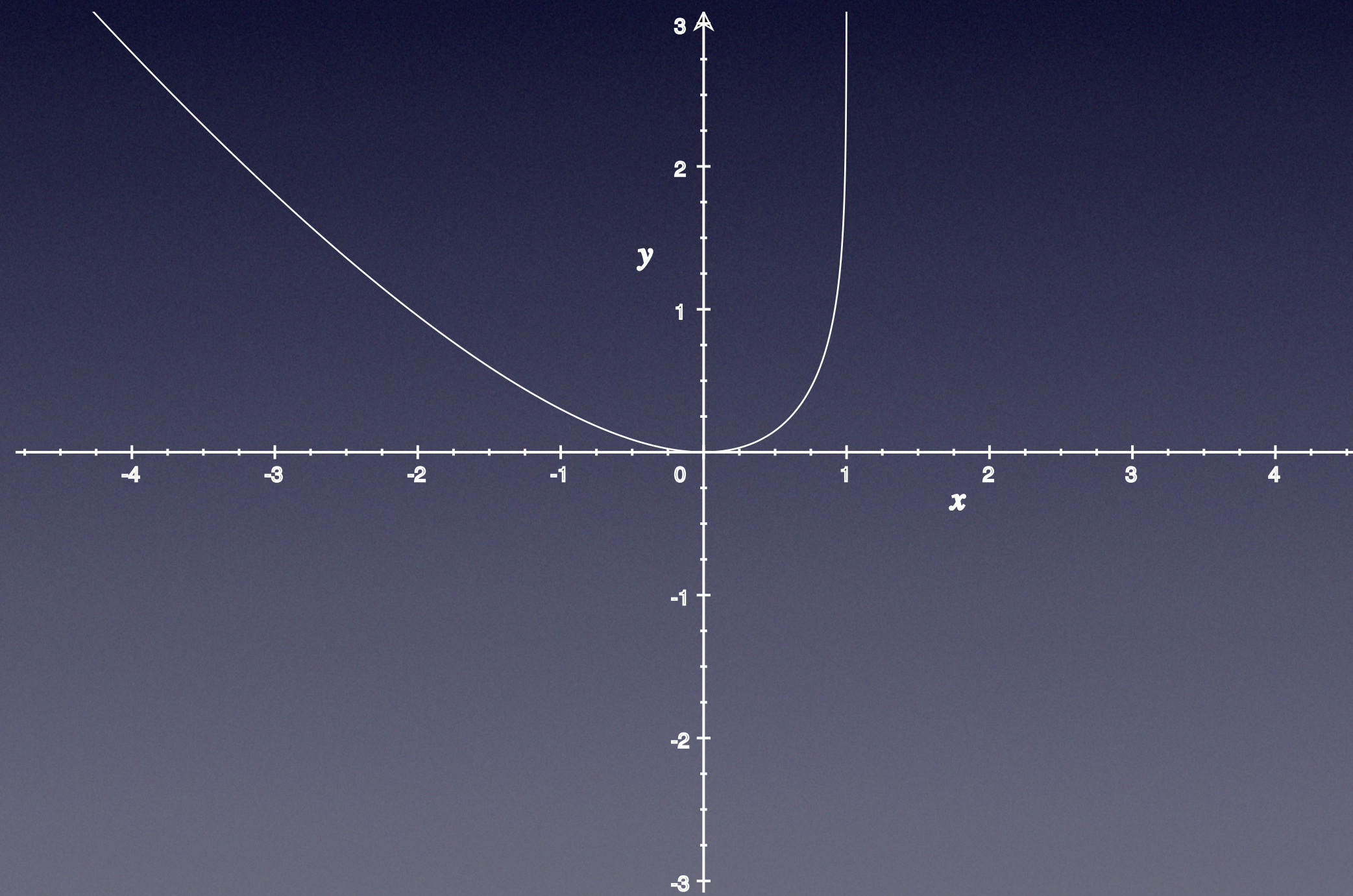
Let  $\gamma_u$  a function from  $[0,1[$  into  $\mathbb{R}$  such that

- $\gamma_u$  is continuously differentiable, strictly convex and non decreasing
- $\lim_{u \uparrow 1} \gamma_u(u) = +\infty$
- $\gamma_u(0) = 0$ ;  $\gamma_u$  is continuously differentiable at 0 with  $\gamma'_u(0) = 0$
- $\gamma'_u(u)$  is locally Lipschitz at  $u = 0$

# Penalty for a static constraint $u \in U^{ad}$

Example

$$\gamma_u(u) = -u \log(1 - u)$$



# Penalty for a static constraint $u \in U^{ad}$

For the constraint  $u(t) \in U^{ad}$  we define the penalty

$$P_u(u) = \int_0^T \gamma_u \circ G_{U^{ad}}(u(t)) dt$$

# Interior penalty methods

(Bonnans et al 2003, Malisani et al 2014)

Given the original control problem (1) and  $\epsilon > 0$  we define the *penalized* problem:

$$\min_{u \in U^{ad}} K(u, \epsilon) = J(u) + \epsilon P_u(u)$$

Theorem 1: we assume that  $x(t)$  is bounded for  $u \in U^{ad}$

We also assume that  $\lim_{\alpha \uparrow 1} \gamma'_u(\alpha) = +\infty$ .

If the penalized problem has a solution, it belongs to the **interior** of  $U^{ad}$ .

# Interior penalty methods

(Bonnans et al 2003, Malisani et al 2014)

Remark 1: why do we keep the constraint  $u(t) \in U^{ad}$  ?

