2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

Null controllability using flatness: a case study of a 1-D heat equation with discontinuous coefficients

Philippe Martin, Lionel Rosier and Pierre Rouchon

Abstract— **The approach recently proposed by the authors for the null controllability of 1-D parabolic equations is applied to the nontrivial case of a heat equation with discontinuous coefficients and subjected to Robin boundary conditions. The control steering the system to zero from a discontinuous initial state is comprehensively derived, together with the corresponding trajectory. Numerical experiments illustrate several features of the theory and demonstrate its effectiveness.**

I. Introduction

The null controllability of parabolic equations has been extensively investigated since several decades. After the pioneering works [1]–[3], mainly concerned with the 1-D case, there has been significant progress in the N-D case by using Carleman estimates, see in particular [4]. More recently, the case of discontinuous coefficients has been studied along similar lines [5]–[7]. A direct alternative approach based on the so-called flatness property [8], [9], was proposed in [10] for the plain heat equation, and extended to the parabolic case with fairly irregular coefficients [11].

In this paper we apply the method of [11] to a physically relevant problem with discontinuous coefficients. We consider the heat conduction in a one-dimensional rod made of two sections with constant thermal properties. Without restriction, we can assume the rod has length 1, with one section of length X and the other of length $1-X$. The evolution of the temperature Θ is given by the heat equation

$$\rho\Theta_t(x,t) = (a\Theta_x)_x(x,t);$$

a and ρ are the piecewise constant functions on $(0,1)$

$$\big(a(x),\rho(x)\big) := \begin{cases}(a_0,\rho_0), & 0<x<X\\ (a_1,\rho_1), & X<x<1,\end{cases}$$

where a_0,a_1,ρ_0,ρ_1 are strictly positive constants. This situation is typical of composite materials. At the 0-end the rod is submitted to the constant ambient temperature Θ_0, and at the 1-end to a time-varying heat source (the control input) of temperature $\Theta_1(t)$. The heat flux $-a\Theta_x$ at the ends obeys the convection conditions

$$\begin{aligned}-(a\Theta_x)(0,t) &= h_0\big(\Theta_0-\Theta(0,t)\big)\\ -(a\Theta_x)(1,t) &= h_1\big(\Theta(1,t)-\Theta_1(t)\big),\end{aligned}$$

with h_0 and h_1 positive constants. Setting $\theta(x,t) := \Theta(x,t)-\Theta_0$ and taking as the control input $u(t) := \Theta_1(t)-\Theta_0$ results in the boundary value problem

$$\rho\theta_t(x,t)-(a\theta_x)_x(x,t) = 0, \qquad (x,t)\in(0,1)\times(0,T) \quad (1)$$

$$\alpha_0\theta(0,t)+\beta_0(a\theta_x)(0,t) = 0 \quad (2)$$

$$\alpha_1\theta(1,t)+\beta_1(a\theta_x)(1,t) = u(t), \quad (3)$$

where the constants $\alpha_0,\beta_0,\alpha_1,\beta_1$ satisfy $\alpha_0^2+\beta_0^2>0$, $\alpha_1^2+\beta_1^2>0$, $\alpha_0\beta_0\le 0$ and $\alpha_1\beta_1\ge 0$. Note the two limiting cases: $\beta_i=0$ (Dirichlet condition), obtained when taking $h_i\to\infty$; $\alpha_i=0$ (Neumann condition), obtained when considering the control input is $u(t) := h_1\big(\Theta(1,t)-\Theta_1(t)\big)$ and taking $h_0=0$. Included in the formulation of the system is the fact that a solution θ and its quasi-derivative $a\theta_x$ are differentiable in particular at $x=X$ (whereas θ_x will in general be discontinuous at X). We could thus rewrite (1) more explicitly as the piecewise constant heat equation

$$\begin{cases}\theta_t(x,t) = \frac{a_0}{\rho_0}\theta_{xx}(x,t), & 0<x<X\\ \theta_t(x,t) = \frac{a_1}{\rho_1}\theta_{xx}(x,t), & X<x<1\end{cases}$$

together with the so-called interface conditions

$$\begin{aligned}\theta(X^-,t) &= \theta(X^+,t)\\ a_0\theta_x(X^-,t) &= a_1\theta_x(X^+,t).\end{aligned}$$

The aim of this paper is to fully derive and numerically test the open-loop control steering the system (1)–(3) from an initial state $\theta_0\in L^2(0,1)$ at time 0 to the final state 0 at time T. We follow the approach proposed in [11] for the null controllability of general 1-D parabolic equations with fairly irregular coefficients. The paper runs as follows: in section II, we recall the main formal steps of the approach on our specific example; in section III, we give the detailed expressions of the quantities involved in the control law; finally in section IV, we present detailed numerical experiments.