Because the penalty is not defined for  $u \notin U^{ad}$  !

But, once the solution is in the interior of  $U^{ad}$ ,  
it can be shown, using a change of variables on the control,  
that solving the problem with  $u(t) \in \text{interior}(U^{ad})$   
is equivalent to an *unconstrained* problem.

Remark 2: The previous example penalty  $\gamma_u$  satisfies the assumptions.

# Interior penalty methods

(Bonnans et al 2003, Malisani et al 2014)

Theorem 2: it is assumed that for  $\epsilon$  in a neighborhood of 0, the penalized problem has a solution  $u^*(\epsilon)$ . Then

- $\lim_{\epsilon \downarrow 0} J(u^*(\epsilon)) = \inf_{u \in ad} J(u)$
- $\lim_{\epsilon \downarrow 0} \epsilon P_u(u^*(\epsilon)) = 0$

This means that the solution of the penalized problem asymptotically solves the original problem.

Under classical convexity assumptions, the optimal control and state converges to a solution of the original problem

# Perturbations of Constrained Optimal Control Problems: Introduction

We wish to solve

$$\min_{u \in U^{ad}} J(u) = \int_0^{t_f} [L_0(x, u, t) + L_1(x, u, t)] dt \quad (1)$$

with the dynamics

$$\frac{dx}{dt} = f_0(x, u, t) + f_1(x, u, t), \quad x(0) = x_0$$

# Regularity assumptions

We assume that  $f_0, f_1, L_0$  and  $L_1$  are

- globally Lipschitz with respect to  $x$  and  $u$
- twice continuously differentiable with respect to  $x$  and  $u$

Since the control and horizon are bounded, we derive that the state is bounded, and that  $f_0, f_1, L_0$  and  $L_1$  and their derivatives are also bounded.



# Orders of magnitude

We assume that there exist  $\varepsilon \in [0,1]$  such that

$$g_1(x, u, t) \leq \varepsilon g_0(x, u, t) \quad (M)$$

where  $g$  stands for  $f$  or  $L$ , or their first and second derivatives, with  $u \in U^{ad}$  and  $x \in X^{ad}$ ,  $X^{ad}$  being a compact domain that will contain all the trajectories driven by  $u \in U^{ad}$ .

Regular perturbation  
for optimal control problems subject  
to control constraints (Maamria et al 2020)

After a rescaling of  $L_1$  and  $f_1$ , our problem becomes

$$\min_{u \in U^{ad}} J(u) = \int_0^{t_f} [L_0(x, u, t) + \varepsilon L_1(x, u, t)] dt \quad (4.3)$$

with the dynamics

$$\frac{dx}{dt} = f_0(x, u, t) + \varepsilon f_1(x, u, t), \quad x(0) = x_0 \quad (4.4)$$

We can solve the unperturbed problem ( $\varepsilon = 0$ ) to obtain an optimal control  $u_0$ . How does  $J(u_0)$  compare to the optimal value of (4.3)?

Regular perturbation  
for optimal control problems subject  
to control constraints (Maamria et al 2020)

If we omit the constraint  $u \in U^{ad}$ ,  
the problem has been solved by Bensoussan  
in a slightly different context  
( $\varepsilon$  being in a neighborhood of 0).

But, if we approximate problem (4.3,4.4) with  
a *penalized* problem, we obtain an interior optimal control,  
and the stationarity condition  $\frac{\partial H}{\partial u} = 0$  still holds.

Regular perturbation  
for optimal control problems subject  
to control constraints (Maamria et al 2020)

Indeed, we shall prove that, for a given penalty parameter  $r$ ,  
the nominal control is optimal up to an error of the form

$$\Delta J \leq K_r \varepsilon^2.$$

We shall then prove that  $K_r$  is bounded when  $r$  tends to zero.

# Perturbed, penalized problems

Consider the family of problems

$$\min_{u \in U^{ad}} J_\varepsilon^r(u) = \int_0^T [L_0(\sigma) + \varepsilon L_1(\sigma) + rP(u)] dt, \quad r > 0 \quad (5)$$

with perturbed dynamics (4.4).

The letter  $\sigma$  denotes the couple  $(x, u)$ ,  
 $\varepsilon$  is the perturbation parameter,  
and  $r$  is the penalty parameter.

The letter  $p$  will be used to denote the adjoint states.

# About the perturbation term $\varepsilon$

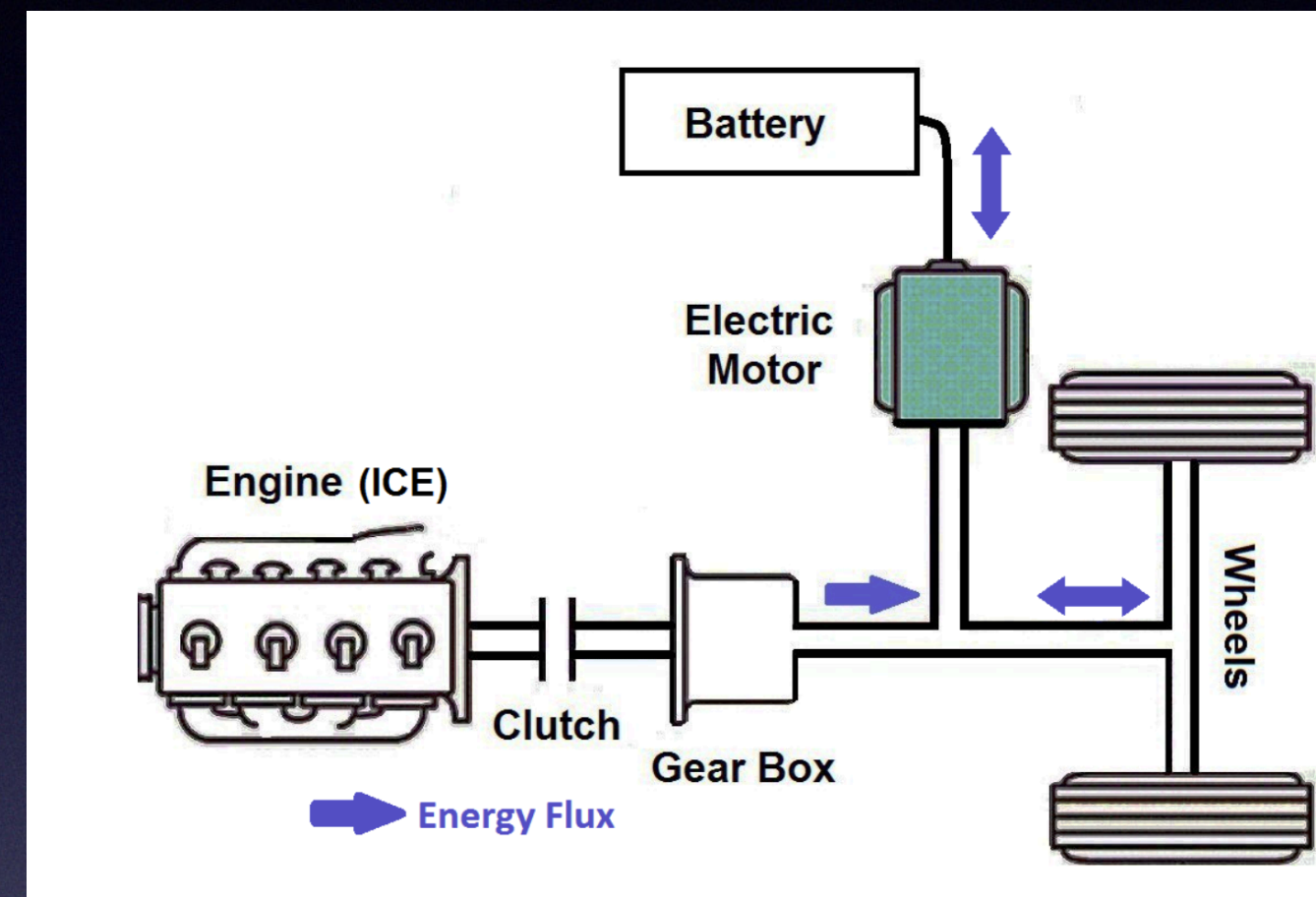
By contrast to the framework used by Bensoussan, we do not assume that  $\varepsilon$  is a parameter that tends to 0.

All that we know is that  $\varepsilon \in [0,1]$  and that it satisfies the inequations featured in (M):

$$g_1(x, u, t) \leq \varepsilon g_0(x, u, t)$$

For instance, if  $|g_1| > 0$  in (M), we cannot have  $\varepsilon \rightarrow 0$ .