II. Outline of the flatness-based approach

In [11] is established a general result for the null controllability of the one-dimensional parabolic system

$$\begin{aligned}&\rho\theta_t-(a\theta_x)_x-b\theta_x-c\theta = 0, \qquad (x,t)\in(0,1)\times(0,T)\\ &\alpha_0\theta(0,t)+\beta_0(a\theta_x)(0,t) = 0\\ &\alpha_1\theta(1,t)+\beta_1(a\theta_x)(1,t) = u(t),\end{aligned}$$

where a,b,c,ρ are functions on $(0,1)$ such that;

- $a(x)>0$ and $\rho(x)>0$ for almost all $x\in(0,1)$
- $\exists K\ge 0$ such that $c(x)\le K\rho(x)$ for almost all $x\in(0,1)$.

Theorem 1: Assume $\frac{1}{a},\frac{b}{a},c,\rho\in L^1(0,1)$ and $a^{1-\frac{1}{p}}\rho\in L^p(0,1)$ for some $p\in(1,\infty]$. Consider an initial state $\theta_0\in L^1_\rho(0,1)$, a final time $T>0$ and $s\in(1,2-\frac{1}{p})$.

P. Martin, L. Rosier and P. Rouchon are with the Centre Automatique et Systèmes, MINES ParisTech, PSL Research University, 75006 Paris, France {philippe.martin, pierre.rouchon, lionel.rosier}@mines-paristech.fr

Then there exists a control $u \in G^s([0,T],\mathbb{R})$ which steers the system from θ_0 to the final state $\theta(\cdot,T)=0$. Moreover $\theta, a\theta_x \in G^s\big([\varepsilon,T],W^{1,1}(0,1)\big)$ for all $\varepsilon \in (0,T)$.

In the theorem, $L^1_\rho(0,1)$ is the space of functions f such that $\int_0^1 |f(x)|\rho(x)dx < \infty$; $G^s([0,T],B) \subset C^\infty([0,T],B)$, where B is a Banach space with norm $\|\cdot\|_B$, is the class of Gevrey functions of order s, i.e. such that for some $M,R>0$,

$$\big\|y^{(i)}(t)\big\|_B \leq M\frac{i!^s}{R^i}, \qquad \forall t\in[0,T],\ \forall i \geq 0.$$

The proof of this result provides an explicit construction of the control u in the form of an infinite series. We outline in the sequel the formal aspects of the construction for our particular case (1)–(3), where $b=c=0$; also $\frac{1}{a},\rho \in L^\infty(0,1)$, which implies $L^2(0,1)\subset L^1_\rho(0,1)$. We refer the reader to [11] for all the technical aspects, in particular proofs of convergence of all the series appearing in the sequel. The method comprises two phases: we first apply a zero control on $[0,\tau]$, where $\tau\in(0,T)$ is some arbitrary intermediate time, to steer the system from the "irregular" initial state $\theta_0 \in L^2$ to a "more regular" intermediate state θ_τ; we then use the flatness property to steer the system from θ_τ to the final state $\theta(T,x)=0$.

A. First phase: null control on $[0,\tau]$

Since $u(t)=0$ here, the boundary condition (3) becomes

$$\alpha_1\theta(1,t)+\beta_1(a\theta_x)(1,t)=0, \tag{4}$$

and we can solve (1), (2) and (4) by the method of separation of variables. Indeed, there exists an orthonormal basis $(e_n)_{n\geq 0}$ of $L^2_\rho(0,1)$ and a sequence $(\lambda_n)_{n\geq0}$ in $\mathbb{R}$ such that

$$-(ae_n')'=\lambda_n\rho e_n \tag{5}$$
$$\alpha_0 e_n(0)+\beta_0(ae_n')(0)=0 \tag{6}$$
$$\alpha_1 e_n(1)+\beta_1(ae_n')(1)=0. \tag{7}$$

$L^2_\rho(0,1)$ is the space of functions f such that $\langle f,f\rangle_\rho<\infty$, endowed with the inner product $\langle f,g\rangle_\rho := \int_0^1 f(x)g(x)\rho(x)dx$. In our case $L^2(0,1)=L^2_\rho(0,1)$ since ρ and $\frac{1}{\rho}$ are bounded.

In other words, the λ_n's and e_n's are the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$-(a\phi')'=\lambda\rho\phi \tag{8}$$
$$\alpha_0\phi(0)+\beta_0(a\phi')(0)=0 \tag{9}$$
$$\alpha_1\phi(1)+\beta_1(a\phi')(1)=0. \tag{10}$$

As we will need this fact in section III-B, we now prove the λ_n's are strictly positive (and by the way rederive that the λ_n's, hence the e_n's, are real). Indeed, consider a (possibly complex) solution λ,ϕ of (8)–(10). On the one hand by (8),

$$\int_0^1 \bar\phi(x)(a\phi')'(x)dx = -\lambda\int_0^1 |\phi(x)|^2\rho(x)dx,$$

where $\bar\phi$ is the complex conjugate of ϕ; on the other hand, integrating (8) by parts gives

$$\begin{aligned}\int_0^1 \bar\phi(x)(a\phi')'(x)dx &= \big[(a\phi'\bar\phi)(x)\big]_0^1 - \int_0^1 (a\phi'\bar\phi')(x)^2dx\\ &= K_0-K_1-\int_0^1 a(x)\left|\phi'(x)\right|^2dx,\end{aligned}$$

where $K_i := \alpha_i\beta_i\frac{|\phi(i)|^2+|(a\phi')(i)|^2}{\alpha_i^2+\beta_i^2}$. As a consequence,

$$\lambda\int_0^1|\phi(x)|^2dx = K_1-K_0+\int_0^1 a(x)\left|\phi'(x)\right|^2dx,$$

which implies λ is real and, because $\alpha_0\beta_0\leq 0$ and $\alpha_1\beta_1\geq0$, strictly positive.

As $(e_n)_{n\geq0}$ is an orthonormal basis of $L^2_\rho(0,1)$, we can expand the initial condition as $\theta_0=\sum_{n\geq0}c_ne_n$ in $L^2_\rho(0,1)$, with coefficients $c_n := \langle\theta_0,e_n\rangle_\rho$ such that $\sum_{n\geq0}|c_n|^2<\infty$. It is then easy to check that

$$\theta(x,t) := \sum_{n\geq0}c_ne^{-\lambda_nt}e_n(x)$$

is the solution of (1), (2), (4) for $t\in(0,\tau]$ starting from θ_0.

B. Second phase: flatness-based control on $[\tau,T]$

We now show the state $\theta_\tau(x):=\theta(x,\tau)$ reached at the end of the first phase can be steered to 0. To this end, we now seek θ in the form

$$\theta(x,t) := \sum_{i\geq0}g_i(x)y^{(i)}(t), \tag{11}$$

where the generating functions g_i are solutions of the sequence of Cauchy problems

$$(ag_0')'=0 \tag{12}$$
$$\alpha_0g_0(0)+\beta_0(ag_0')(0)=0 \tag{13}$$
$$\beta_0g_0(0)-\alpha_0(ag_0')(0)=K \tag{14}$$

for some $K\neq0$, and

$$(ag_i')'=\rho g_{i-1} \tag{15}$$
$$\alpha_0g_i(0)+\beta_0(ag_i')(0)=0 \tag{16}$$
$$\beta_0g_i(0)-\alpha_0(ag_i')(0)=0 \tag{17}$$

for $i\geq0$. It can then be shown that for some $C,R>0$

$$\sup_{x\in[0,1]}|g_i(x)|\leq\frac{C}{R^i(i!)^2},\quad \forall i\geq0,$$

so that the series (11) converges as soon as $y\in G^s([\tau,T],\mathbb{R})$ with $s<2$.

On the other hand, each eigenfunction e_n can be expanded on the g_i's as $e_n=\zeta_n\sum_{i\geq0}(-\lambda_n)^ig_i$, with

$$\zeta_n := \frac{\beta_0e_n(0)-\alpha_0(ae_n')(0)}{\beta_0g_0(0)-\alpha_0(ag_0')(0)}.$$

Indeed, setting $f_n:=\zeta_n\sum_{i\geq0}(-\lambda_n)^ig_i$ we find

$$\begin{aligned}(af_n')' &= \zeta_n\sum_{i\geq0}(-\lambda_n)^i(ag_i')'\\ &= \zeta_n\sum_{i\geq1}(-\lambda_n)^i(\rho g_{i-1})\\ &= -\lambda_n\rho\zeta_n\sum_{j\geq0}(-\lambda_n)^jg_j\\ &= -\lambda_n\rho f_n\\ \alpha_0f_n(0)+\beta_0(af_n')(0) &= \zeta_n\big(\alpha_0g_0(0)+\beta_0(ag_0')(0)\big)=0\\ \beta_0f_n(0)-\alpha_0(af_n')(0) &= \zeta_n\big(\beta_0g_0(0)-\alpha_0(ag_0')(0)\big)\\ &= \beta_0e_n(0)-\alpha_0(ae_n')(0).\end{aligned}$$

Therefore $e_n - f_n$ satisfies a Cauchy problem with zero initial conditions, implying $e_n - f_n = 0$.

It is now easy to check that the control defined by

$$y(t) := \phi_s\Big(\frac{t-\tau}{T-\tau}\Big)\sum_{n\geq 0} c_n\zeta_n e^{-\lambda_n t} \tag{18}$$

$$u(t) := \sum_{i\geq 0}\big(\alpha_1 g_i(1) + \beta_1 g_i'(1)\big)y^{(i)}(t), \tag{19}$$

where ϕ_s is the "Gevrey step" of section III-C, steers the system from θ_τ at time τ to 0 at time T. Indeed, (11), (18), (19) is clearly the solution of (1)–(3); as $\phi_s^{(i)}(1) = 0$ for $i \geq 0$,

$$\theta(T,x) = \sum_{i\geq 0} y^{(i)}(T)g_i(x) = 0;$$

and as $\phi_s(0) = 1$ and $\phi_s^{(i)}(0) = 0$ for $i \geq 1$,

$$\begin{aligned}\theta_\tau(x) &= \sum_{n\geq 0} c_n e^{-\lambda_n\tau}e_n(x)\\ &= \sum_{n\geq 0} c_n e^{-\lambda_n\tau}\zeta_n\sum_{i\geq 0}(-\lambda_n)^i g_i(x)\\ &= \sum_{i\geq 0}\Big(\sum_{n\geq 0} c_n\zeta_n e^{-\lambda_n\tau}(-\lambda_n)^i\Big)g_i(x)\\ &= \sum_{i\geq 0} y^{(i)}(\tau)g_i(x).\end{aligned}$$