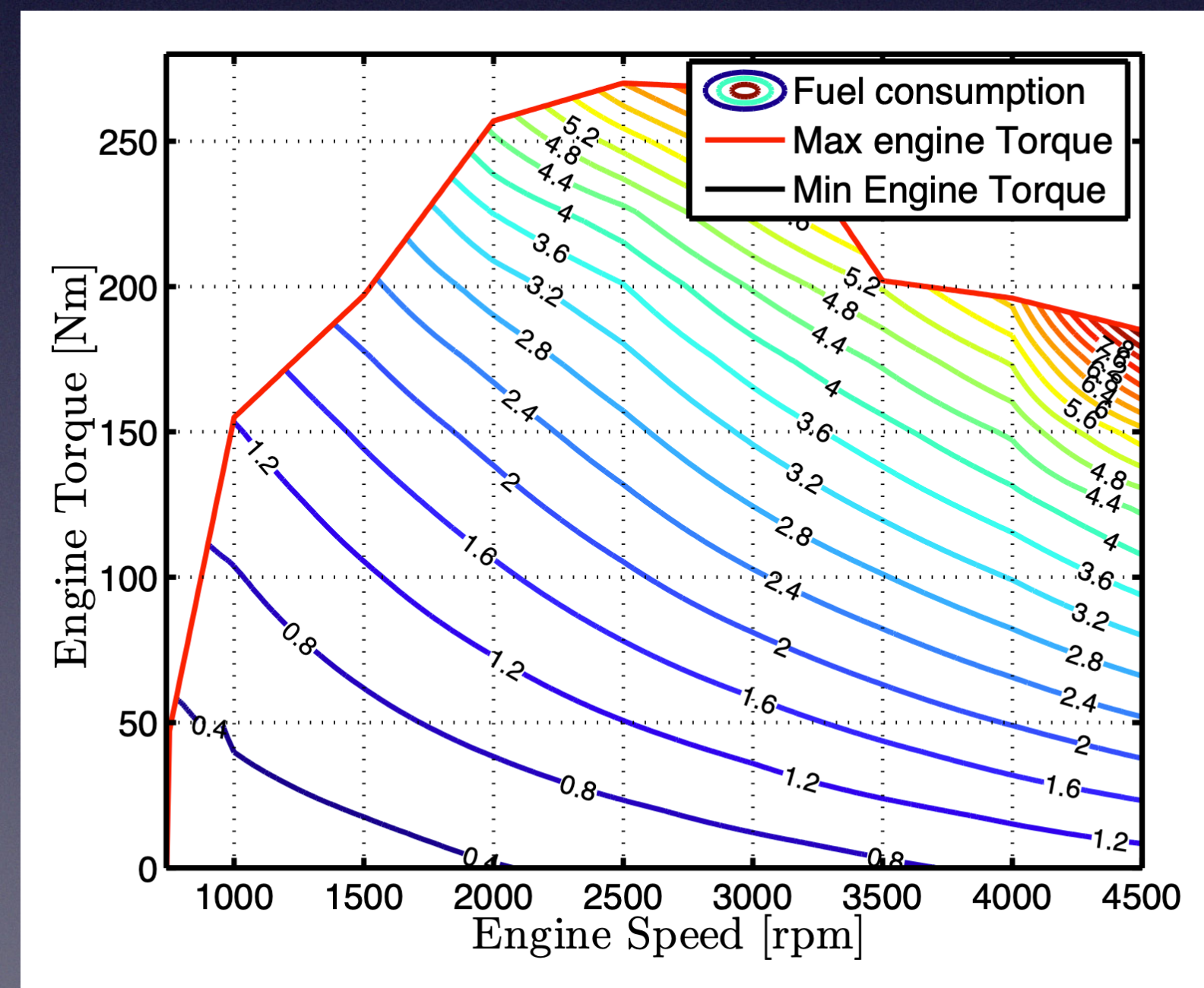
# A practical example



For a given path, we wish to minimize the consumption of a car engine. The model of the engine consumption is obtained after a series of benchmarks. Sample measures are obtained, and we assume that we have some relation between the number of samples and the precision of the model.

# A practical example

Here is a typical chart of the relation between the engine speed and the fuel consumption





# A practical example

The optimal control problem depends heavily on this modelization.

For a given number of measures, how can we make sure that the derived « optimal » control is good enough with respect to the best consumption model we *might* have?

Here we cannot play with the number of samples.

Therefore we cannot have  $\varepsilon \rightarrow 0$  in the problem formulation, because we have no access to the exact continuous consumption model.

# Notations

For  $\varepsilon = 0$ , define the Hamiltonian  $H_0^r(\sigma, p) = H_0(\sigma, p) + rP(u)$ .

For  $\varepsilon > 0$ , the **Hamiltonian** is

$$H_\varepsilon^r(\sigma, p) = L_0(\sigma) + \varepsilon L_1(\sigma) + p^T [f_0(\sigma) + \varepsilon f_1(\sigma)] + rP(u) = H_0^r(\sigma, p) + \varepsilon H_1(\sigma, p).$$

and  $u_\varepsilon^r$ ,  $x_\varepsilon^r$  et  $p_\varepsilon^r$  denote the optimal control, state and adjoint state.

For any  $x$  and  $u$ , we denote

$$\delta x^r = x - x_0^r, \quad \delta u^r = u - u_0^r, \quad \delta \sigma^r = \sigma - \sigma_0^r$$

and, for the optimal variables,

$$\delta x_\varepsilon^r = x_\varepsilon^r - x_0^r, \quad \delta u_\varepsilon^r = u_\varepsilon^r - u_0^r, \quad \delta \sigma_\varepsilon^r = \sigma_\varepsilon^r - \sigma_0^r.$$

The reference is the solution of the unperturbed, penalized problem.

# Requirement on the estimation terms

In what follows, we shall obtain some bounds of the form

$$K\varepsilon^n$$

We shall require that  $K$  only depends on

- the formulation and solution of the *unperturbed* problem,
- estimates on the magnitude of the various derivatives of  $f_1$  and  $L_1$ .

If so, we shall say that  $K$  is a *good bound*.

# Second order expansion of the cost with integral remainder

For **any**  $u$ , the following expansion holds along the direction  $\delta\sigma^r = \sigma - \sigma_0^r$  (nominal point is the unperturbed, nominal optimum) :

$$J_\varepsilon^r(u) = \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt + \varepsilon \int_0^T [N^0(t) \cdot \delta u^r + N^1(t) \cdot \delta x^r] dt \\ + \int_0^T \int_0^1 \int_0^1 [\delta\sigma^r] \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r) [\delta\sigma^r]^T \lambda d\lambda d\mu dt$$

with

$$N^0(t) = \partial_u H_1(\sigma_0^r, p_0^r) , N^1(t) = \partial_x H_1(\sigma_0^r, p_0^r)$$

# Error bounds on the trajectories driven by $u_0^r$ with respect to perturbations

In the previous expansion, the 0 order term involves  
the Hamiltonian  $H_\varepsilon^r$  and  
 $x_0^r$ , the trajectory driven by  $u_0^r$  with unperturbed dynamics ( $\varepsilon = 0$ ).

Let  $X_\varepsilon^r$  the trajectory driven by the *same* control  $u_0^r$   
but with a perturbed dynamics ( $\varepsilon \neq 0$ ). Then

$$\| X_\varepsilon^r(t) - x_0^r(t) \| \leq \varepsilon F_1$$

where  $F_1$  is a good bound.

# A bound on the linear and quadratic terms

In what follows, for a given penalty parameter  $r$ , we wish to compare the performances of  $u_0^r$  and  $u_\varepsilon^r$ .

Define

$$M_0 \triangleq J_\varepsilon^r(u_0^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt.$$

One can prove that

$$|M_0| \leq m_0 \varepsilon^2$$

where  $m_0$  is a good bound.