III. Explicit expressions

We now provide explicit expressions for the generating functions g_i, the eigenfunctions e_n and the derivatives of the "Gevrey step" ϕ_s. We set $\kappa_0 := \sqrt{\frac{\rho_0}{a_0}}$ and $\kappa_1 := \sqrt{\frac{\rho_1}{a_1}}$.

A. Generating functions

It is easily seen that the solution of (12)–(17) is

$$g_i(x) = \begin{cases} g_{i,0}(x), & 0 < x < X\\ g_{i,1}(x), & X < x < 1,\end{cases}$$

where

$$g_{i,0}(x) = K\Big(a_0\beta_0 - \frac{\alpha_0 x}{2i+1}\Big)\frac{(\kappa_0 x)^{2i}}{(2i)!}$$

$$g_{i,1}(x) = \sum_{j=0}^{i}\Big[g_{i-j,0}(X) + \frac{a_0}{a_1}\frac{(x-X)g'_{i-j,0}(X)}{2j+1}\Big]\frac{\big(\kappa_1(x-X)\big)^{2j}}{(2j)!}.$$

The quasiderivatives are $(ag'_{0,0})(x) = (ag'_{0,1})(x) = -a_0K\alpha_0$, and for $i \geq 1$,

$$(ag'_{i,0})(x) = a_0\kappa_0 K\Big(a_0\beta_0 - \frac{\alpha_0 x}{2i}\Big)\frac{(\kappa_0 x)^{2i-1}}{(2i-1)!}$$

$$\begin{aligned}(ag'_{i,1})(x) = (ag)'_{i,0}(x) + \kappa_1\sum_{j=1}^{i}\Big[&a_1 g_{i-j,0}(X)\\ &+ \frac{(x-X)(ag)'_{i-j,0}(X)}{2j}\Big]\frac{\big(\kappa_1(x-X)\big)^{2j-1}}{(2j-1)!}.\end{aligned}$$

Clearly, the initial conditions (13), (14), (16) and (17) are satisfied, as well as the interface conditions $g_{i,0}(X) = g_{i,1}(X)$ and $(ag'_{i,0})(X) = (ag'_{i,1})(X)$.

B. Eigenvalues and eigenfunctions

Since we have proved the Sturm-Liouville problem (8)–(10) may have nonzero solutions only for $\lambda > 0$, we set $\mu := \sqrt{\lambda}$. A candidate solution ϕ then reads

$$\phi(x) = \begin{cases}\phi_0(x), & 0 < x < X\\ \phi_1(x), & X < x < 1,\end{cases}$$

with

$$\begin{aligned}\phi_0(x) &= C_0\cos(\mu\kappa_0 x) + D_0\sin(\mu\kappa_0 x)\\ \phi_1(x) &= C_1\cos\big(\mu\kappa_1(x-X)\big) + D_1\sin\big(\mu\kappa_1(x-X)\big).\end{aligned}$$

The interface conditions $\phi_0(X) = \phi_1(X)$ and $(a\phi_0)'(X) = (a\phi_1)'(X)$ then yield (using $\mu \neq 0$)

$$C_1 = C_0\cos(\mu\kappa_0 X) + D_0\sin(\mu\kappa_0 X) \tag{20}$$

$$D_1 = \frac{a_0\kappa_0}{a_1\kappa_1}\big(D_0\cos(\mu\kappa_0 X) - C_0\sin(\mu\kappa_0 X)\big). \tag{21}$$

The boundary conditions (9), (10) read

$$\alpha_0 C_0 + \beta_0\mu\kappa_0 D_0 = 0 \tag{22}$$

$$(\alpha_1 c_1 - \mu\beta_1 a_1\kappa_1 s_1)C_1 + (\alpha_1 s_1 + \mu\beta_1 a_1\kappa_1 c_1)D_1 = 0, \tag{23}$$

where we have set

$$\begin{aligned}c_0 &:= \cos(\mu\kappa_0 X)\\ s_0 &:= \sin(\mu\kappa_0 X)\\ c_1 &:= \cos\big(\mu\kappa_1(1-X)\big)\\ s_1 &:= \sin\big(\mu\kappa_1(1-X)\big).\end{aligned}$$

Injecting (20), (21) into (22), (23) gives the linear system

$$\begin{pmatrix}\alpha_0 & \beta_0\mu\kappa_0\\ F(\mu) & G(\mu)\end{pmatrix}\begin{pmatrix}C_0\\ D_0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix},$$

which has a nonzero solution iff the determinant $f(\mu) := \alpha_0 G(\mu) - \beta_0\mu\kappa_0 F(\mu)$ is zero. Since

$$\begin{aligned}f(\mu) = (&\alpha_0\alpha_1 + \mu^2\beta_0\beta_1 a_0\rho_0)s_0c_1\\ &+ \Big(\alpha_0\alpha_1\frac{a_0\kappa_0}{a_1\kappa_1} + \mu^2\beta_0\beta_1 a_0\rho_0\frac{a_1\kappa_1}{a_0\kappa_0}\Big)c_0s_1\\ &+ \mu a_0\kappa_0\Big[(\alpha_0\beta_1 - \alpha_1\beta_0)c_0c_1\\ &- \Big(\alpha_0\beta_1\frac{a_1\kappa_1}{a_0\kappa_0} - \alpha_1\beta_0\frac{a_0\kappa_0}{a_1\kappa_1}\Big)s_0s_1\Big]\end{aligned}$$

is odd, and $\mu = 0$ is excluded, we are interested only in the strictly positive roots of $f(\mu) = 0$. There is no closed-form expression for these roots, so they must be determined numerically. In the simulations of section IV, we have for instance used the function `roots` from the open-source package Chebfun [12], which is very handy to find all the roots of a function in a given interval.

To each strictly positive root μ_n of f then corresponds the eigenfunction

$$e_n(x) = \begin{cases} e_{n,0}(x), & 0 < x < X\\ e_{n,1}(x), & X < x < 1,\end{cases}$$

with

$$e_{n,0}(x) = M_n\left[\beta_0 a_0 \cos(\mu_n\kappa_0 x) - \alpha_0\frac{\sin(\mu_n\kappa_0 x)}{\mu_n\kappa_0 x}\right]$$
$$e_{n,1}(x) = e_{n,0}(X)\cos(\mu_n\kappa_0 x) + \frac{a_0}{a_1}e'_{n,0}(X)\frac{\sin\big(\mu_n\kappa_1(x-X)\big)}{\mu_n\kappa_1};$$

the coefficient M_n is chosen so that $\langle e_n, e_n\rangle_\rho = 1$. The quasi-derivatives are

$$(ae'_{n,0})(x) = -M_n\big[\beta_0 a_0\mu_n\kappa_0\sin(\mu_n\kappa_0 x) + \alpha_0\cos(\mu_n\kappa_0 x)\big]$$
$$(ae'_{n,1})(x) = -e_{n,0}(X)\mu_n\kappa_0\sin(\mu_n\kappa_0 x)) + \frac{1}{a_1}(ae'_{n,0})(X)\cos\big(\mu_n\kappa_1(x-X)\big).$$

Clearly, the boundary conditions (6), (7) are satisfied, as well as the interface conditions $e_{n,0}(X) = e_{n,1}(X)$ and $(ae'_{n,0})(X) = (ae'_{n,1})(X)$.

Fig. 1 displays some eigenfunctions; though not obvious at first sight, these eigenfunctions are indeed orthogonal.

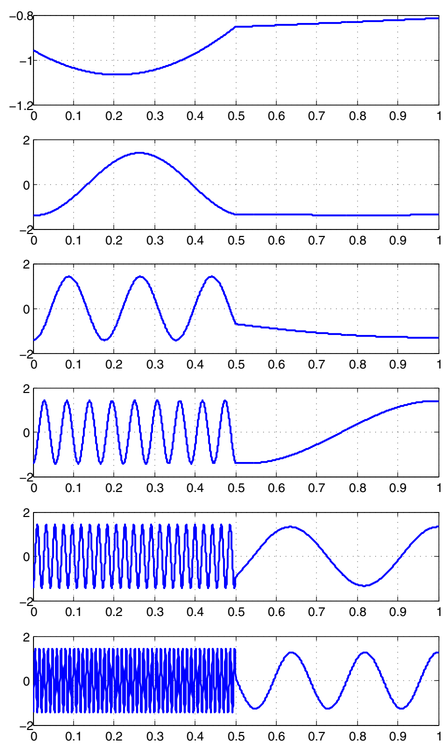

Fig. 1. Eigenfunctions $e_1, e_3, e_7, e_{20}, e_{50}, e_{100}$.

C. The Gevrey "step function" and its derivatives

It it well-known that the "bump function"

$$\varphi_s(t) := \begin{cases} 0 & \text{if } t \notin (0,1) \\ \exp\left(\frac{-1}{Mt^k(1-t)^k}\right) & \text{if } t \in (0,1), \end{cases}$$

where $k = (s-1)^{-1}$ and $M > 0$, is Gevrey of order s; and so is the Gevrey "step function"

$$\phi_s(t) := \begin{cases} 1 & \text{if } t \le 0 \\ 0 & \text{if } t \ge 1 \\ 1 - \frac{\int_0^t \varphi_s(\rho)d\rho}{\int_0^1 \varphi_s(\rho)d\rho} & \text{if } t \in (0,1), \end{cases}$$

used in (18) to design the flatness-based control. It is readily checked that $\phi_s(0) = 1$, $\phi_s^{(i)}(0) = 0$ for $i \ge 1$, and $\phi_s^{(i)}(1) = 0$ for $i \ge 0$.