# Proof

We use the Taylor expansion for  $u = u_0^r$ . In this case, we have  $\delta u^r = u - u_0^r = 0$ .

Since we have shown that

$$\| X_\varepsilon^r(t) - x_0^r(t) \| \leq \varepsilon F_1,$$

the quadratic term is proportional to  $\varepsilon^2 F_1^2$  because

$$\partial_{xx} H_\varepsilon^r = \partial_{xx} (H_0^0 + \varepsilon H_1)$$
 does not depend on  $r$ .

The linear term is proportional to  $\varepsilon^2 F_1 n_1$ , where  $n_1$  is a bound on  $\| N_1 \|$ .

$F_1^2$  and  $F_1 n_1$  are good bounds.

# Remark

The previous result is false if we have state penalties,  
i.e. penalized state constraints.



# Comparison with the optimal cost

Since  $J_\varepsilon^r(u_\varepsilon^r) \leq J_\varepsilon^r(u_0^r)$ , we derive

$$J_\varepsilon^r(u_\varepsilon^r) - \int_0^T [H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r] dt \leq M_0 \leq m_0 \varepsilon^2.$$

Inserting the Taylor expansion of  $J_\varepsilon^r(u_\varepsilon^r)$  yields

$$m_0 \varepsilon^2 \geq \varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt + \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_\varepsilon^r(\sigma_0^r + \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt. \quad (6)$$

where

$$\partial_{\sigma\sigma} H_\varepsilon^r = \boxed{\partial_{\sigma\sigma} H_0^r} + \varepsilon \partial_{\sigma\sigma} H_1$$

# Investigation of the quadratic term with $\partial_{\sigma\sigma}H_0^r$

Define

$$z(\lambda, \mu, t) \triangleq \delta u_\varepsilon^r + \left[ \partial_{uu}H_0^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r) \right]^{-1} \partial_{ux}H_0^r(\sigma_0^r + \lambda\mu\delta\sigma^r, p_0^r) \delta x_\varepsilon^r$$

Inserting  $z$  into the the quadratic term that involves  $\partial_{\sigma\sigma}H_0^r$  yields

$$\partial_{\sigma\sigma}H_0^r(\cdot)(\delta\sigma_\varepsilon^r)^2 = z^T \partial_{uu}H_0^r(\cdot)z + \delta x_\varepsilon^{rT} \left[ \partial_{xx}H_0^r - \partial_{xu}H_0^r[\partial_{uu}H_0^r]^{-1}\partial_{ux}H_0^r \right](\cdot)\delta x_\varepsilon^r.$$

# Legendre-Clebsch assumptions

We assume that, for the *unperturbed, unpenalized* problem

- $\partial_{uu}H_0(\sigma, p_0^*) \geq \beta I$  uniformly with respect to  $\sigma$  (h1)
- $(\partial_{xx}H_0 - \partial_{xu}H_0[\partial_{uu}H_0]^{-1}\partial_{ux}H_0)(\sigma, p_0^*) \geq 0$  uniformly with respect to  $\sigma$  (h2)

Observe that

- $H_0^r$  is the sum of  $H_0$  and a strong convex penalty, hence (h1) still holds for  $H_0^r$
- We have  $\partial_x H_0^r = \partial_x H_0$ , hence (h2) still holds for  $H_0^r$

From this we derive

$$\partial_{\sigma\sigma}H_0^r(\cdot)(\delta\sigma_\varepsilon^r)^2 \geq \beta \left\| z(\lambda, \mu, t) \right\|^2 \quad (7)$$

# Summary

From (6) and (7), and since  $H_\varepsilon^r = H_0^r + \varepsilon H_1$ , we derive

$$\begin{aligned} (c_0 F_1^2 + c_1 N_1^2) \varepsilon^2 &\geq \varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt + \beta R \\ + \varepsilon \int_0^T \int_0^1 \int_0^1 &\partial_{\sigma\sigma} H_1 (\sigma_0^r + \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt. \end{aligned} \quad (8)$$

$$\text{where } R = \int_0^T \int_0^1 \int_0^1 \|z(\lambda, \mu, t)\|^2 \lambda d\lambda d\mu dt.$$

# Intermediary estimates on $\delta u$ and $\delta x$

The definition of  $z$  can be rewritten as  $\delta u_\varepsilon^r = z - [\partial_{uu}H_0^r(\cdot)]^{-1}\partial_{ux}H_0^r(\cdot)\delta x_\varepsilon^r$ .

From the Legendre-Clebsch conditions we derive

$$\|\delta u_\varepsilon^r\| \leq \|z(\lambda, \mu, t)\| + \gamma_1 \|\delta x_\varepsilon^r\| \quad (9)$$

where  $\gamma$  is a constant. Moreover, one can easily prove that

$$\|\delta x_\varepsilon^r(t)\| \leq \Gamma \int_0^t \left[ \|\delta x_\varepsilon^r(t)\| + \|\delta u_\varepsilon^r(t)\| \right] dt + \varepsilon F_1 t \quad (10)$$

From (10) and (9) we derive

$$\|\delta x_\varepsilon^r(t)\| \leq \Gamma(1 + \gamma_1) \int_0^t \|\delta x_\varepsilon^r(t)\| dt + \Gamma \int_0^t \|z(\lambda, \mu, s)\| ds + \varepsilon F_1 t \quad (11)$$

# A bound on $\delta x_\varepsilon^r$ and $\delta u_\varepsilon^r$

Using the Grönwall lemma, we derive

$$\| \delta x_\varepsilon^r(t) \|^2 \leq \alpha_1(t)R + \alpha_2(t)F_1^2\varepsilon^2 \quad (12).$$

where  $\alpha_1$  and  $\alpha_2$  are good bounds.