A practical problem when implementing the control (19) is to evaluate sufficiently many derivatives of ϕ_s, i.e. of φ_s. This can be done easily as follows. We first note that

$$p^{k+1}\dot\varphi_s = k\dot p\varphi_s, \tag{24}$$

where $p(t) := M^{\frac{1}{k}}t(1-t)$ is a polynomial of degree 2 (hence its derivatives of order > 2 are zero). We then apply the general Leibniz rule

$$(uv)^{(i)} = \sum_{j=0}^{i}\binom{i}{j}u^{(i)}v^{(i-j)}$$

to both sides of (24), yielding

$$P\varphi_s^{(i+1)} + \sum_{j=1}^{i}\binom{i}{j}P^{(j)}\varphi_s^{(i+1-j)} = k\big(\dot p\varphi_s^{(i)} + i\ddot p\varphi_s^{(i-1)}\big). \tag{25}$$

This is a recursion formula giving $\varphi_s^{(i+1)}$ in function of $\varphi_s^{0)}, \dots, \varphi_s^{i)}$ and the derivatives of $P := p^{k+1}$. The derivatives of P are obtained in the same manner, by applying the Leibniz rule to both sides of

$$p\dot P = (k+1)\dot pP,$$

yielding the recursion formula

$$pP^{(i+1)} = (k+1-i)\dot pP^{(i)} + \frac{i}{2}(2k+3-i)\ddot pP^{(i-1)}. \tag{26}$$

To avoid computing ratios of very large numbers, it is in practice better to use recursion formulas for $\tilde P^{(i)}) := \frac{P^{(i)}}{i!}$ and $\tilde\varphi_s^{(i)}) := \frac{\varphi_s^{(i)}}{(2i)!}$. From (26) and (25), we find

$$p\tilde P^{(i+1)} = \frac{1}{i+1}\left[(k+1-i)\dot p\tilde P^{(i)} + \frac{2k+3-i}{2}\ddot p\tilde P^{(i-1)}\right]$$
$$\tilde P\tilde\varphi_s^{(i+1)} = \frac{k}{(2i+2)(2i+1)}\left[\dot p\tilde\varphi_s^{(i)} + \frac{\ddot p\tilde\varphi_s^{(i-1)}}{2(2i-1)}\right] - \sum_{j=1}^{i}d_{i+1,j}\tilde P^{(j)}\tilde\varphi_s^{(i+1-j)}$$
$$d_{i+1,0} = 1$$
$$d_{i+1,j} = \frac{i-j+1}{(2i-2j+4)(2i-2j+3)}d_{i+1,j-1}, \quad j = 1,\cdots,i.$$

Using this procedure, about 140 derivatives can be efficiently determined with Matlab double-precision arithmetics.

IV. Numerical experiments

We now show some numerical results, using as parameters

X	a_0	ρ_0	a_1	ρ_1	α_0	β_0	α_1	α_1
$\frac{1}{2}$	$\frac{10}{19}$	$\frac{15}{8}$	10	$\frac{1}{8}$	$\cos\frac{\pi}{3}$	$-\sin\frac{\pi}{3}$	$\cos\frac{\pi}{4}$	$\sin\frac{\pi}{4}$

The initial condition $\theta_0 \in L^2(0,1)$ is the step function $\theta_0(x) = -\frac{1}{2}$ on $(0,\frac{1}{2})$ and $\theta_0(x) = \frac{1}{2}$ on $(\frac{1}{2},1)$; θ_0 not being continuous, its coefficients c_n decay slowly. The final time is $T = 0.35$; several values of the intermediate time τ are used, namely $\tau = 0.01, 0.05, 0.1, 0.15$, to see its influence. Finally $s = 1.65$ and $M = 2$ for the base case (recall s and M are the coefficients in the Gevrey "step function"); theses values are later changed one at a time to see their influences. The series for u and y in (18) and (19) were truncated at a "large enough" order for a good accuracy, namely $\bar{i} = 130$ and $\bar{n} = 60$; a fairly large $\bar{i}$ is needed here because $(T-\tau)\frac{a_0}{\rho_0}$ is rather small. The error on the trajectory due to these truncations is expected to behave well, provided that the uniform estimates proved in [10] in a much simpler situation are generalizable. Also note that the control effort, as well as the truncation index $\bar{i}$ needed to ensure a good accuracy of u, grows rapidly as T decreases.