Inserting this into (9) yields

$$\| \delta u_\varepsilon^r \|^2 \leq 2 \| z(\lambda, \mu, t) \|^2 + 2\gamma_1^2 \| \delta x_\varepsilon^r \|^2$$

Together with (12), this implies

$$\| \delta u_\varepsilon^r(t) \|_{L^2}^2 dt \leq \alpha_3 R + \alpha_4 F_1^2 \varepsilon^2 \quad (13)$$

where  $\alpha_3$  and  $\alpha_4$  are good bounds.

# Remark

If we can bound  $R$  by some  $k\varepsilon^2$ , the main result will easily follow!

To do so, we come back to equation (8):

$$\begin{aligned} (c_0 F_1^2 + c_1 N_1^2) \varepsilon^2 &\geq \varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt + \beta R \\ &+ \varepsilon \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_1(\sigma_0^r + \mu \delta \sigma_\varepsilon^r, p_0^r) (\delta \sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt. \end{aligned} \quad (8)$$

# Investigation of the **linear** term

We want a lower bound on  $\varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt$ .

Using the Young inequality and the bounds on  $\delta x$  and  $\delta u$ , we obtain a lower bound of the form

$$-(n_0^2 + n_1^2) \frac{\varepsilon^2}{2m} - \alpha_5 \left( F_1^2 \frac{m}{2} \varepsilon^2 - \frac{m}{2} R \right)$$

where  $m$  is any positive number,  $n_0$  and  $n_1$  are good bounds, and  $\alpha_5$  is derived from the estimates (12) and (13).



# Investigating the quadratic term

The quadratic term involves  $\delta x$ ,  $\delta u$  and  $\partial_{\sigma\sigma}H_1(\sigma, p_0^r)$ .

Observe that  $\partial_{\sigma\sigma}H_1(\sigma, p_0^r)$  depend on  $r$  only through  $p_0^r$ .

Since the control and states are bounded, and that the dynamics of the adjoint state does not involve the penalty,

$p_0^r$  is bounded and we can define, for a bounded  $r$ ,

$$\bar{H} = \inf_{\sigma, r} \left\| \partial_{\sigma\sigma}H_1(\sigma, P_0^r) \right\|, \text{ which is a good bound.}$$

Using (12) and (13) yields the following lower bound on the quadratic term:

$$-\frac{\bar{H}}{2} [F_1^2 \alpha_8 \varepsilon^2 + \varepsilon \alpha_9 R]$$

# Estimate on $R$

Inserting the previous estimates into (8) yield

$$(c_0 F_1^2 + c_1 N_1^2) \varepsilon^2 \geq -\varepsilon^2 \left[ \frac{1}{2m} \alpha_5 + \frac{m}{2} F_1^2 \alpha_6 \right] + R \left[ \beta - \frac{m}{2} \alpha_7 \right] - \frac{\bar{H}}{2} \left[ F_1^2 \alpha_8 \varepsilon^2 + \varepsilon \alpha_9 R \right]$$

Reordering this inequality yields

$$R \left[ \beta - \frac{m}{2} \alpha_7 - \varepsilon \frac{\alpha_9}{2} \bar{H} \right] \leq F_1^2 \varepsilon^2 \left[ c_0 + \alpha_6 \frac{m}{2} + \frac{\alpha_8}{2} \bar{H} \right] + \varepsilon^2 \left[ c_1 + \alpha_5 \frac{1}{2m} \right] \quad (15)$$

We will have an upper bound on  $R$  if

$$\gamma \triangleq \left[ \beta - \frac{m}{2} \alpha_7 - \varepsilon \frac{\alpha_9}{2} \bar{H} \right] > 0$$

# Investigating the coefficient of $R$

$$\text{We wish } \gamma \triangleq \left[ \beta - \frac{m}{2}\alpha_7 - \varepsilon \frac{\alpha_9}{2}\bar{H} \right] > 0.$$

To do so, we *choose*  $m = \frac{\beta}{\alpha_7}$ . Then  $\gamma = \frac{\beta}{2} - \varepsilon \frac{\alpha_9}{2}\bar{H}$ .

We have  $\gamma > 0$  if and only if the following holds:

$$\varepsilon\bar{H} < \frac{\beta}{\alpha_9} \quad (\text{h3})$$

Denote  $\rho = \frac{\varepsilon\alpha_9\bar{H}}{\beta} < 1$ . Then

$$\gamma = (1 - \rho)\frac{\beta}{2}.$$

# Interpretation of (h3)

Assumption (h3) requires essentially that the *perturbation* parameter  $\varepsilon$  be reasonably smaller than the strong convexity constant  $\beta$  of the *nominal* Hamiltonian  $H_0$  with respect to  $u$ .