Fig. 2 shows the evolution of the control $u(t)$ and Fig. 3 the evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$; it appears that the control effort increases with τ (for a given T), with a more oscillatory behavior. Fig. 4 shows the resulting temperature θ (for $\tau = 0.05$ only); the discontinuity of θ_x at $x = X$ is clearly visible. Fig. 5 and Fig. 6 show the evolution of the control and of its energy when $M = 1$, and so do Fig. 7 and Fig. 8 when $s = 2.5$; it appears that the control effort decreases as M increases, but at the expense of a more oscillatory behavior (and a higher $\bar{i}$ for a good accuracy of u). Finally, Fig. 9 and Fig. 10 show the evolution of the control and of the control energy when $s = 1.45$, and so do Fig. 11 and Fig. 12 when $s = 1.5$; it appears that the control effort decreases as s increases, but at the expense of a more oscillatory behavior (and a higher $\bar{i}$ for a good accuracy of u). Of course these few qualitative remarks should be taken with caution, as they are yet to be backed by theory.

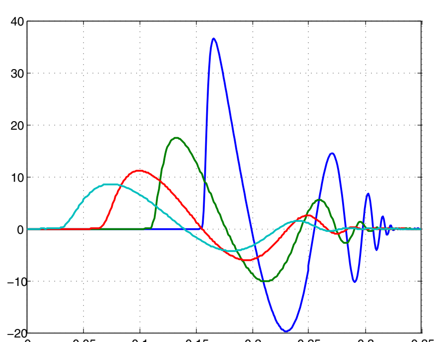

Fig. 2. Base case: evolution of the control $u(t)$.

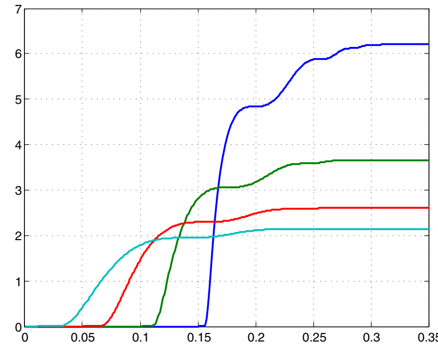

Fig. 3. Base case: evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$.

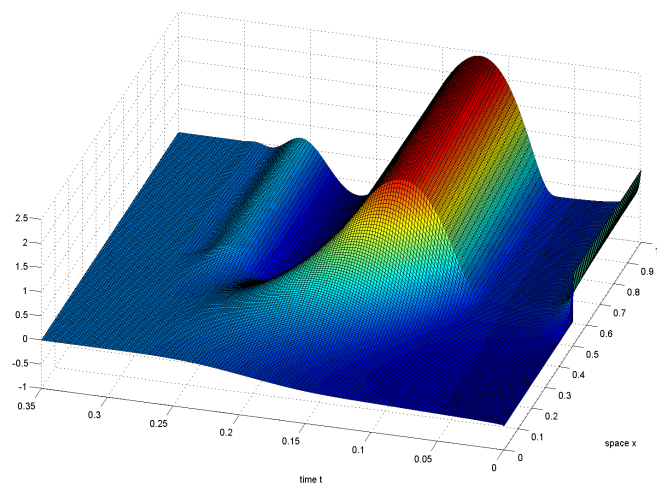

Fig. 4. Evolution of the temperature $\theta(t,x)$.

References

[1] H. Fattorini and D. Russell, "Exact controllability theorems for linear parabolic equations in one space dimension," *Arch. Rational Mech. Anal.*, vol. 43, no. 4, pp. 272–292, 1971.

[2] B. Jones Jr., "A fundamental solution for the heat equation which is supported in a strip," *J. Math. Anal. Appl.*, vol. 60, no. 2, pp. 314–324, 1977.

[3] W. A. J. Luxemburg and J. Korevaar, "Entire functions and Müntz-Szász type approximation," *Trans. Amer. Math. Soc.*, vol. 157, pp. 23–37, 1971.

[4] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, ser. Lecture Notes Series. Seoul National University, Research Institute of Mathematics, 1996, vol. 34.

[5] G. Alessandrini and L. Escauriaza, "Null-controllability of one-dimensional parabolic equations," *ESAIM Control Optim. Calc. Var.*, vol. 14, no. 2, pp. 284–293, 2008.

[6] A. Benabdallah, Y. Dermenjian, and J. Le Rousseau, "Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem," *J. Math. Anal. Appl.*, vol. 336, no. 2, pp. 865–887, 2007.

[7] E. Fernández-Cara and E. Zuazua, "On the null controllability of the one-dimensional heat equation with BV coefficients," *Comput. Appl. Math.*, vol. 21, no. 1, pp. 167–190, 2002.

[8] B. Laroche, P. Martin, and P. Rouchon, "Motion planning for the heat equation," *Int J Robust Nonlinear Control*, vol. 10, no. 8, pp. 629–643, 2000.

[9] T. Meurer, *Control of Higher–Dimensional PDEs: Flatness and Backstepping Designs*. Springer, 2012.

[10] P. Martin, L. Rosier, and P. Rouchon, "Null controllability of the heat equation using flatness," *Automatica*, vol. 50, no. 12, pp. 3067–3076, 2014.

[11] ——, "Null controllability of one-dimensional parabolic equations using flatness," 2014, arXiv:1410.2588, http://arxiv.org/abs/1410.2588.

[12] T. A. Driscoll, N. Hale, and L. N. Trefethen, Eds., *Chebfun Guide*. Oxford: Pafnuty Publications, 2014, http://www.chebfun.org/.

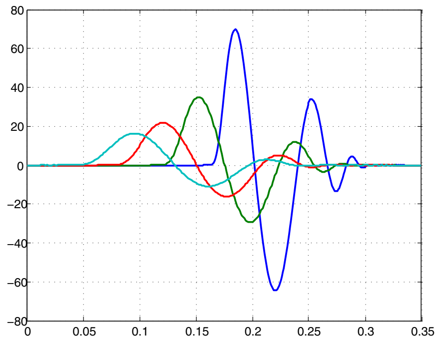

Fig. 5. $M = 1$: evolution of the control $u(t)$.

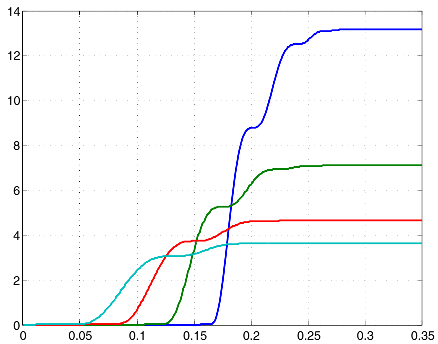

Fig. 6. $M = 1$: evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$.

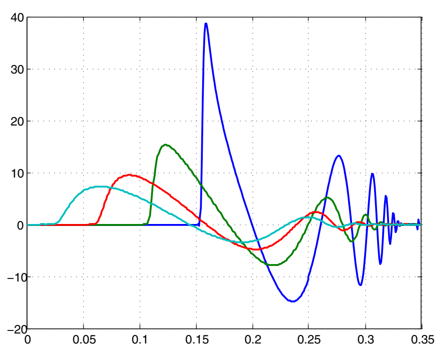

Fig. 7. $M = 2.5$: evolution of the control $u(t)$.

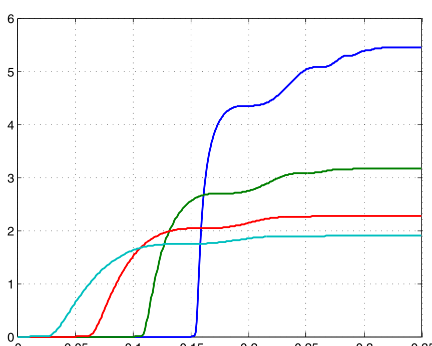

Fig. 8. $M = 2.5$: evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$.

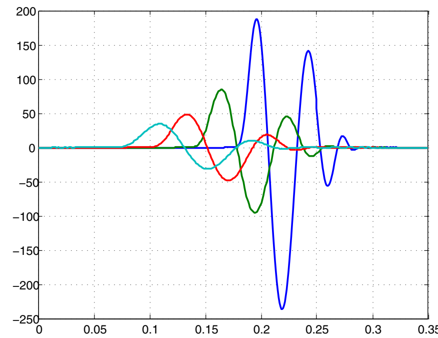

Fig. 9. $s = 1.45$: evolution of the control $u(t)$.

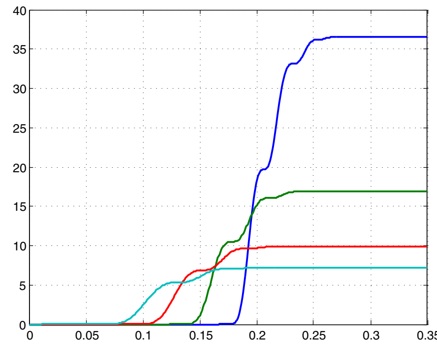

Fig. 10. $s = 1.45$: evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$.

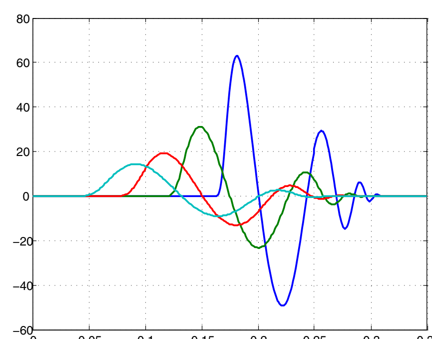

Fig. 11. $s = 1.55$: evolution of the control $u(t)$.

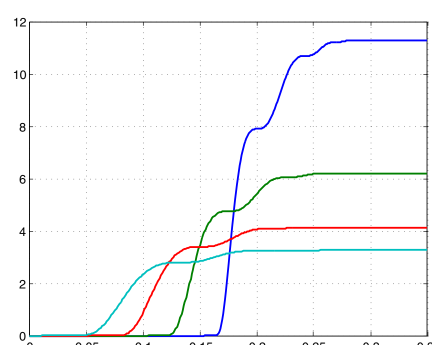

Fig. 12. $s = 1.55$: evolution of the control energy $\left(\int_0^t u^2(s)ds\right)^{\frac{1}{2}}$.