# Estimate on $R$

Assuming that (h3) holds, equation (15) yields

$$R(1 - \rho)\frac{\beta}{2} \leq F_1^2 \varepsilon^2 \left[ c_0 + \alpha_6 \frac{m}{2} + \frac{\alpha_8}{2} \bar{H} \right] + \varepsilon^2 \left[ c_1 N_1^2 + \alpha_5 \frac{1}{2m} \right] \text{ with } m = \frac{\beta}{\alpha_7}.$$

and hence

$$R \leq \varepsilon^2 \frac{2}{(1 - \rho)\beta} \left[ F_1^2 \left( c_0 + \frac{\alpha_6}{2} \frac{\beta}{\alpha_7} + \rho \frac{\alpha_8}{2} \frac{\beta}{\alpha_9} \right) + \left( c_1 N_1^2 + \frac{\alpha_5}{2} \frac{\alpha_7}{\beta} \right) \right]$$

which is conservative but highlights the importance of  $\beta$  and  $\rho = \frac{\varepsilon \alpha_9 \bar{H}}{\beta}$ .

Since  $\beta$  is a convexity estimate, we can always assume that it is not too large.

Problems arise when  $\beta$  is too small with respect to the perturbation terms.

# Estimates on $\delta x_\varepsilon^r$ and $\delta u_\varepsilon^r$

$$\text{Let } s_{2a} = c_0 + \frac{\alpha_6}{2} \frac{\beta}{\alpha_7} + \rho \frac{\alpha_8}{2} \frac{\beta}{\alpha_9} \text{ and } s_{2b} = c_1 N_1^2 + \frac{\alpha_5}{2} \frac{\alpha_7}{\beta}.$$

The bound on  $R$  can be written  $R \leq \varepsilon^2 \frac{2}{(1-\rho)\beta} [s_{2a} F_1^2 + s_{2b}]$

From (12) and (13) we directly derive

$$\| \delta x_\varepsilon^r(t) \|^2 \leq \varepsilon^2 \left[ F_1^2 \left( \alpha_2(t) + \frac{2}{(1-\rho)\beta} \alpha_1(t) s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_1(t) s_{2b} \right]$$

and

$$\int_0^T \| \delta u_\varepsilon^r(t) \|^2 dt \leq \varepsilon^2 \left[ F_1^2 \left( \alpha_4 + \frac{2}{(1-\rho)\beta} \alpha_3 s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_3 s_{2b} \right]$$

# Now what?

We have estimates on the control and state,  
so it seems that our problem is solved.

# Now what?

We have estimates on the control and state,  
so it seems that our problem is solved.  
However, we have to investigate the cost itself  
because of the penalty parameter  
 $rP(u)$ .



# Sub optimality

The following holds

$$0 \leq J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \leq |M_0| - M_1,$$

where  $M_0$  satisfies:

$$|M_0| \leq (c_0 F_1^2 + c_1 N_1^2) \varepsilon^2$$

and  $M_1$  is defined by

$$M_1 = J_\varepsilon^r(u_\varepsilon^r) - \int_0^T \left[ H_\varepsilon^r(\sigma_0^r, p_0^r) - p_0^{rT} \dot{x}_0^r \right] dt$$

# The simple case

If  $M_1$  is not negative, we directly have

$$0 \leq J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \leq (c_0 F_1^2 + c_1 N_1^2) \varepsilon^2$$

and the result is proved.

# Second order expansion for $M_1$ negative

If  $M_1$  is *negative*, we rewrite  $-M_1 > 0$  as

$$\begin{aligned} -M_1 = & -\varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt - \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_0^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \\ & -\varepsilon \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_1^r(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \end{aligned}$$

# Second order expansion for $M_1$ negative

Since the penalty is *convex*, we have  $\partial H_{\sigma\sigma} H_0 \leq \partial H_{\sigma\sigma} H_0^r$   
and hence  $-\partial H_{\sigma\sigma} H_0^r \leq -\partial H_{\sigma\sigma} H_0$ .

As a consequence, we have

$$\begin{aligned} -M_1 \leq & -\varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt - \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_0(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \\ & -\varepsilon \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \end{aligned}$$

# Second order expansion for $M_1$ negative

Since the penalty is *convex*, we have  $\partial H_{\sigma\sigma} H_0 \leq \partial H_{\sigma\sigma} H_0^r$

and hence  $-\partial H_{\sigma\sigma} H_0^r \leq -\partial H_{\sigma\sigma} H_0$ .

$r$  has vanished

As a consequence, we have

$$\begin{aligned} -M_1 \leq & -\varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt - \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_0(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \\ & -\varepsilon \int_0^T \int_0^1 \int_0^1 \partial_{\sigma\sigma} H_1(\sigma_0^r + \lambda\mu\delta\sigma_\varepsilon^r, p_0^r) (\delta\sigma_\varepsilon^r)^2 \lambda d\lambda d\mu dt \end{aligned}$$

# Investigation of the **linear** term

Using the Young inequality and the bounds on  $\delta x$  and  $\delta r$ ,  
we can bound the first order term

$$-\varepsilon \int_0^T [N^0 \delta u_\varepsilon^r + N^1 \delta x_\varepsilon^r] dt$$

by

$$\varepsilon^2 \left[ \frac{n_0^2 + n_1^2}{2m_1} + m_1 \left( F_1^2 \left( \alpha_{10} + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2b} \right) \right]$$

for any  $m_1 > 0$ .

# Investigation of the quadratic terms

We have the following estimate on the vector  $\delta\sigma_\varepsilon^r$ :

$$\|\delta\sigma_\varepsilon^r\|^2 \leq \varepsilon^2 \left[ F_1^2 \left( \alpha_{10} + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2b} \right].$$

If we bound the quadratic form by  $\bar{H}_{\sigma,\sigma} = \max_{i=1,2 \text{ et } t \in [0,T]} \|\partial_{\sigma\sigma} H_i\|$

the quadratic term is bounded

$$2\varepsilon^2 \bar{H}_{\sigma,\sigma} \left[ F_1^2 \left( \alpha_{10} + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2b} \right].$$

# $M_1$ summary

We have the bound

$$0 \leq J_\varepsilon^r(u_0^r) - J_\varepsilon^r(u_\varepsilon^r) \leq |M_0| - M_1$$

with

$$-M_1 \leq \varepsilon^2 \left[ \frac{\bar{N}^2}{2m_1} + (m_1 + 2\bar{H}_{\sigma,\sigma}) \left( F_1^2 \left( \alpha_{10} + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2b} \right) \right]$$

where  $m_1$  is a parameter that remains to be determined.



# Sub optimality

Gathering the bounds on  $M_0$  and  $M_1$  yields

$$0 \leq J'_\varepsilon(u'_0) - J'_\varepsilon(u'_\varepsilon) \leq K_r \varepsilon^2 \quad (15)$$

with

$$K_r = (c_0 F_1^2 + c_1 \|N_1\|_\infty^2) + \frac{\bar{N}^2}{2m_1} + (m_1 + 2\bar{H}_{\sigma,\sigma}) \left( F_1^2 \left( \alpha_{10} + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2a} \right) + \frac{2}{(1-\rho)\beta} \alpha_{11} s_{2b} \right)$$

which is a good bound, independent of  $\varepsilon$ , and bounded if  $r$  is bounded.

# Main result

Going to the limit  $r \rightarrow 0$  in (15) yields  $K_0 \geq 0$  such that the following bound holds

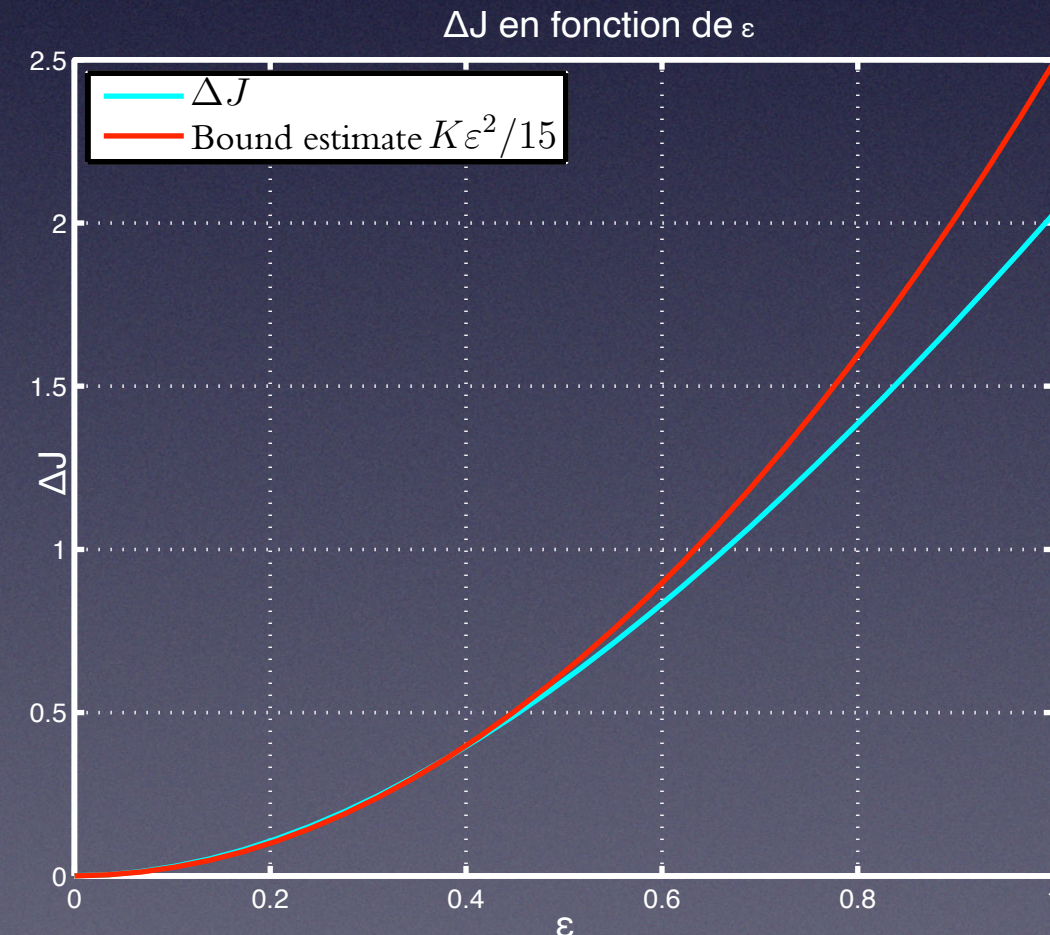
$$0 \leq J_\varepsilon(u^*(0)) - \inf_{u \in U^{ad}} J_\varepsilon(u) \leq K_0 \varepsilon^2$$

# Numerical evaluation

As a numerical example, all the estimates are computed on an LQ problem, using the numerical values and the stability of the chosen system.

We also compute the cost of the perturbed problem.

We obtain the following figure



We see that there is a ratio of 15 between the conservative estimate and the actual error

# Conclusion

The sub optimality of the nominal control, subject to model perturbations, has been extended to the control constrained case.

It is also a *robustness* result.

If some model *uncertainties* can be bounded (relatively to the nominal data), then the optimality of the nominal control will be robust to these uncertainties (with an error which is the *square* of their relative magnitude).

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and the speaker's notes on the course by Alain Bensoussan.